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# Uniqueness of Steady Symmetric Deep-Water Waves with Vorticity

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**Abstract**

Given a steady and symmetric deep-water wave we prove that the surface profile and the vorticity distribution determine the wave motion completely throughout the fluid.

## 1 Introduction

It is known that the open sea to a large extent is characterized by deep-water waves propagating on wind-generated currents [7, 8, 10, 11, 12, 13]. Looking at such a wave one may wonder to what extent the motion underneath the surface is in fact determined by the surface. The case of shallow water waves propagating over a flat bed was recently investigated [9]. The result of this note is that if the motion disappears reasonably fast as we approach large depths, then a symmetric wave is determined if we know not only its surface, but also the vorticity in the water. The condition that practically nothing happens deep down is supported by empirical data [7, 8, 10, 11]. Moreover, in our analysis we find that there is an invariant connected to the amount of water passing any vertical line in the water. For deep-water waves this entity corresponds to what in the setting of finite depth is called mass flux [4, 5].

Section 2 presents the setting in a more detailed manner and Section 3 includes the actual proof together with a comment on the invariant.

## 2 Formulation

We are looking at a wave on the open sea, periodic in the $x$-variable and traveling with speed $c > 0$. Deep-water waves are described within the setting of infinite depth, with no motion at great depths [1, 7, 11, 14]. We therefore require the horizontal and vertical velocity, $u$ and $v$ respectively, to vanish deep down. Moreover, if a wave is not near breaking, the horizontal fluid velocity within the water is considerably smaller than the wave speed [8]. For this reason, we assume $u < c$ throughout the fluid. Furthermore, we assume that the density of the water is constant.

We define the *stream function* $\psi(x, y)$ and *vorticity* $\omega$ by

$$\psi_x = -v, \quad \psi_y = u - c, \quad -\omega = \triangle\psi = u_y - v_x.$$

In our setting, the governing equations for water waves are equivalent to the following system (see e.g. [3]):

$$\begin{cases} \Delta\psi &= -\gamma(\psi) & \text{in} \quad -\infty < y < \eta(x), \\ |\nabla\psi|^2 + 2gy &= C & \text{on} \quad y = \eta(x), \\ \psi &= 0 & \text{on} \quad y = \eta(x), \\ \nabla\psi &\to (0, -c) & \text{as} \quad y \to -\infty \quad \text{uniformly for } x \in \mathbb{R}, \end{cases} \tag{2.1}$$

to be satisfied for $\eta \in C^3(\mathbb{R})$ and $\psi \in C^3(\overline{D}_\eta)$, both $L$-periodic in the $x$-variable. Here, $C$ is a constant, $\gamma(\psi) = \omega$, and $\psi$ is given by the explicit formula [1]

$$\psi(x, y) = \psi_0 - \int_0^x v(\xi, -d)\, d\xi + \int_{-d}^y [u(x, \xi) - c]\, d\xi.$$

where $\psi_0 \in \mathbb{R}$ is a constant and $d > 0$ is chosen so that the horizontal line $y = -d$ lies entirely within the fluid domain. Also, $\psi_0$ is chosen so that $\psi$ satisfies $\psi = 0$ on the surface. An explicit calculation shows that the expression $\Big(\psi(x + L, y) - \psi(x, y)\Big) = -\int_x^{x+L} v(\xi, -d)\, d\xi$ is a constant throughout the fluid, so that $\psi$ is periodic in the $x$-variable.

We require that the vorticty $\omega$ is non-increasing with depth, i.e.

$$\partial_y\omega \geq 0. \tag{2.2}$$

Recall that by assumption

$$\partial_y\psi = u - c < 0. \tag{2.3}$$

We also assume that the horizontal velocity vanishes fast with increasing depth or, for some $M > 0$ and $n_0 \in \mathbb{R}$,

$$|\ \partial_y\psi(x, y) + c\ | = |\ u(x, y)\ | \leq Me^y \quad \text{for all } x \in \mathbb{R} \quad \text{if } y < -n_0. \tag{2.4}$$

Note that the last condition in (1) encompasses the fact that $v$ vanishes at great depths.

## 3 Uniqueness of symmetric deep-water waves

**Proposition 1.** *Assume that $(\eta, u, v)$ is a symmetric deep-water wave. Also, assume that the vorticity decreases as depth increases and that the horizontal velocity vanishes exponentially fast at some depth, i.e. (1)-(4) hold. Then, under the condition that the amount of water passing some vertical line is given, $(\eta, \gamma)$ determines $(u, v)$ uniquely.*

[1] Using the homogeneity and the incompressibility assumptions on the fluid: $u_x + v_y = 0$.

**Proof.** $\psi$ is given by $\psi(x,y) = \psi_0 - \int_0^x v(\xi,-d)\,d\xi + \int_{-d}^y [u(x,\xi) - c]\,d\xi$.
In particular

$$\psi(0,y) + cy = \psi_0 + \int_{-d}^{y} u(0,\xi)\,d\xi - cd \to \underbrace{\psi_0 + \int_{-d}^{-\infty} u(0,\xi)\,d\xi - cd}_{\alpha}$$

as $y \to -\infty$, since $u(0,y)$ is integrable on $(-\infty,\eta)$ by assumption.

Then by periodicity of $\psi$, the Mean value theorem and properties of $v$ we have that

$$| \psi(x,y) - (\alpha - cy) | \leq | \psi(x,y) - \psi(0,y) | + | \psi(0,y) - (\alpha - cy) | =$$
$$= \underbrace{| \psi_x(\xi,y) |}_{-v(\xi,y)} \underbrace{| x - 0 |}_{\leq L} + | \psi(0,y) - (\alpha - cy) | \to 0 \text{ uniformly in } x \text{ as } y \to -\infty.$$

What is this $\alpha = \psi_0 + \int_{-d}^{-\infty} u(0,\xi)\,d\xi - cd$? A straightforward calculation shows that $\int_{-d}^{\eta(x)} u(x,y)\,dy - c\eta(x) = \int_{-d}^{\eta(x)} \big(\psi_y(x,y) + c\big)\,dy - c\eta(x) = \big[\psi(x,y)\big]_{y=-d}^{y=\eta(x)} + cd \to -\alpha$ as $d \to \infty$ by choice of $\psi$, so that

$$\int_{-\infty}^{\eta(x)} u(x,y)\,dy - c\eta(x) = -\alpha, \quad x \in \mathbb{R}.$$

$\alpha$ is consequently determined by the surface and the horizontal velocity component in the fluid, and is invariant of $x$. It is a measure of the amount of water passing any vertical line in the fluid.

Now suppose that both $\psi$ and $\tilde{\psi}$ satisfy (1-4) of Section 2. Since the amount of water passing at least some vertical line in the water is determined, and since this has been shown to be the same for all such lines, $y = \eta(x)$ being given yields $\alpha = \tilde{\alpha}$. We have $\psi - \tilde{\psi} = 0$ on the free surface $y = \eta(x)$ and using the above asymptotic behaviour valid at great depths,by triangle inequality $| \psi - \tilde{\psi} | < \epsilon$ if $y < -n$ for some large $n$. Also, $\triangle(\psi - \tilde{\psi}) = -(\gamma(\psi) - \gamma(\tilde{\psi})) = -\gamma'(\xi)(\psi - \tilde{\psi})$ so that if $\psi - \tilde{\psi} \neq 0$, $\psi - \tilde{\psi}$ attains its maximum in an interior point of $\big\{(x,y); -L < x < L, -\infty < y < \eta(x)\big\}$, interior since $\psi - \tilde{\psi}$ is periodic in the $x$-variable. Since $\gamma' \leq 0$ by (2), applying the strong Maximum principle (see e.g. [6]) we attain $\psi \equiv \tilde{\psi}$, and thus $(u,v)$ is determined by $(\eta,\gamma)$. ■

**Remark. i)** We remark that $\alpha$ corresponds to the notion of mass flux, which in the case of finite depth is the expression $\int_{-d}^{\eta(x)} (u(x,y) - c)\,dy$, constant in the $x$-variable[2] (see [4, 5]).

**ii)** The symmetry assumption on the wave is by no means necessary for our considerations. The reason why it is included is that under minimal prerequisits on the shape of the surface, the waves are bound to be symmetric (see [2, 3]).

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[2] Here $y = -d$ is the flat bottom.

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