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The Structure of Gelfand-Levitan-Marhenko Type Equations for Delsarte Transmutation Operators of Linear Multidimensional Differential Operators and Operator Pencils. Part 1.

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Abstract

An analog of Gelfand-Levitan-Marchenko integral equations for multi- dimensional Delsarte transmutation operators is constructed by means of studying their differential-geometric structure based on the classical Lagrange identity for a formally conjugated pair of differential operators. An extension of the method for the case of affine pencils of differential operators is suggested.

1 Introduction

Consider the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m; \mathbb{C}^N)$, $m, N \in \mathbb{Z}_+$, and the correspondingly conjugated pair $\mathcal{H}^* \times \mathcal{H}$ on which one can define the natural scalar product

$$(\varphi, \psi) = \int_{\mathbb{R}^m} dx < \varphi, \psi >:= \int_{\mathbb{R}^m} dx \bar{\varphi}^\intercal(x)\psi(x), \tag{1.1}$$

where $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, the sign " $-$ " denotes the complex conjugation and the sign " $\intercal$ " denotes the standard matrix transposition. Take also two linear densely defined differential operators L and $\tilde{L} : \mathcal{H} \to \mathcal{H}$ and some two closed functional subspaces $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0 \subset \mathcal{H}_-$, where $\mathcal{H}_-$ is the negative Hilbert space from a Gelfand triple

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \tag{1.2}$$

of the correspondingly Hilbert-Schmidt rigged [22, 1] Hilbert space $\mathcal{H}$. We will use further on the following definition.

Definition 1. (see J. Delsarte and J. Lions [2, 3]). A linear invertible operator $\mathbf{\Omega}$, defined in $\mathcal{H}$ and acting from $\mathcal{H}_0$ onto $\tilde{\mathcal{H}}_0$, is called a Delsarte transmutation operator for a pair of linear differential operators L and $\tilde{L} : \mathcal{H} \longrightarrow \mathcal{H}$, if the following two conditions hold:
- the operator $\mathbf{\Omega}$ and its inverse $\mathbf{\Omega}^{-1}$ are continuous in $\mathcal{H}$;
- the operator identity

$$\tilde{L}\mathbf{\Omega} = \mathbf{\Omega}L \tag{1.3}$$

is satisfied.

Such transmutation operators were introduced in [2, 3] for the case of one-dimensional differential operators. In particular, for the Sturm-Liouville and Dirac operators the complete structure of the corresponding Delsarte transmutation operators was described in [5, 6, 4], where also extensive applications to spectral theory were done. As it was shown in [5, 7, 4], for the case of one-dimensional differential operators, an important part in the theory of Delsarte transmutation operators is played by special integral Gelfand-Levitan-Marchenko (GLM) equations [20, 4, 5], whose solutions are kernels of the corresponding Delsarte transmutation operators. Some results for two-dimensional Dirac and Laplace type operators, were also obtained in [17, 7] .

In the present work, based on the results of [10, 9, 11, 14], we shall construct for a pair of multi-dimensional differential operators acting in a Hilbert space $\mathcal{H}$ a special pair of conjugated Delsarte transmutation operators $\mathbf{\Omega}_+$ and $\mathbf{\Omega}_-$ in $\mathcal{H}$ and a pair $\mathbf{\Omega}_+^{\circledast}$ and $\mathbf{\Omega}_-^{\circledast}$ in $\mathcal{H}^*$ parametrized by two pairs of closed subspaces $\mathcal{H}_0$, $\tilde{\mathcal{H}}_0 \subset \mathcal{H}_-$ and $\mathcal{H}_0^*$, $\tilde{\mathcal{H}}_0^* \subset \mathcal{H}_-^*$, so that the operators $\Phi(\mathbf{\Omega}) := \mathbf{\Omega}_+^{-1}\mathbf{\Omega}_- - \mathbf{1}$ from $\mathcal{H}$ to $\mathcal{H}$ and $\Phi(\mathbf{\Omega})^{\circledast} := \mathbf{\Omega}_+^{\circledast,-1}\mathbf{\Omega}_-^{\circledast} - \mathbf{1}$ from $\mathcal{H}^*$ to $\mathcal{H}^*$ are compact ones of Hilbert-Shmidt type, thereby determining via the equalities

$$\mathbf{\Omega}_+(1 + \mathbf{\Phi}(\mathbf{\Omega})) = \mathbf{\Omega}_-, \quad \mathbf{\Omega}_+^{\circledast}(1 + \mathbf{\Phi}(\mathbf{\Omega})^{\circledast}) = \mathbf{\Omega}_-^{\circledast} \tag{1.4}$$

the corresponding analogs of GLM-equations, taking into account that supports of both kernels of integral operators $\mathbf{\Omega}_+, \mathbf{\Omega}_-$ and $\mathbf{\Omega}_+^{\circledast}$, $\mathbf{\Omega}_-^{\circledast}$ are correspondingly disjoint. Moreover, the following important expressions

$$\begin{aligned} \mathbf{\Omega}_+ L \mathbf{\Omega}_+^{-1} &= \tilde{L} = \mathbf{\Omega}_- L \mathbf{\Omega}_-^{-1}, \\ (\mathbf{1} + \mathbf{\Phi}(\mathbf{\Omega}))L &= L(\mathbf{1} + \mathbf{\Phi}(\mathbf{\Omega})), \quad (\mathbf{1} + \mathbf{\Phi}(\mathbf{\Omega})^{\circledast})L^* = L^*(\mathbf{1} + \mathbf{\Phi}(\mathbf{\Omega})^{\circledast}) \end{aligned} \tag{1.5}$$

hold. As in the classical case [4, 5, 20], the solutions to this GLM-equation also give rise to kernels of the corresponding Delsarte transmutation operators $\mathbf{\Omega}_\pm$ in $\mathcal{H}$, that are very important [1, 21] for diverse applications.

Another trend of this work is related with a similar problem of constructing Delsarte transmutation operators and corresponding integral GLM-equations for affine pencils of linear multi-dimensional differential operators in $\mathcal{H}$, having important applications, in particular, for the inverse spectral problem and feedback control theory [8].

2 Generalized Lagrangian identity, its differential-geometric structure and Delsarte transmutation operators

Consider a multi-dimensional differential operator $L : \mathcal{H} \longrightarrow \mathcal{H}$ of order $n(L) \in \mathbb{Z}_+$:

$$L(x;\partial) := \sum_{|\alpha|=0}^{n(L)} a_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{2.1}$$

defined on a dense subspace $D(L) \subset \mathcal{H}$, where, as usual, one assumes that coefficients $a_\alpha \in \mathcal{S}(\mathbb{R}^m; End\mathbb{C}^N)$, $\alpha \in \mathbb{Z}_+^m$ is a multi-index, $|\alpha| = \overline{0, n(L)}$, and $x \in \mathbb{R}^m$. The operator formally conjugated to (2.1) $L^* : \mathcal{H}^* \longrightarrow \mathcal{H}^*$ is of the form

$$L^*(x;\partial) := \sum_{|\alpha|=0}^{n(L)} (-1)^{|\alpha|}(\frac{\partial^{|\alpha|}}{\partial x^\alpha} \cdot \bar{a}_\alpha(x)), \tag{2.2}$$

$x \in \mathbb{R}^m$, and the dot " $\cdot$ " above denotes the usual composition of operators.

As to the standard semilinear form $< \cdot, \cdot >$ on $\mathbb{C}^N \times \mathbb{C}^N$ one can write down easily the following generalized Lagrangian identity:

$$< L^*\varphi, \psi > - < \varphi, L\psi >= \sum_{i=1}^{m}(-1)^{i+1}\frac{\partial}{\partial x_i}Z_i[\varphi, \psi], \tag{2.3}$$

where for any pair $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$ the expressions $Z_i[\varphi, \psi]$, $i = \overline{1, m}$, being semi-linear on $\mathcal{H}^* \times \mathcal{H}$. Multiplying (2.3) by the oriented Lebesgue measure $dx := \underset{j=\overrightarrow{1,m}}{\wedge} dx_j$, we easily get that

$$[< L^*\varphi, \psi > - < \varphi, L\psi >]dx = dZ^{(m-1)}[\varphi, \psi], \tag{2.4}$$

where

$$Z^{(m-1)}[\varphi, \psi] := \sum_{i=1}^{m} dx_1 \wedge dx_2 \wedge ... \wedge dx_{i-1} \wedge Z_i[\varphi, \psi]dx_{i+1} \wedge ... \wedge dx_m \tag{2.5}$$

is a $(m-1)$-differential form [12, 13] on $\mathbb{R}^m$ with values in $\mathbb{C}$.

Consider now a pair $(\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0 \subset \mathcal{H}_-^* \times \mathcal{H}_-$, where

$$\begin{aligned} \mathcal{H}_0 &:= \{\psi(\xi) \in \mathcal{H}_- : L\psi(\xi) = 0, \quad \psi(\xi)|_\Gamma = 0,\ \xi \in \Sigma \subset \mathbb{C}^p\}, \\ \mathcal{H}_0^* &:= \{\varphi(\eta) \in \mathcal{H}_-^* : L^*\varphi(\eta) = 0, \quad \varphi(\eta)|_\Gamma = 0,\ \eta \in \Sigma \subset \mathbb{C}^p\}, \end{aligned} \tag{2.6}$$

with $\Sigma \subset \mathbb{C}^p$ being some "spectral" parameter space, $\Gamma \subset \mathbb{R}^m$ being some $(m-1)$-dimensional hypersurface piecewise smoothly imbedded into $\mathbb{R}^m$, and $\mathcal{H}_-^* \supset \mathcal{H}^*$, $\mathcal{H}_- \supset \mathcal{H}$, being as before the correspondingly Hilbert-Schmidt rigged [1, 22, 20] Hilbert spaces, containing so called generalized eigenfunctions of the operators L^* and L. Thereby, for any pair $(\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0$ one gets from (2.4) that the differential $(m-1)$-form $Z^{(m-1)}[\varphi, \psi]$ is closed in the Grassmann algebra $\Lambda(\mathbb{R}^m; \mathbb{C})$. As a result, from the Poincaré lemma [12, 13] one finds that there exists an $(m-2)$-differential form $\Omega^{(m-2)}[\varphi, \psi] \in \Lambda^{m-2}(\mathbb{R}^m; \mathbb{C})$ semilinearly depending on $\mathcal{H}_0^* \times \mathcal{H}_0$, such that

$$Z^{(m-1)}[\varphi, \psi] = d\Omega^{(m-2)}[\varphi, \psi]. \tag{2.7}$$

Now making use of the expression (2.7), Stokes' theorem [12, 13] implies that

$$\begin{aligned} &\int_{\mathcal{S}_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\eta), \psi(\xi)] = \\ &\int_{\sigma_x^{(m-2)}} \Omega^{(m-2)}[\varphi(\eta), \psi(\xi)] - \int_{\sigma_{x_0}^{(m-2)}} \Omega^{(m-2)}[\varphi(\eta), \psi(\xi)] := \\ &\Omega_x(\eta, \xi) - \Omega_{x_0}(\eta, \xi), \end{aligned} \tag{2.8}$$

for all $(\eta, \xi) \in \Sigma \times \Sigma$, where an $(n-1)$-dimensional hypersurface $\mathcal{S}_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \subset \mathbb{R}^m$ with the boundary $\partial\mathcal{S}_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) = \sigma_x^{(m-2)} - \sigma_{x_0}^{(m-2)}$ is defined as a film spanning, in some way, between two $(m-2)$-dimensional homological nonintersecting each other cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \subset \mathbb{R}^m$, parametrized, respectively , by some arbitrary but fixed points x and $x_0 \in \mathbb{R}^m$. The quantities $\Omega_x(\eta, \xi)$ and $\Omega_{x_0}(\eta, \xi)$, $(\eta, \xi) \in \Sigma \times \Sigma$, obtained above have to be considered naturally as the corresponding kernels [1, 19, 20] of bounded Hilbert-Schmidt type integral operators $\Omega_x, \Omega_{x_0} : H \to H$, where $H := L_2^{(\rho)}(\Sigma; \mathbb{C})$ is a Hilbert space of functions on Σ measurable with respect to a finite Borel measure ρ on Borel subsets of Σ, and satisfying the following weak regularity condition

$$\lim_{x \to x_0} \Omega(\eta, \xi)] = \Omega_{x_0}(\eta, \xi) \tag{2.9}$$

for any pair $(\varphi(\eta), \psi(\xi)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $(\eta, \xi) \in \Sigma \times \Sigma$.

Now we are just as with [9, 10, 11] in a position to construct the corresponding pair of spaces $\tilde{\mathcal{H}}_0^*$ and $\tilde{\mathcal{H}}_0 \subset \mathcal{H}$, related with a Delsarte transformed linear differential operator $\tilde{L} : \mathcal{H} \longrightarrow \mathcal{H}$ and its conjugated expression $\tilde{L}^* : \mathcal{H}^* \longrightarrow \mathcal{H}^*$,

$$\tilde{L}(x; \partial) := \sum_{|\alpha|=0}^{n(\tilde{L})} \tilde{a}_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{2.10}$$

with coefficients $\tilde{a}_\alpha \in \mathcal{S}(\mathbb{R}^m; End\mathbb{C}^N)$, $\alpha \in \mathbb{Z}_+^m$ is a multi-index, $|\alpha| = \overline{0, n(\tilde{L})}$, $x \in \mathbb{R}^m$, under the condition that $n(\tilde{L}) = n(L) \in \mathbb{Z}_+$ be fixed. Namely, let closed subspaces $\tilde{\mathcal{H}}_0^* \subset \tilde{\mathcal{H}}_-^*$ and $\tilde{\mathcal{H}}_0 \subset \tilde{\mathcal{H}}_-$ be defined as

$$\begin{aligned} &\tilde{\mathcal{H}}_0 := \{\tilde{\psi}(\xi) \in \mathcal{H}_- : \tilde{\psi}(\xi) = \psi(\xi) \cdot \Omega_x^{-1}\Omega_{x_0}, \\ &\quad (\varphi(\eta), \psi(\xi)) \in \mathcal{H}_0^* \times \mathcal{H}_0, \ (\eta, \xi) \in \Sigma \times \Sigma\}, \\ &\tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}(\eta) \in \mathcal{H}_-^* : \tilde{\varphi}(\eta) = \varphi(\eta) \cdot \Omega_x^{\circledast, -1}\Omega_{x_0}^{\circledast}, \\ &\quad (\varphi(\eta), \psi(\xi)) \in \mathcal{H}_0^* \times \mathcal{H}_0, \ (\eta, \xi) \in \Sigma \times \Sigma\}\}. \end{aligned} \tag{2.11}$$

Here, similarly to (2.8), we defined the kernels of bounded invertible integral operators $\Omega_x^{\circledast}$ and $\Omega_{x_0}^{\circledast} : H \longrightarrow H$ as follows:

$$\begin{aligned}
&\int_{\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[\varphi(\eta),\psi(\xi)] \\
&= \int_{\sigma_x^{(m-2)}} \bar{\Omega}^{(m-2),\intercal}[\varphi(\eta),\psi(\xi)] - \int_{\sigma_{x_0}^{(m-2)}} \bar{\Omega}^{(m-2),\intercal}[\varphi(\eta),\psi(\xi)] \\
&:= \Omega_x^{\circledast}(\eta,\xi) - \Omega_{x_0}^{\circledast}(\eta,\xi)
\end{aligned} \tag{2.12}$$

for all $(\eta,\xi) \in \Sigma \times \Sigma$, where homological $(m-2)$-cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \subset \mathbb{R}^m$ are the same as taken above. Thereby, making use of the classical method of variation of constants as in [10, 14, 9], one gets easily from (2.12) that for any $(\varphi(\eta),\psi(\xi)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $(\eta,\xi) \in \Sigma \times \Sigma$,

$$\tilde{\psi}(\xi) = \mathbf{\Omega}_+\psi(\xi), \qquad \tilde{\varphi}(\eta) = \mathbf{\Omega}_+^{\circledast}\varphi(\eta), \tag{2.13}$$

where the integral expressions

$$\begin{aligned}
\mathbf{\Omega}_+ &:= \mathbf{1} - \int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\tilde{\psi}(\xi)\Omega_{x_0}^{-1}(\xi,\eta)\int_{\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\eta),\cdot], \\
\mathbf{\Omega}_+^{\circledast} &:= \mathbf{1} - \int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\tilde{\varphi}(\eta)\Omega_{x_0}^{\circledast,-1}(\xi,\eta)\int_{\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[\cdot,\psi(\xi)]
\end{aligned} \tag{2.14}$$

are bounded Delsarte transmutation operators of Volterra type defined on the whole spaces $\mathcal{H}$ and $\mathcal{H}^*$, respectively.

Now, based on operator expressions (2.14) and the definition (1.3), one easily obtaines the expressions for the Delsarte transformed operators $\tilde{L}$ and $\tilde{L}^*$:

$$\begin{aligned}
\tilde{L} &= \mathbf{\Omega}_+ L\mathbf{\Omega}_+^{-1} = L + [\mathbf{\Omega}_+, L]\mathbf{\Omega}_+^{-1}, \\
\tilde{L}^* &= \mathbf{\Omega}_+^{\circledast} L\mathbf{\Omega}_+^{\circledast,-1} = L^* + [\mathbf{\Omega}_+^{\circledast}, L^*]\mathbf{\Omega}_+^{\circledast,-1}.
\end{aligned} \tag{2.15}$$

Here also note that the transformations similar to the above in the one-dimensional case were studied in [4, 20, 5, 6]. They satisfy evidently the following easily found conditions:

$$\tilde{L}\tilde{\psi} = 0, \qquad \tilde{L}^*\tilde{\varphi} = 0 \tag{2.16}$$

for any pair $(\tilde{\varphi},\tilde{\psi}) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, which can be specified by constraints

$$\tilde{\psi}|_{\tilde{\Gamma}} = 0, \qquad \tilde{\varphi}|_{\tilde{\Gamma}^*} = 0 \tag{2.17}$$

for some hypersurface $\tilde{\Gamma} \subset \mathbb{R}^m$, related with the previously chosen hypersurface $\Gamma \subset \mathbb{R}^m$ and the homological pair of $(m-2)$-dimensional cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \subset \mathbb{R}^m$. Thereby, the closed subspaces $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_0^*$ can be re-defined similarly to (2.6) :

$$\begin{aligned}
\tilde{\mathcal{H}}_0 &:= \{\tilde{\psi}(\xi) \in \mathcal{H}_- : \tilde{L}\psi(\xi) = 0, \quad \tilde{\psi}(\xi)|_{\tilde{\Gamma}} = 0,\ \xi \in \Sigma \subset \mathbb{C}^p\}, \\
\tilde{\mathcal{H}}_0^* &:= \{\tilde{\varphi}(\eta) \in \mathcal{H}_-^* : \tilde{L}^*\tilde{\varphi}(\eta) = 0, \quad \tilde{\varphi}(\eta)|_{\tilde{\Gamma}} = 0,\ \eta \in \Sigma \subset \mathbb{C}^p\}
\end{aligned} \tag{2.18}$$

Moreover, the following lemma, based on a pseudo-differential operators technique from [1, 20, 14], holds.

Lemma 1. *The Delsarte transformed operators (2.14) obtained via (2.13), are differential if the operator $L:\mathcal{H}\longrightarrow\mathcal{H}$ is taken differential.*

As a simple consequence of the structure of the Delsarte transformed operators (2.14) one finds that for any pair $(\tilde{\varphi},\tilde{\psi})\in\tilde{\mathcal{H}}_0^*\times\tilde{\mathcal{H}}_0$ the following differential forms equality holds:

$$\tilde{Z}^{(m-1)}[\tilde{\varphi},\tilde{\psi}]=d\tilde{\Omega}^{(m-2)}[\tilde{\varphi},\tilde{\psi}], \tag{2.19}$$

where, by definition, a pair $(\tilde{\varphi},\tilde{\psi})\in\tilde{\mathcal{H}}_0^*\times\tilde{\mathcal{H}}_0$ is fixed and the equality

$$\Big(<\tilde{L}^*\tilde{\varphi},\tilde{\psi}>-<\tilde{\varphi},\tilde{L}\tilde{\psi}>\Big)\,dx=d\tilde{Z}^{(m-1)}[\tilde{\varphi},\tilde{\psi}] \tag{2.20}$$

holds. The equality (2.18) makes it possible to construct the corresponding kernels

$$\begin{aligned}&\tilde{\Omega}_x(\eta,\xi):=\int_{\sigma_x^{(m-2)}}\tilde{\Omega}^{(m-2)}[\tilde{\varphi}(\eta),\tilde{\psi}(\xi)],\\ &\tilde{\Omega}_{x_0}(\eta,\xi)]:=\int_{\sigma_{x_0}^{(m-2)}}\tilde{\Omega}^{(m-2)}[\tilde{\varphi}(\eta),\tilde{\psi}(\xi)]\end{aligned} \tag{2.21}$$

of bounded integral invertible Hilbert-Schmidt operators $\tilde{\Omega}_x,\tilde{\Omega}_{x_0}:H\rightarrow H$, and corresponding kernels

$$\begin{aligned}&\tilde{\Omega}_x^{\circledast}(\eta,\xi):=\int_{\sigma_x^{(m-2)}}\bar{\tilde{\Omega}}^{(m-2),\intercal}[\tilde{\varphi}(\eta),\tilde{\psi}(\xi)],\\ &\tilde{\Omega}_{x_0}^{\circledast}(\eta,\xi):=\int_{\sigma_{x_0}^{(m-2)}}\bar{\tilde{\Omega}}^{(m-2,\intercal)}[\tilde{\varphi}(\eta),\tilde{\psi}(\xi)]\end{aligned} \tag{2.22}$$

of bounded integral invertible Hilbert-Schmidt operators $\tilde{\Omega}_x,\tilde{\Omega}_{x_0}:H^*\rightarrow H^*$. Then the following equalities hold for all mutually related pairs $(\varphi,\psi)\in\mathcal{H}_0^*\times\mathcal{H}_0$ and $(\tilde{\varphi},\tilde{\psi})\in\tilde{\mathcal{H}}_0^*\times\tilde{\mathcal{H}}_0$:

$$\begin{aligned}&\psi(\xi)=\tilde{\psi}(\xi)\cdot\tilde{\Omega}_x^{-1}\tilde{\Omega}_{x_0},\\ &\varphi(\eta)=\tilde{\varphi}(\eta)\cdot\tilde{\Omega}_x^{\circledast,-1}\tilde{\Omega}_{x_0}^{\circledast}\end{aligned} \tag{2.23}$$

where $(\eta,\xi)\in\Sigma\times\Sigma$. Thus, based on the symmetry property between relations (2.10) and (2.22), one easily finds from expression (2.23), that expressions

$$\begin{aligned}&\mathbf{\Omega}_+^{-1}:=\mathbf{1}-\int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\psi(\xi)\tilde{\Omega}_{x_0}^{-1}(\xi,\eta)\int_{\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\tilde{Z}^{(m-1)}[\tilde{\varphi}(\eta),\cdot],\\ &\mathbf{\Omega}_+^{\circledast,-1}:=\mathbf{1}-\int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\varphi(\eta)\tilde{\Omega}_{x_0}^{\circledast,-1}(\xi,\eta)\int_{\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\bar{\tilde{Z}}^{(m-1),\intercal}[\cdot,\tilde{\psi}(\xi)]\end{aligned} \tag{2.24}$$

for some homological $(m-2)$-dimensional cycles $\tilde{\sigma}_x^{(m-2)},\tilde{\sigma}_{x_0}^{(m-2)}\subset\mathbb{R}^m$ are Delsarte transmutation integral operators of Volterra type inverse to (2.14), satisfying the following relationships:

$$\psi(\xi)=\mathbf{\Omega}_+^{-1}\cdot\tilde{\psi}(\xi),\qquad\varphi(\eta)=\mathbf{\Omega}_+^{*,-1}\cdot\tilde{\varphi}(\eta) \tag{2.25}$$

for all arbitrary but fixed pairs of functions $(\varphi(\eta),\psi(\xi))\in\mathcal{H}_0^*\times\mathcal{H}_0$ and $(\tilde{\varphi}(\eta),\tilde{\psi}(\xi))\in\tilde{\mathcal{H}}_0^*\times\tilde{\mathcal{H}}_0$, $(\eta,\xi)\in\Sigma\times\Sigma$. Thus, one can formulate the following characterization of the constructed Delsarte transmutation operators.

Theorem 1. *Let a matrix multi-dimensional differential operator (2.1) acting in a Hilbert space* $\mathcal{H} = L_2(\mathbb{R}^m; \mathbb{C}^N)$ *and its formally adjoint operator (2.2) acting in a Hilbert space* $\mathcal{H}^* = L_2^*(\mathbb{R}^m; \mathbb{C}^N)$*, possess, respectively, a pair of closed spaces* $\mathcal{H}_0$ *and* $\mathcal{H}_0^*$ *(2.6) of their generalized kernel eigenfunctions parametrized by some set* $\Sigma \subset \mathbf{C}^p$*. Then there exist bounded invertible Delsarte transmutation integral operators* $\mathbf{\Omega}_+ : \mathcal{H} \longrightarrow \mathcal{H}$ *and* $\Omega_+^{\circledast} : \mathcal{H}^* \longrightarrow \mathcal{H}^*$*, so that for this pair* $(\mathcal{H}_0, \mathcal{H}_0^*)$ *of closed subspaces (2.6) and their dual (2.18) the bounded invertible mappings (2.14)* $\mathbf{\Omega}_+ : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0$ *and* $\mathbf{\Omega}_+^{\circledast} : \mathcal{H}_0^* \to \tilde{\mathcal{H}}_0^*$ *are compatibly defined. Moreover, the operator expressions (2.15) are also differential, acting in the corresponding spaces* $\mathcal{H}$ *and* $\mathcal{H}^*$.

The above structure of the Delsarte transmutation operators (2.14) makes it possible to understand more deeply their properties by means of deriving new integral equations that are multi-dimensional analogs of the well known Gelfand-Levitan-Marchenko equations [4, 5, 20, 7, 6] to be a topic of the next section. We will only mention here that our approach devised above is a special case of the general De Rham-Hodge-Skrypnik theory [15, 16] of differential complexes on metric spaces recently successfully adapted to construction of Delsarte transmutaion operators in multi-dimension.

3 Multi-dimensional Gelfand-Levitan-Marchenko type integral equations

While investigating the inverse scattering problem for a three-dimensional perturbed Laplace operator

$$L(x; \partial) = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + q(x), \tag{3.1}$$

with $q \in W_2^2(\mathbb{R}^3)$, $x \in \mathbb{R}^3$, in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C})$, L.D. Faddeev [17] suggested studing the structure of corresponding Delsarte transmutation operators $\mathbf{\Omega}_\gamma : \mathcal{H} \to \mathcal{H}$ of Volterra type, based on a priori chosen half-spaces $S_{\pm\gamma,x}^{(3)} = \{y \in \mathbb{R}^3 :< y - x, \pm\gamma > > 0\}$, parametrized by unity vectors $\gamma \in \mathbb{S}^2$, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the standard two-dimensional sphere a three-dimensional analog of the integral GLM-equation, whose solution gives rise to the kernel of the corresponding Delsarte transmutation operator for (3.1). But two important problems related with this approach were not discussed in detail: the first one concerns the question whether the Delsarte transformed operator $\tilde{L} = \mathbf{\Omega}_\gamma L \hat{\mathbf{\Omega}}_\gamma^{-1}$ is also a differential operator of Laplace type, and the second one concerns the question of the existence of Delsarte transmutation operators in the Faddeev form and their interior spectral structure.

Below we will study our multi-dimensional Delsarte transmutation operators (2.14), parametrized by a hypersurface $S_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})$ piecewise smoothly embedded into $\mathbb{R}^m$.

Consider now some $(m-2)$-dimensional homological cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \subset \mathbb{R}^m$ and two $(m-1)$-dimensional smooth hypersurfaces

$$S_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}), \quad S_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})$$

spanned between them in such a way that the whole hypersurface $\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})$ $\cup\mathcal{S}_-(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})$ is closed. Then as in Section 2 one can naturally define two pairs of Delsarte transmutation operators for a given pair of multi-dimensional differential operators (2.1) and (2.9), namely the operators $\mathbf{\Omega}_+ : \mathcal{H} \rightleftharpoons \mathcal{H}$, $\mathbf{\Omega}_+^{\circledast} : \mathcal{H}^* \rightleftharpoons \mathcal{H}^*$, defined by (2.14) and the operators $\mathbf{\Omega}_-^{\circledast} : \mathcal{H}^* \rightleftharpoons \mathcal{H}^*$, $\mathbf{\Omega}_-^{\circledast} : \mathcal{H}^* \rightleftharpoons \mathcal{H}^*$, where, by definition,

$$\begin{aligned}
\mathbf{\Omega}_- &:= \mathbf{1} - \int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\tilde{\psi}(\xi)\Omega_{x_0}^{-1}(\xi,\eta)\int_{\mathcal{S}_-(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\eta),\cdot],\\
\mathbf{\Omega}_-^{\circledast} &:= \mathbf{1} - \int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\tilde{\varphi}(\eta)\Omega_{x_0}^{\circledast,-1}(\xi,\eta)\int_{\mathcal{S}_-(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[\cdot,\psi(\xi)]
\end{aligned}\tag{3.2}$$

As to the Delsarte transmutation operators (2.13)) and (3.2) the following operator relationships

$$\tilde{L} = \mathbf{\Omega}_\pm L\mathbf{\Omega}_\pm^{-1}, \qquad \mathbf{\Omega}_\pm^{\circledast}L_\pm^{\circledast}\mathbf{\Omega}_\pm^{\circledast,-1} = \tilde{L}^* \tag{3.3}$$

hold. As in the theory of classical GLM-equations [4, 5, 20, 6], we can now construct linear integral compact operators $\Phi(\mathbf{\Omega})$: $\mathcal{H} \to \mathcal{H}$ and $\Phi(\mathbf{\Omega})^{\circledast} : \mathcal{H}^* \to \mathcal{H}^*$, so that the expressions

$$\mathbf{1} + \mathbf{\Phi}(\mathbf{\Omega}) := \mathbf{\Omega}_+^{-1}\cdot\mathbf{\Omega}_-, \qquad \mathbf{1} + \mathbf{\Phi}(\mathbf{\Omega})^{\circledast} := \mathbf{\Omega}_+^{\circledast,-1}\cdot\mathbf{\Omega}_-^{\circledast} \tag{3.4}$$

are Fredholmian [18] operators. Making use of the expressions (3.4), one easily gets a pair of linear integral GLM-equations

$$\mathbf{\Omega}_+\cdot(\mathbf{1} + \Phi(\mathbf{\Omega})) = \mathbf{\Omega}_-, \qquad \mathbf{\Omega}_+^{\circledast}\cdot(\mathbf{1} + \Phi(\mathbf{\Omega})^{\circledast}) = \mathbf{\Omega}_-^{\circledast}, \tag{3.5}$$

whose solution is a pair of the corresponding Volterra type kernels for the Delsarte transmutation operators $\mathbf{\Omega}_+$ and $\mathbf{\Omega}_+^{\circledast}$. Thus, the problem of constructing Delsarte transmutation operators for a given pair of differential operators (2.1) and (2.9) is reduced to describing a suitable class of linear Fredholm type operators (3.4) in the Hilbert space $\mathcal{H}$, satisfying the following natural conditions: operators $(\mathbf{1}+\Phi(\mathbf{\Omega})) : \mathcal{H} \longrightarrow \mathcal{H}$ and $(\mathbf{1}+\Phi^{\circledast}(\mathbf{\Omega}))$: $\mathcal{H}^* \longrightarrow \mathcal{H}^*$ are onto, bounded and invertible and, moreover,

$$(\mathbf{1} + \Phi(\mathbf{\Omega}))L = L(\mathbf{1} + \Phi(\mathbf{\Omega})), \quad (\mathbf{1} + \Phi^{\circledast}(\mathbf{\Omega}))L^* = L^*(\mathbf{1} + \Phi^{\circledast}(\mathbf{\Omega})) \tag{3.6}$$

owing to (3.5) and (2.14). This problem is very important for applications of the theory devised here to find diverse spectral properties of a given pair of Delsarte transformed differential operators (3.3). We plan to study this in detail in another place.

4 The structure of Delsarte transmutation operators for affine pencils of multidimensional differential expressions

Consider in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m;\mathbb{C}^N)$ a pencil of multi-dimensional differential operators defined in terms of an affine polynomial in $\lambda \in \mathbb{C}$, having the form:

$$L(x;\partial|\lambda) := \sum_{i=0}^{r(L)} \lambda^i L_i(x;\partial), \tag{4.1}$$

where $x \in \mathbb{R}^m$, $ordL_i(x;\partial) = n_i \in \mathbb{Z}_+$, $i = \overline{1, r(L)}$, the order $r(L) \in \mathbb{Z}_+$ is fixed and

$$L_i(x;\partial) := \sum_{|\alpha_i|=0}^{n_i} a_{i,\alpha_i}(x) \frac{\partial^{|\alpha_i|}}{\partial x^{\alpha_i}} \tag{4.2}$$

are differential expressions with smooth coefficients $a_{i,\alpha_i} \in \mathcal{S}(\mathbb{R}^m; End\mathbb{C}^N)$, $i = \overline{1, r(L)}$. The pencil (4.1) can be, in particular, characterized by its spectrum

$$\sigma(L) = \{\lambda \in \mathbb{C} : \exists\ \psi(x;\lambda) \in \mathcal{H}_-, \quad L(x;\partial|\lambda)\psi(x;\lambda) = 0\}. \tag{4.3}$$

As it was demonstrated in [8] the transformations of the pencil (4.1) which preserve a part of the spectrum $\sigma(L)$ and simultaneously change in a prescribed way the rest of the spectrum (so called an assignment spectrum problem [8]) are very important for feedback control theory and its applications in different fields of mechanics.

We shall interpret these "spectrum assignment" transformations as ones of Delsarte transmutation type, satisfying some additional special conditions. Thus, we look for such transformation $\mathbf{\Omega} : \mathcal{H} \longrightarrow \mathcal{H}$ of the pencil (4.1) into a similar pencil

$$\tilde{L}(x;\partial|\lambda) = \sum_{i=1}^{r(L)} \lambda^i \tilde{L}_i(x;\partial), \qquad \tilde{L}_i(x;\partial) := \sum_{|\alpha_i|=0}^{n_i} \tilde{a}_{i,\alpha_i}(x) \frac{\partial^{|\alpha_i|}}{\partial x^{\alpha_i}} \tag{4.4}$$

with $\tilde{a}_{i,\alpha_i} \in S(\mathbb{R}^m; End\mathbb{C}^N)$, $i = \overline{1, r(L)}$, $\lambda \in \mathbb{C}$, of the same polynomial and differential orders, so that

$$\tilde{L} = L + [\mathbf{\Omega}, L]\mathbf{\Omega}^{-1} = \mathbf{\Omega} L \mathbf{\Omega}^{-1}. \tag{4.5}$$

For such an operator $\mathbf{\Omega} : \mathcal{H} \to \mathcal{H}$ to be constructed, we suggest an extension of the polynomial pencil of differential operators (4.1) to a pure differential operator $L_\tau := L(x;\partial|\partial/\partial\tau)$, $x \in \mathbb{R}^m$, $\tau \in \mathbb{R}$, with $\mathbb{R} \ni \tau$-independent coefficients and acting suitably in the parametric functional space $\mathcal{H}_{(\tau)} := L_1(\mathbb{R}_\tau; \mathcal{H})$. Thereby we get to the same situation, which was studied before in [11]. For completeness, we shall give a short derivation of the corresponding affine expression for the Delsarte transmutation operator $\mathbf{\Omega} : \mathcal{H} \longrightarrow \mathcal{H}$.

Let a pair of functions $(\varphi_{(\tau)}, \psi_{(\tau)}) \in \mathcal{H}^*_{(\tau)} \times \mathcal{H}_{(\tau)}$ be arbitrary and consider the following semi-linear scalar form on $\mathcal{H}^*_{(\tau)} \times \mathcal{H}_{(\tau)}$:

$$(\varphi_{(\tau)}, \psi_{(\tau)}) := \int_{\mathbb{R}_\tau} d\tau \int_{\mathbb{R}^m} dx \bar{\varphi}^{\intercal}_{(\tau)}(x) \psi_{(\tau)}(x). \tag{4.6}$$

Then as to the interior semi-linear form $< \cdot, \cdot >$ on $\mathbb{C}^N \times \mathbb{C}^N$ one can write down for the operator $L_{(\tau)} : \mathcal{H}_{(\tau)} \longrightarrow \mathcal{H}_{(\tau)}$ and any pair $(\varphi_{(\tau)}, \psi_{(\tau)}) \in \mathcal{H}^*_{(\tau)} \times \mathcal{H}_{(\tau)}$ the following Lagrangian identity:

$$\Big[< L^*_{(\tau)} \varphi_{(\tau)}, \psi_{(\tau)} > - < \varphi_{(\tau)}, L_{(\tau)} \psi_{(\tau)} > \Big] d\tau \wedge dx = dZ^{(m)}_{(\tau)}[\varphi, \psi], \tag{4.7}$$

where $Z^{(m)}_{(\tau)}[\varphi, \psi] \in \Lambda^m(\mathbb{R}_\tau \times \mathbb{R}^m; \mathbb{C})$ is the corresponding differential m-form with values in $\mathbb{C}$, parametrically depending on $\tau \in \mathbb{R}$. Thus, for defining the closed subspaces $\mathcal{H}^*_{(\tau),0} \subset$

$\mathcal{H}^*_{(\tau),-}$ one can write down, respectively, the following expressions:

$$\begin{aligned} &\mathcal{H}_{(\tau),0} := \{\psi_{(\tau)}(\xi) \in \mathcal{H}_{(\tau),-} : L_{(\tau)}\psi_{(\tau)}(\xi) = 0,\\ &\quad \tau \in \mathbb{R},\ \psi_{(\tau)}(\xi)|_\Gamma = 0,\ \xi \in \Sigma \subset \mathbb{C}^p\},\\ &\mathcal{H}^*_{(\tau),0} := \{\varphi_{(\tau)}(\eta)(\in \mathcal{H}^*_{(\tau),-} : L^*_{(\tau)}\varphi_{(\tau)}(\eta) = 0,\\ &\quad \tau \in \mathbb{R},\ \ \varphi_{(\tau)}(\eta)|_\Gamma = 0,\ \eta \in \Sigma \subset \mathbb{C}^p\}, \end{aligned} \tag{4.8}$$

where $\Gamma \subset \mathbb{R}^m$ is some piecewise smooth boundary hypersurfaces in $\mathbb{R}^m$. One can also write down similar expressions for the Delsarte transformed operator expression $\tilde{L} : \mathcal{H}_{(\tau)} \longrightarrow \mathcal{H}_{(\tau)}$:

$$\begin{aligned} &\tilde{\mathcal{H}}_{(\tau),0} := \{\tilde{\psi}_{(\tau)}(\xi) \in \mathcal{H}_{(\tau),-} : \tilde{L}_{(\tau)}\tilde{\psi}_{(\tau)}(\xi) = 0,\\ &\quad \tau \in \mathbb{R},\ \tilde{\psi}_{(\tau)}(\xi)|_{\tilde{\Gamma}} = 0,\ \xi \in \Sigma \subset \mathbb{C}^p\},\\ &\tilde{\mathcal{H}}^*_{(\tau),0} := \{\tilde{\varphi}_{(\tau)}(\eta) \in \mathcal{H}^*_{(\tau),-} : \tilde{L}^*_{(\tau)}\tilde{\varphi}_{(\tau)}(\eta) = 0,\\ &\quad \tau \in \mathbb{R},\ \ \tilde{\varphi}_{(\tau)}(\eta)|_{\tilde{\Gamma}} = 0,\ \eta \in \Sigma \subset \mathbb{C}^p\}, \end{aligned} \tag{4.9}$$

where $\tilde{\Gamma} \subset \mathbb{R}^3$ is some piecewise smooth boundary hypersurface in $\mathbb{R}^m$.

Making use of the expressions (4.6) and (4.7), we easily find that the differential m-form $Z^{(m)}_{(\tau)}[\varphi,\psi] \in \Lambda^m(\mathbb{R}_\tau \times \mathbb{R}^m;\mathbb{C})$ is exact for any pair $(\varphi,\psi) \in \mathcal{H}^*_{(\tau),0} \times \mathcal{H}_{(\tau),0}$. This means due to the Poincaré lemma [12, 13], that there exists a differential $(m-1)$-form $\Omega^{(m-1)}_\tau[\varphi,\psi] \in \Lambda^{m-1}(\mathbb{R}_\tau \times \mathbb{R}^m;\mathbb{C})$, so that

$$Z^{(m)}_{(\tau)}[\varphi,\psi] = d\Omega^{(m-1)}_\tau[\varphi,\psi] \tag{4.10}$$

for all pairs $(\varphi_{(\tau)},\psi_{(\tau)}) \in \mathcal{H}^*_{(\tau),0} \times \mathcal{H}_{(\tau),0}$. Now we are in a position to begin our definition of the corresponding Delsarte transmutation operators $\mathbf{\Omega}_{(\tau)} : \mathcal{H}_{(\tau),0} \to \tilde{\mathcal{H}}_{(\tau),0}$ and $\mathbf{\Omega}^{\circledast}_{(\tau)}$: $\mathcal{H}^*_{(\tau),0} \to \tilde{\mathcal{H}}^*_{(\tau),0}$:

$$\begin{aligned} &\tilde{\psi}_{(\tau)}(\xi) = \mathbf{\Omega}_{(\tau)} \cdot \psi_{(\tau)}(\xi) := \psi_{(\tau)}(\xi) \cdot \Omega^{-1}_{(x,\tau)}\Omega_{(x_0,\tau)} =\\ &(\mathbf{1} - \tilde{\psi}_{(\tau)}\Omega^{-1}_{(x_0,\tau)} \textstyle\int_{\mathcal{S}(\sigma^{(m-1)}_x,\sigma^{(m-1)}_{x_0})} Z^{(m)}_{(\tau)}[\varphi_{(\tau)},\cdot])\psi_{(\tau)}(\xi),\\ &\tilde{\varphi}_{(\tau)}(\eta) = \mathbf{\Omega}^{\circledast}_{(\tau)} \cdot \varphi_{(\tau)}(\eta) := \varphi_{(\tau)}(\eta) \cdot \Omega^{\circledast,-1}_{(x,\tau)}\Omega^{\circledast}_{(x_0,\tau)} =\\ &(\mathbf{1} - \tilde{\varphi}_{(\tau)}\Omega^{\circledast,-1}_{(x_0,\tau)} \textstyle\int_{\mathcal{S}(\sigma,^{(m-1)}_x,\sigma^{(m-1)}_{x_0})} \bar{Z}^{(m),\intercal}_{(\tau)}[\cdot,\psi_{(\tau)}])\varphi_{(\tau)}(\eta). \end{aligned} \tag{4.11}$$

Here $(\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)) \in \mathcal{H}^*_{(\tau),0} \times \mathcal{H}_{(\tau),0}$, $\xi,\eta \in \Sigma$, and, due to (4.9), for kernels $\Omega_{(x,\tau)}(\eta,\xi)$, $\Omega_{(x_0,\tau)}(\eta,\xi) \in H \otimes H$ and $\Omega^{\circledast}_{(x,\tau)}(\eta,\xi)$, $\Omega^{\circledast}_{(x_0,\tau)}(\eta,\xi) \in H^* \otimes H^*$ of the corresponding integral operators $\Omega_{(x,\tau)}$, $\Omega_{(x_0,\tau)} : H \to H$ and $\Omega^{\circledast}_{(x,\tau)}$, $\Omega^{\circledast}_{(x_0,\tau)} : H^* \to H^*$ one has

$$\begin{aligned}&\int_{\mathcal{S}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)})} Z^{(m)}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] = \int_{\sigma_x^{(m-1)}} \Omega^{(m-1)}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] \\ &- \int_{\sigma_{x_0}^{(m-1)}} \Omega^{(m-1)}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] := \Omega_{(x,\tau)}(\eta,\xi) - \Omega_{(x_0,\tau)}(\eta,\xi),\end{aligned} \tag{4.12}$$

$$\begin{aligned}&\int_{\mathcal{S}(\sigma_{,x}^{(m-1)},\sigma_{x_0}^{(m-1)})} \bar{Z}^{(m),\intercal}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] = \int_{\sigma_x^{(m-1)}} \bar{\Omega}^{(m-1),\intercal}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] \\ &- \int_{\sigma_{x_0}^{(m-1)}} \bar{\Omega}^{(m-1),\intercal}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] := \Omega^{\circledast}_{(x,\tau)}(\eta,\xi) - \Omega^{\circledast}_{(x_0,\tau)}(\eta,\xi),\end{aligned}$$

where, as before, $S(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)}) \subset \mathbb{R}^m$ is a smooth hypersurface in the configuration space $\mathbb{R}^m$, respectively spanning between two arbitrary but fixed nonintersecting each other homological $(m-1)$-dimensional cycles $\sigma_x^{(m-1)}$and $\sigma_{x_0}^{(m-1)} \subset \mathbb{R}^m$, parametrized by points x, $x_0 \in \mathbb{R}^m$. As a result of the construction above, the Volterra type integral operators

$$\mathbf{\Omega}_{(\tau)} := \mathbf{1} - \int_\Sigma d\rho(\xi) \int_\Sigma d\rho(\eta)\tilde{\psi}_{(\tau)}(\xi)\Omega^{-1}_{(x_0,\tau)}(\xi,\eta) \int_{\mathcal{S}_-(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}_{(\tau)}[\varphi_{(\tau)}(\eta),\cdot], \tag{4.13}$$

and

$$\mathbf{\Omega}^{\circledast}_{(\tau)} := \mathbf{1} - \int_\Sigma d\rho(\xi) \int_\Sigma d\rho(\eta)\tilde{\varphi}_{(\tau)}(\eta)\Omega^{\circledast,-1}_{(x_0,\tau)}(\xi,\eta) \int_{\mathcal{S}_-(\sigma_x^{(m-2)},\sigma_{x_0}^{()m-2)})} \bar{Z}^{(m-1),\intercal}_{(\tau)}[\cdot,\psi_{(\tau)}(\xi)] \tag{4.14}$$

being bounded and invertible act, correspondingly, in the spaces $\mathcal{H}_{(\tau)}$ and $\mathcal{H}^*_{(\tau)}$. Moreover, the Delsarte transformed operator $\tilde{L}_{(\tau)} : \mathcal{H}_{(\tau)} \longrightarrow \mathcal{H}_{(\tau)}$ can be written down as

$$\tilde{L}_{(\tau)} = \mathbf{\Omega}_{(\tau)} L_{(\tau)} \mathbf{\Omega}^{-1}_{(\tau)} = L_{(\tau)} + [\mathbf{\Omega}_{(\tau)}, L_{(\tau)}]\mathbf{\Omega}^{-1}_{(\tau)}, \tag{4.15}$$

being, due to reasoning as in [14, 9], also a differential multi-dimensional operator in $\mathcal{H}_{(\tau)}$.

Now we can make the pullback reduction of our τ-dependent objects, recalling, that our operator (4.1) is independent of the parameter $\tau \in \mathbb{R}$. In particular, from (4.8) one obtains for any $(\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)) \in \mathcal{H}^*_{(\tau),0} \times \mathcal{H}_{(\tau),0}$, $\xi,\eta \in \Sigma$,

$$\psi_{(\tau)}(\xi) = \psi_\lambda(\xi)e^{\lambda\tau}, \qquad \varphi_{(\tau)}(\eta) = \varphi_\lambda(\eta)e^{-\bar{\lambda}\tau} \tag{4.16}$$

with $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$ and for any pair $(\varphi_\lambda(\xi),\psi_\lambda(\xi)) \in \mathcal{H}^*_0 \times \mathcal{H}_0$, $\xi,\eta \in \Sigma_\sigma$,

$$\begin{aligned}&\mathcal{H}_0 := \{\psi_\lambda(\xi) \in \mathcal{H}_- : L(x;\partial|\lambda)\psi_\lambda(\xi) = 0, \\ &\quad \psi_\lambda(\xi)|_\Gamma = 0,\ (\lambda;\xi) \in \sigma(L) \cap \bar{\sigma}(L^*)\times\Sigma_\sigma\}, \\ &\mathcal{H}^*_0 := \{\varphi_\lambda(\eta) \in \mathcal{H}^*_- : L^*(x;\partial|\lambda)\varphi_\lambda(\eta) = 0, \\ &\quad \varphi_\lambda(\eta)|_\Gamma = 0,\ (\lambda;\eta) \in \sigma(L) \cap \bar{\sigma}(L^*)\times\Sigma_\sigma\},\end{aligned} \tag{4.17}$$

where, by definition, $\Sigma_\sigma \times \mathbb{C} \subset \Sigma$ is some "spectral" set of parameters. With respect to the closed subspaces $\mathcal{H}_0 \subset \mathcal{H}_-$ and $\mathcal{H}_0^* \subset \mathcal{H}_-^*$ the corresponding Delsarte transmutation operators $\mathbf{\Omega} : \mathcal{H} \to \mathcal{H}$ and $\mathbf{\Omega}^{\circledast} : \mathcal{H}^* \to \mathcal{H}^*$ can be retrieved easily by substituting the expressions (4.16) into (4.13) and (4.14) :

$$\begin{aligned} \mathbf{\Omega} := \mathbf{1} - \int_{\sigma(L)\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \tilde{\psi}_\lambda(\xi) \\ \times \Omega_{x_0}^{-1}(\lambda;\xi,\eta) \int_{\mathcal{S}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)})} Z^{(m)}[\varphi_\lambda(\eta),\cdot], \\ \mathbf{\Omega}^{\circledast} := \mathbf{1} - \int_{\sigma(L)\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \tilde{\varphi}_\lambda(\eta) \\ \times \Omega_{x_0}^{\circledast,-1}(\lambda;\xi,\eta) \int_{\mathcal{S}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)})} Z^{(m)}[\cdot,\psi_\lambda(\xi)], \end{aligned} \tag{4.18}$$

where $d\rho_\sigma \times d\rho_{\Sigma_\sigma}$ is the corresponding finite Borel measure on Borel subsets of $\sigma(L) \cap \bar{\sigma}(L^*) \times \Sigma_\sigma \subset \Sigma$,

$$\begin{aligned} \tilde{\psi}_\lambda(\xi) := \psi_\lambda(\xi) \cdot \Omega_x^{-1} \Omega_{x_0} \\ \tilde{\varphi}_\lambda(\eta) := \varphi_\lambda(\eta) \cdot \Omega_x^{\circledast,-1} \Omega_{x_0}^{\circledast}. \end{aligned} \tag{4.19}$$

Owing to semi-linearity, the expressions for kernels

$$\begin{aligned} \Omega_x(\lambda;\xi,\eta) &:= \Omega_{(x,\tau)}[\varphi_\lambda e^{-\bar{\lambda}\tau}, \psi_\lambda e^{\lambda\tau}], \\ \Omega_{x_0}(\lambda;\xi,\eta) &:= \Omega_{(x_0,\tau)}[\varphi_\lambda e^{-\bar{\lambda}\tau}, \psi_\lambda e^{\lambda\tau}], \\ Z^{(m)}[\varphi_\lambda,\psi_\lambda] &:= Z^{(m)}_{(\tau)}[\varphi_\lambda e^{-\bar{\lambda}\tau}, \psi_\lambda e^{\lambda\tau}], \end{aligned} \tag{4.20}$$

from the Hilbert space $L_2^{(\rho)}(\Sigma_\sigma;\mathbb{C}) \otimes L_2^{(\rho)}(\Sigma_\sigma;\mathbb{C})$ don't depend on the whole on the parameter $\tau \in \mathbb{R}_\tau$ but only on the spectral parameter $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$. Now we write down the differential m-form $Z^{(m)}[\varphi_\lambda,\psi_\lambda] \in \Lambda^m(\mathbb{R}_\tau \times \mathbb{R}^m;\mathbb{C})$, $(\varphi_\lambda(\xi),\psi_\lambda(\xi)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\xi,\eta \in \Sigma_\sigma$, as

$$\begin{aligned} Z^{(m)}[\varphi_\lambda,\psi_\lambda] \;=\; & \sum_{i=1}^m dx_1 \wedge dx_2 \wedge \dots \wedge dx_{i-1} \wedge Z_i[\varphi_\lambda,\psi_\lambda] d\tau \wedge dx_{i+1} \wedge \\ & \dots \wedge dx_m + Z_0[\varphi_\lambda,\psi_\lambda] dx. \end{aligned} \tag{4.21}$$

Then, owing to the specially chosen $(m-1)$-dimensional homological cycles $(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)})$ and the corresponding closed m-dimensional surface $S(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)}) = \mathbb{R}^m$ at which $d\tau = 0$, the differential m-forms $Z^{(m)}[\varphi_\lambda,\psi_\lambda] \in \Lambda^m(\mathbb{R}^m;\mathbb{C})$, $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, satisfies the following expressions:

$$Z^{(m)}[\varphi_\lambda,\psi_\lambda] = Z_0[\varphi_\lambda,\psi_\lambda] dx, \tag{4.22}$$

since for any $i = \overline{0,m}$

$$Z_i[\varphi_\lambda,\psi_\lambda] := Z_{i,(\tau)}[\varphi_\lambda e^{-\bar{\lambda}\tau}, \psi_\lambda e^{\lambda\tau}], \tag{4.23}$$

being not dependent on the parameter $\tau \in \mathbb{R}$ but only on $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$. Thus, due to (4.21) and (4.22) one can finally write down Delsarte transmutation operators (4.12) and (4.13) as the following invertible and bounded of Volterra type integral expressions:

$$\begin{aligned} \mathbf{\Omega} &:= \mathbf{1} - \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \tilde{\psi}_\lambda(\xi) \\ &\times \Omega_{x_0}^{-1}(\lambda;\xi,\eta) \int_{\mathcal{S}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)})} Z_{(0)}[\varphi_\lambda(\eta),\cdot]dx, \\ \mathbf{\Omega}^{\circledast} &:= \mathbf{1} - \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \tilde{\varphi}_\lambda(\eta) \\ &\times \Omega_{x_0}^{\circledast,-1}(\lambda;\xi,\eta) \int_{\mathcal{S}(\sigma_x^{(m-1)},\sigma_{x_0}^{(m-1)})} Z_{(0)}[\cdot,\psi_\lambda(\xi)]dx, \end{aligned} \tag{4.24}$$

where, by definition, $(\varphi_\lambda, \psi_\lambda) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $(\tilde{\varphi}_\lambda, \tilde{\psi}_\lambda) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$ and $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$. The operator expressions (4.24) were defined on closed subspaces of generalized eigenfunctions $\mathcal{H}_0$ and $\mathcal{H}_0^*$. When these spaces are dense, respectively, in ambient spaces $\mathcal{H}_-$ and $\mathcal{H}_-^*$, the kernels of the operator expressions (4.24) can be naturally extended to their corresponding Hilbert spaces $\mathcal{H}_-$ and $\mathcal{H}_-^*$ defining invertible integral operators of Volterra type in their corresponding spaces $\mathcal{H}$ and $\mathcal{H}^*$ due to duality [22, 1] between Hilbert spaces $\mathcal{H}_-$ and $\mathcal{H}_+$, where the latter space is dense in $\mathcal{H}$. Just as above in Section 3 one can construct the corresponding pair (3.5) of Gelfand-Levitan-Marchenko integral equations for an affine polynomial pencil (4.1) of multi-dimensional differential operators in the Hilbert space $\mathcal{H}$. This completes our present analysis of the structure of Delsarte transmutation operators for pencils of multidimensional differential operators. As for their natural applications to the inverse spectral problem and related problems of feedback control theory mentioned before, we plan to study them in more detail later.

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