

# Some Group Theoretical Aspects of Nonlinear Quantal Oscillators

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## Abstract

We investigate the algebraic properties of the time-dependent Schrödinger equations of certain nonlinear oscillators introduced by Calogero and Graffi (Calogero F & Graffi S, On the quantisation of a nonlinear Hamiltonian oscillator *Physics Letters A* **313** (2003) 356-362; Calogero F, On the quantisation of two other nonlinear harmonic oscillators *Physics Letters A* **319** (2003) 240-245; Calogero F, On the quantisation of yet another two nonlinear harmonic oscillators *Journal of Nonlinear Mathematical Physics* **11** (2004) 1-6). Although all of the corresponding classical Hamiltonians are characterised by the Lie algebra  $sl(2, R)$ , we find that the algebras in the quantal case are not unique and depend upon the choice of parameters made in the quantisation process.

## 1 Introduction

In a series of papers Calogero and Graffi [1] and Calogero [2, 3] have discussed the quantisation and interrelationships of a set of nonlinear oscillators characterised by the Hamiltonians

$$H_1 = \frac{1}{2} \left\{ \frac{p^2 q^3}{c^2} + c \left( q + \frac{1}{q} \right) \right\} \quad (1.1)$$

$$H^{(s)} = \frac{1}{2} \left\{ \frac{p^2 q}{c} + c \left( q + \frac{s}{q} \right) \right\} \quad (1.2)$$

$$H_{EP} = \frac{1}{2} \left\{ \frac{p^2}{c} + c \left( \frac{q^2}{4} + \frac{4}{q^2} \right) \right\} \quad (1.3)$$

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$$H_2 = \frac{1}{2} \left\{ \frac{p^2 \sin^2 q \sin 2q}{2c} + \frac{2c}{\sin 2q} \right\} \quad (1.4)$$

$$H_3 = \frac{1}{2} \left\{ \frac{p^2 \sin^2 q \sin 2q}{2c} + 2c \cot 2q \right\} \quad (1.5)$$

in which  $c > 0$  is a parameter which plays no role in the Newtonian equations of motion derived from the application of Hamilton's Principle to the Action Integral for each Hamiltonian. In principle the parameter  $s$  takes the values  $\pm 1$  [3], but the value  $s = -1$  is subsequently shown not to be physically acceptable. Here we write  $H_{EP}$  for what Calogero and Graffi describe as  $\mathcal{H}$  [1] [eq (38)], with  $c = 1$  and the coefficient of  $q^{-2}$  written as  $g$ , since the corresponding Newtonian equation of motion, *videlicet*

$$\ddot{q} + \frac{1}{4}q = \frac{4}{q^3}, \quad (1.6)$$

is well-known in Physics as the Ermakov-Pinney equation after the mathematicians Ermakov [4] and Pinney [12] who provided some basic results for the more general equation

$$\ddot{q} + \omega^2(t)q = \frac{h^2}{q^3}. \quad (1.7)$$

Calogero and Graffi and Calogero provide time-independent Schrödinger equations for each of the Hamiltonians (1.1) – (1.5) corresponding to the time-dependent forms

$$2ic \frac{\partial u}{\partial t} + x^3 \frac{\partial^2 u}{\partial x^2} + 3x^2 \frac{\partial u}{\partial x} + \left\{ (1 + \rho - c^2)x - \frac{c^2}{x} \right\} u = 0 \quad (1.8)$$

$$2ic \frac{\partial u}{\partial t} + x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - c^2 \left\{ x + \frac{s}{x} \right\} u = 0 \quad (1.9)$$

$$2ic \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - c^2 \left\{ \frac{x^2}{4} + \frac{4}{x^2} \right\} u = 0 \quad (1.10)$$

$$2ic \frac{\partial u}{\partial t} + \sin^3 x \cos x \frac{\partial^2 u}{\partial x^2} + \sin^2 x (4 \cos^2 x - 1) \frac{\partial u}{\partial x} + \left\{ \frac{1}{4}(\beta + \gamma) (8 \cos^2 x - 5) - \frac{\beta \gamma \sin x}{16 \cos x} (4 \cos^2 x - 1)^2 \right\} u - \frac{c^2}{\sin x \cos x} u = 0 \quad (1.11)$$

$$2ic \frac{\partial u}{\partial t} + \sin^3 x \cos x \frac{\partial^2 u}{\partial x^2} + \sin^2 x (4 \cos^2 x - 1) \frac{\partial u}{\partial x} + \left\{ \frac{1}{4}(\beta + \gamma) (8 \cos^2 x - 5) - \frac{\beta \gamma \sin x}{16 \cos x} (4 \cos^2 x - 1)^2 \right\} u - 2 \frac{c^2}{\cot 2x} u = 0 \quad (1.12)$$

([1] [eq (14)], [3] [eq (17)], [1] [adapted from (36a)], [2] [eq (10a) with (7) and (10b)] and [2] [same as for (1.11) with the 'potential' adjusted to that of (1.5)] respectively) in which  $\rho$  is a parameter which arises in the quantisation of  $H_1$  and  $\beta$  and  $\gamma$  are parameters which arise in the quantisation of  $H_2$  and  $H_3$ .

A central theme of the papers of Calogero and Graffi [1] and Calogero [2, 3] is the practical demonstration that the two processes of quantisation and nonlinear canonical transformation need not commute [14]. Furthermore the papers report ground state energy levels for the Schrödinger equations (1.8) – (1.11) (in Calogero [3] the results for the

equation (1.12) are left as an exercise for the diligent reader) which differ and depend upon the parameter  $c$  and the quantisation procedure, *ie* the value of  $\rho$  in the case of (1.8) and the values of  $\beta$  and  $\gamma$  in the case of (1.11) (and by implication (1.12)), although the classical Hamiltonians (1.1) – (1.5) are related by autonomous canonical transformations and the corresponding Newtonian equations are free of the parameter  $c$ .

The interface between Classical Mechanics and Quantum Mechanics represented by the results reported by Calogero and Graffi and Calogero are not the concern of this paper. Indeed we are not concerned with the classical aspects at all. Our concern is the investigation of the group theoretical properties of the Schrödinger equations (1.8) – (1.12). We determine the Lie point symmetries of each of the Schrödinger equations and note some unusual features. We recall the use of the Lie point symmetries to construct the solutions of the time-dependent Schrödinger equation [6] and in the case of three of these Hamiltonians find some anomalous results which suggest that these nonlinear quantal oscillators present further problems of interpretation than the already serious questions addressed in the papers of Calogero and Graffi and Calogero.

## 2 Lie point symmetries of the time-dependent Schrödinger equation

The Lie point symmetries of the time-dependent Schrödinger equations (1.8) – (1.10) are easily calculated using LIE [5, 13] and we simply quote the results. In the case of (1.8) there are two possibilities. For unconstrained  $\rho$  and  $c$  the symmetries are

$$\begin{aligned}\Gamma_1 &= i\partial_t \\ \Gamma_{2\pm} &= e^{\pm it} \left\{ \pm i\partial_t + x\partial_x + \left( \pm \frac{c}{x} - \frac{1}{2} \right) u\partial_u \right\} \\ \Gamma_4 &= u\partial_u \\ \Gamma_5 &= f(t, x)\partial_u,\end{aligned}\tag{2.1}$$

where  $f(t, x)$  is any solution of (1.8). The symmetries  $\Gamma_4$  and  $\Gamma_5$  are generic to homogeneous linear partial differential equations and we do not repeat them below for the remaining Schrödinger equations. The subalgebra consisting of the symmetries  $\Gamma_1$  and  $\Gamma_{2\pm}$  is  $sl(2, R)$ . The presence of this subalgebra is common to all of the corresponding Newtonian equations of motion for the Schrödinger equations considered in this paper.

When the quantisation parameter  $\rho$  is fixed according to  $\rho = c^2 - 1/16$ , the symmetries listed in (2.1) are supplemented by

$$\Gamma_{3\pm} = x^{1/2} e^{\pm it/2} \left\{ x\partial_x + \left[ \pm \frac{c}{x} - \frac{3}{4} \right] u\partial_u \right\}.\tag{2.2}$$

With  $\Gamma_4$  the symmetries  $\Gamma_{3\pm}$  constitute the subalgebra  $A_{3,1}$  in the Mubarakzyanov classification scheme [7, 8, 9] and so is a representation of the Weyl algebra. Note that in the case of unconstrained  $\rho$  its value does not appear in the coefficient functions of the symmetries listed in (2.1) save through a solution of (1.8) in  $\Gamma_5$ .

The nongeneric symmetries of (1.9) are

$$\Gamma_1 = i\partial_t$$

$$\Gamma_{2\pm} = e^{\pm it} \left\{ \pm i\partial_t - x\partial_x + \left[ \pm cx + \frac{1}{2} \right] u\partial_u \right\} \quad (2.3)$$

and of (1.10) are

$$\begin{aligned} \Gamma_1 &= i\partial_t \\ \Gamma_{2\pm} &= e^{\pm it} \left\{ \pm i\partial_t - \frac{1}{2}x\partial_x + \frac{1}{4} [1 \pm cx^2] u\partial_u \right\}. \end{aligned} \quad (2.4)$$

In (2.3) and (2.4) the subalgebra of the listed symmetries is  $sl(2, R)$ .

For each of the Schrödinger equations (1.8) – (1.10) the algebra is  $\{sl(2, R) \oplus A_1\} \oplus_s \{\infty A_1\}$ , where the third subalgebra is the infinite-dimensional subalgebra of the solution symmetries. In the case of the constraint  $\rho = c^2 - 1/16$  the algebra is  $\{sl(2, R) \oplus_s A_{3,1}\} \oplus_s \{\infty A_1\}$  where  $A_{3,1}$  is the Weyl subalgebra.

In the cases of (1.11) and (1.12) we are not able to compute directly the symmetries using LIE due to the complexity of the coefficients<sup>3</sup>. Fortunately the situation is partially resolved for us by Calogero [2] who has provided the transformation of the autonomous equivalent of (1.11) to a polynomial form. Under the transformation

$$u(t, x) = (y^2 (1 + y^4))^{-1/4}, \quad \tan x = y^{-2}, \quad t = t \quad (2.5)$$

(1.11) takes the form ([2] [(21a, 21b adapted)])

$$8ic \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial y^2} - V(y)v = 0, \quad (2.6)$$

where

$$V(y) = 4c^2 \left( y^2 + \frac{1}{y^2} \right) - \frac{\beta\gamma + 1}{4y^2} \left( \frac{1 - 3y^4}{1 + y^4} \right)^2 - (\beta + \gamma - 2)y^2 \frac{(3y^4 - 5)}{(1 + y^4)^2}. \quad (2.7)$$

It follows that for general values of  $\beta$  and  $\gamma$ , (1.11), which is related to (2.6) by means of a point transformation and so preserves Lie point symmetries, can have only the two generic symmetries,  $\Gamma_4$  and  $\Gamma_5$  of (2.1), plus  $\Gamma_1$  representing invariance under time translation. Only in the case  $\beta\gamma + 1 = 0$  and  $\beta + \gamma - 2 = 0$ , *ie*  $\beta = 1 \pm \sqrt{2}$  and  $\gamma = 1 \mp \sqrt{2}$ , in which the potential reduces to that of the Ermakov-Pinney equation, (1.10), are additional symmetries found.

A similar story applies to (1.12). The same transformation, (2.5), gives the potential (2.7) with  $4c^2y^{-2}$  replaced with  $-4c^2y^{-2}$ .

### 3 Construction of the Wave Functions

We can use the symmetries listed in (2.1) and (2.4) to construct formal solutions to (1.8) and (1.10). The formal solutions become acceptable when the physical conditions are satisfied. The standard procedure is to construct a basic solution and then use the property of Lie symmetries that they map solutions into solutions [6]. In each case of equations (1.8) – (1.12) we can use the symmetries  $\Gamma_{2\pm}$  to obtain nontrivial results. We illustrate the procedure with (1.8) and its  $\Gamma_{2\pm}$  symmetries given in (2.1).

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<sup>3</sup>Probably the interactive code of Nucci [10, 11] would be successful, but we did not have access to it at the time this work was performed.

We obtain a similarity solution of (1.8) by determining the invariants of  $\Gamma_{2\pm}$  in (2.1) and using these invariants as new variables to reduce (1.8) to an ordinary differential equation. The Lie method has reduction at its core whereas the method of separation of variables relies on the existence of a constant which plays a role analogous to that of a first integral for ordinary differential equations.

The invariants of  $\Gamma_{2\pm}$  (2.1b) are found from the solution of the equation  $\Gamma_{2\pm}g(t, x, u) = 0$  for which the associated Lagrange's system is

$$\frac{dt}{\pm i} = \frac{dx}{x} = \frac{du}{\left(\pm \frac{c}{x} - \frac{1}{2}\right)u} \quad (3.1)$$

after the common exponential term has been removed. From the first and second components of (3.1) we obtain the invariant  $v = x \exp[\pm it]$  and from the second and third components of (3.1) with  $v$  taken into account to avoid an  $x^{1/2}$  in the characteristic we find the second invariant to be  $w = u \exp[\pm(\frac{c}{x} - \frac{1}{2}it)]$ . Consequently the similarity solution has the structure

$$u = \exp\left[\mp \frac{c}{x} \pm \frac{1}{2}it\right] h(v), \quad v = x \exp[\pm it]. \quad (3.2)$$

This we substitute into (1.8) to obtain the ordinary differential equation

$$\begin{aligned} x^2 e^{\pm 2it} h'' + 3x e^{\pm it} h' + (1 + \rho - c^2) h &= 0 \\ \Leftrightarrow v^2 h'' + 3v h' + (1 + \rho - c^2) h &= 0, \end{aligned} \quad (3.3)$$

where the prime denotes differentiation with respect to the similarity variable  $v$ , for  $h(v)$ . Equation (3.3) is an Euler equation with the solution set

$$\{h_1, h_2\} = \{v^{-1+\alpha}, v^{-1-\alpha}\}, \quad (3.4)$$

where  $\alpha = \sqrt{c^2 - \rho}$ . Consequently the solution set of (1.8) invariant under  $\Gamma_{2\pm}$  is

$$\begin{aligned} \{u : \Gamma_{2\pm}u = 0\} = \left\{ \exp\left[\mp \left(\frac{c}{x} - \frac{1}{2}it\right)\right] (xe^{\pm it})^{-1+\alpha}, \right. \\ \left. \exp\left[\mp \left(\frac{c}{x} - \frac{1}{2}it\right)\right] (xe^{\pm it})^{-1-\alpha} \right\}. \end{aligned} \quad (3.5)$$

Solutions of (1.8) are required to be square integrable over  $x \in (0, \infty)$  and for proper behaviour at  $x = 0$  we must take the upper, negative, sign in (3.5). Thus we have

$$u_{\alpha\pm} = \exp\left[-\frac{c}{x} - \left(\frac{1}{2} \mp \alpha\right) it\right] x^{-1\pm\alpha}. \quad (3.6)$$

With the change of variable  $x = \eta^{-1}$  the norm over  $(0, \infty)$  can be written as

$$\|u_{\alpha\pm}\|^2 = \int_0^\infty \eta^{\mp 2\alpha} \exp[-2c\eta] d\eta. \quad (3.7)$$

which is manifestly convergent for  $u_{\alpha-}$ . In the case of  $u_{\alpha+}$  the integral in (3.7) is improper. However, it is convergent for  $2\alpha < 1$ . For (1.8) to be well-posed it is necessary for  $\rho \leq c^2$ , *ie* there is a constraint on the quantisation procedure as was already noted by Calogero

and Graffi [1]. Hence the  $u_{\alpha+}$  solution has finite norm for  $c^2 - \frac{1}{4} < \rho \leq c^2$ . Nevertheless the solution fails to be acceptable since the Schrödinger equation (1.8) is not self-adjoint as also was already noted by Calogero and Graffi [1]. Hence there is just the single acceptable solution of (1.8),

$$u_0 = \exp \left[ -\frac{c}{x} - \left( \frac{1}{2} + \alpha \right) it \right] x^{-1-\alpha}, \quad (3.8)$$

corresponding to the symmetry  $\Gamma_{2+}$ .

The solution (3.8) has no zero within  $(0, \infty)$  and so represents a proper ground state. The energy eigenvalue is given by

$$\begin{aligned} \Gamma_1 u_0 &= E_0 u_0 \\ E_0 &= \alpha + \frac{1}{2}. \end{aligned} \quad (3.9)$$

We can create further solutions by the use of the Lie Bracket of  $\Gamma_{2-}$  with  $\Gamma_5$  which is

$$\begin{aligned} [\Gamma_{2-}, \Gamma_5]_{LB} &= \left[ e^{-it} \left( -i\partial_t + x\partial_x + \left( -\frac{1}{2} - \frac{c}{x} \right) u\partial_u \right), f\partial_u \right]_{LB} \\ &= e^{-it} \left\{ -i\frac{\partial f}{\partial t} + x\frac{\partial f}{\partial x} + \left( \frac{1}{2} + \frac{c}{x} \right) f \right\} u\partial_u \end{aligned}$$

so that

$$f_{\text{new}} = e^{-it} \left\{ -i\partial_t + x\partial_x + \frac{1}{2} + \frac{c}{x} \right\} f_{\text{old}}. \quad (3.10)$$

Thus for example we obtain

$$u_1 = (2c - (1 + 2\alpha)x) x^{-2-\alpha} \exp \left[ -\frac{c}{x} - \frac{1}{2}(3 + 2\alpha)it \right] \quad (3.11)$$

$$E_1 = \alpha + \frac{3}{2}. \quad (3.12)$$

Higher states are similarly constructed. In general we have

$$E_n = \frac{1}{2} + n + \alpha, \quad (3.13)$$

ie the increment in the energy levels is one. The parameter  $c$  of the Hamiltonian (1.1) and the parameter  $\rho$  of the quantisation procedure used to obtain (1.8) occur only through the ground state. That this interesting phenomenon occurs for each of the Schrödinger equations (1.8) – (1.11) has been reported by Calogero and Graffi [1] and Calogero [2, 3] and is inferred for (1.12).

In a similar manner the solutions for (1.9) and (1.10) can be constructed by using the corresponding  $\Gamma_{2\pm}$  symmetries in (2.3) and (2.4). The base solution is constructed using  $\Gamma_{2+}$  and subsequent solutions from the Lie Bracket of  $\Gamma_{2-}$  and  $\Gamma_5$ . The energy is the eigenvalue of the equation

$$\Gamma_1 u_n = E_n u_n. \quad (3.14)$$

As there is no methodological development beyond that given for (1.8) above, we simply quote the results. In all formulæ we assume that the parameter  $c$  is a positive real number. For (1.9) we find that

$$u_0 = x^c \exp \left[ - \left( cx + \left( \frac{1}{2} + c \right) it \right) \right]$$

$$\begin{aligned} u_n &= \left\{ e^{-it} \left[ i\partial_t + x\partial_x + \left(\frac{1}{2} - cx\right) \right] \right\}^n u_0 \\ E_n &= \frac{1}{2} + n + c \end{aligned} \quad (3.15)$$

in the case that  $s = 1$ . The case  $s = -1$  is not physical. In the case of (1.10) we obtain

$$\begin{aligned} u_0 &= x^{(1+\beta)/2} \exp \left[ -\frac{1}{4} (cx^2 + (2 + \beta)it) \right] \\ u_n &= \left\{ e^{-it} \left[ i\partial_t + \frac{1}{2}x\partial_x - \frac{1}{4}(1 - cx^2) \right] \right\}^n u_0 \\ E_n &= \frac{1}{2} + n + \frac{1}{4}\beta, \end{aligned} \quad (3.16)$$

where  $\beta = \sqrt{1 + 16c^2}$ . In the case that  $\beta < 5/64$  a second base solution

$$u_0^- = x^{(1-\beta)/2} \exp \left[ -\frac{1}{4} (cx^2 + (2 - \beta)it) \right]$$

exists for which  $\|u_0\| < \infty$ . However, as Calogero and Graffi [1] have observed, the equation ceases to be self-adjoint.

In the case of general values of  $\beta$  and  $\gamma$  equations (1.11) and (1.12) do not have easily accessible wave functions as can be inferred from the comment of Calogero [2] [paragraph after (24)]. For the specific choice of the parameter as stated in §2 the results are as given in (3.16).

## 4 The exceptional case

We turn now to the exceptional case of (1.8) in which  $\rho = c^2 - 1/16$  and the additional symmetries,  $\Gamma_{3\pm}$  of (2.2), occur. The  $\Gamma_{2\pm}$  solutions are now

$$u_{\frac{1}{4}+} = x^{-\frac{3}{4}} \exp \left[ -\frac{c}{x} - \frac{1}{4}it \right] \quad (4.1)$$

$$u_{\frac{1}{4}-} = x^{-\frac{5}{4}} \exp \left[ -\frac{c}{x} - \frac{3}{4}it \right]. \quad (4.2)$$

The associated Lagrange's system for  $\Gamma_{3\pm}$  is

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{\left( \pm \frac{c}{x} - \frac{3}{4} \right) u} \quad (4.3)$$

for which the invariants are evidently  $v = t$  and  $w = ux^{3/4}e^{\pm c/x}$ . We seek a similarity solution of the form

$$u_{\pm} = x^{-3/4} \exp \left[ \mp \frac{c}{x} \right] h(t). \quad (4.4)$$

Substitution of (4.4) into (1.8) with  $\rho = c^2 - 1/16$  yields the first-order equation

$$2ich \mp \frac{1}{2}ch = 0$$

so that the similarity solution is

$$u_{\pm} = x^{-\frac{3}{4}} \exp \left[ \mp \left( \frac{c}{x} + \frac{1}{4}it \right) \right]. \quad (4.5)$$

The norm is given by

$$\begin{aligned} \|u_{\pm}\|^2 &= \int_0^{\infty} x^{-\frac{3}{2}} \exp\left[\mp \frac{2c}{x}\right] dx \\ &= \int_0^{\infty} \eta^{-\frac{1}{2}} \exp[\mp 2c\eta] d\eta, \quad \eta = x^{-1}. \end{aligned}$$

The integral is convergent with the upper sign and so we have a replication of the solution  $u_{\frac{1}{4}+}$  given in (4.1) and this is the solution corresponding to  $\Gamma_{3+}$ .

The technique of using the Lie Bracket of  $\Gamma_{3-}$  with  $\Gamma_5$  gives

$$f_{\text{new}} = x^{1/2} \exp\left[-\frac{1}{2}it\right] \left\{ x\partial_x + \left(\frac{3}{4} \mp \frac{c}{x}\right) \right\} f_{\text{old}}. \quad (4.6)$$

Using  $\Gamma_{3-}$  with  $f_{\text{+old}} = u_{\frac{1}{4}+}$  we obtain

$$f_{\text{new}} = 2cx^{-\frac{5}{4}} \exp\left[-\left(\frac{c}{x} + \frac{3}{4}it\right)\right]$$

which is just  $u_{\frac{1}{4}-}$ .

For  $\rho = c^2 - 1/16$  the basic solution is

$$u_0 = x^{-\frac{3}{4}} \exp\left[-\frac{c}{x} - \frac{1}{4}it\right] \quad (4.7)$$

$$E_0 = \frac{1}{4}, \quad (4.8)$$

which corresponds to the  $\Gamma_{2+}$  solution of (3.8) when  $\rho = c^2 - 1/16$  and  $\alpha = -\frac{1}{4}$ . Further solutions are generated by the action of  $\Gamma_{3-}$  on the initial solution derived from the use of  $\Gamma_{3+}$ .

## 5 The exceptional case and the simple harmonic oscillator

The algebra of the Lie point symmetries of (1.8) in the exceptional case for which  $\rho = c^2 - 1/16$  is  $\{sl(2, R) \oplus_s Weyl\} \oplus_s \{\infty A_1\}$  which has the two additional symmetries  $\Gamma_{3\pm}$  of (2.2). These two symmetries with  $\Gamma_1$  constitute the Weyl algebra which is the algebra of the Dirac creation and annihilation operators and the identity element in the standard treatment of the quantal simple harmonic oscillator. Indeed the algebra  $\{sl(2, R) \oplus_s Weyl\} \oplus_s \{\infty A_1\}$  is the algebra of the Lie point symmetries of the time-dependent Schrödinger equation for the simple harmonic oscillator [6]. The algebras of (1.8) in the nonexceptional case, (1.9) and (1.10) are characteristic of the time-dependent Schrödinger equations for potentials of Ermakov-Pinney type.

The time-dependent Schrödinger equation for a simple harmonic oscillator with an Hamiltonian written in the spirit of (1.1) – (1.5), *videlicet*

$$H_{SH0} = \frac{1}{2} \left\{ \frac{p^2}{c} + \frac{c}{4}q^2 \right\}, \quad (5.1)$$



is, *cf* (1.10),

$$2ic \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - \frac{1}{4}c^2 x^2 u = 0. \quad (5.2)$$

Equation (5.2) possesses the Lie point symmetries

$$\begin{aligned} \Gamma_1 &= i\partial_t \\ \Gamma_{2\pm} &= e^{\pm it} \left[ \pm i\partial_t - \frac{1}{2}x\partial_x + \frac{1}{4}(1 \pm cx^2)u\partial_u \right] \\ \Gamma_{3\pm} &= e^{\pm \frac{1}{2}it} \left[ -\partial_x \pm \frac{1}{2}cxu\partial_u \right] \\ \Gamma_4 &= u\partial_u \\ \Gamma_5 &= f(t, x)\partial_u, \end{aligned} \quad (5.3)$$

where  $f(t, x)$  is a solution of (5.2), in which the numbering of the symmetries follows the convention established above. We note that  $\Gamma_1$ ,  $\Gamma_{2\pm}$ ,  $\Gamma_4$  and  $\Gamma_5$  coincide with the symmetries of (1.10). The solution of the time-dependent Schrödinger equation (5.2) is found using the symmetries  $\Gamma_{3\pm}$  in (5.3) as we have demonstrated above. We find that

$$\begin{aligned} u_0 &= \exp \left[ -\frac{1}{4}(cx^2 + it) \right] \\ u_n &= \left\{ \exp \left[ -\frac{1}{2}it \right] \left( -\partial_x + \frac{1}{2}cx \right) \right\}^n u_0 \\ E_n &= \frac{1}{4}(2n + 1), \end{aligned} \quad (5.4)$$

in which  $u_n$  follows from repeatedly taking the Lie Bracket of  $\Gamma_{3-}$  and  $\Gamma_5$  with  $f = u_0$ .

We note that  $E_n$  is independent of  $c$ .

The symmetries of (2.2) and (5.3c) can be related by means of a point transformation of rather special type. It is evident that time is unchanged, space maps to space and the wavefunction transforms linearly. We use uppercase letters for (5.2) and its symmetries (5.3) and lowercase letters for (1.8). It is a simple matter to show that the transformation connecting (2.2) and (5.3c) is

$$t = T, \quad x = 4X^{-2}, \quad u = X^{\frac{3}{2}}U. \quad (5.5)$$

Under this transformation (1.8) reduces to

$$2ic \frac{\partial U}{\partial T} + \frac{\partial^2 U}{\partial X^2} + \left\{ \left[ 4(1 + \rho - c^2) - \frac{15}{4} \right] X^{-2} - \frac{c^2}{4} X^2 \right\} U = 0 \quad (5.6)$$

which becomes

$$2ic \frac{\partial U}{\partial T} + \frac{\partial^2 U}{\partial X^2} - \frac{1}{4}c^2 X^2 U = 0, \quad (5.7)$$

*ie* (5.2), when  $\rho$  takes the exceptional value of  $c^2 - 1/16$ .

We have the peculiar result that a nonlinear oscillator,  $H$  (1.1), constructed from the Ermakov-Pinney Hamiltonian (1.3) [2], becomes a linear oscillator under a specific choice of quantisation procedure. This quantisation has the effect of a transformation of the variables, both independent and dependent, which is nonlocal since the number of Lie

point symmetries changes. A quantisation scheme which changes the underlying Physics is disturbing and is more than worthy of further investigation from the viewpoint of its physical meaning in contrast to the mathematical approach of this paper.

In this treatment of the exceptional case for (1.8), *videlicet*  $\rho = c^2 - 1/16$ , we see that in the general case the equation transforms to

$$2ic \frac{\partial U}{\partial T} + \frac{\partial^2 U}{\partial X^2} - \frac{1}{4} \left[ c^2 X^2 - \frac{(1 + 16(\rho - c^2))}{X^2} \right] U = 0 \quad (5.8)$$

which coincides with (1.10) only if  $\rho = -1/16$  in which case the energy levels of (1.8) and (1.10) are the same as they are equivalent descriptions of the same system.

## 6 Conclusion

In this paper we have considered a number of quantal nonlinear oscillators, introduced by Calogero and Graffi [1] and Calogero [2, 3] (apart from the Ermakov-Pinney system) from a perspective of their curious variations as quantal systems from classical behaviour, from the viewpoint of the Lie algebras of their time-dependent Schrödinger equations. In general all systems have the same algebraic structure and we have demonstrated how to construct their wave functions using standard algebraic techniques. During the course of the analysis of (1.8) we found an increase in the number of point symmetries when two parameters were related in a specific way. This indicated a fundamental change in the nature of the underlying physical problem since one went from a potential with a repulsive centrifugal term to that of a simple harmonic oscillator even though this was not evident in the time-dependent Schrödinger equation, (1.8), describing the system. Apart from the slightly disconcerting lack of obvious difference between  $\rho = c^2 - 1/16$  and  $\rho \neq c^2 - 1/16$  there are sufficient instances in the literature of critical values of parameters for this not to be a surprise. What is disconcerting in this case, however, is that the parameter in question occurs in a quantisation procedure, *ie* the choice of the parameter in the procedure can change the nature of the physical problem under consideration.

In a similar vein the analyses of (1.11) and (1.12) showed that for a general quantisation scheme there were only the symmetries expected for an arbitrary autonomous potential. Again a particular choice of the parameters of the quantisation scheme lead to an increase in the number of nontrivial symmetries so that the nontrivial algebra was  $sl(2, R)$ .

This is not good Physics! It does, however, provide an impetus to study again the process of quantisation to see if a consistent procedure can be established to avoid problems such as the one illustrated here. In the papers of Calogero and Graffi and Calogero particular point was made of the general noncommutivity of nonlinear canonical transformations and quantisation. Here we have found an anomaly even within the quantisation procedure. We are indebted to Francesco Calogero for opening this line of research to reveal a succession of fundamental questions to be answered.

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