

Gerschgorin Variations of Complex Matrices

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Abstract—We present various “additive” sufficient conditions for the nonsingularity of a complex partitioned matrix and an irreducible partitioned matrix. As the application, a new criteria of the positive stable matrix is given. The obtained results generalize some corresponding results.

Keywords- Nonsingularity; Gerschgorin disc; Stable matrix

I. INTRODUCTION

Matrix theory which is a fundamental tool in Mathematics has been an important tool in some technologic fields. And estimate of eigenvalue which is a significant part in Matrix theory plays an important role in some relevant fields such as Statistics, Numerical analysis, System engineering, Stability theory, Computation methods, etc.

In this paper, first of all, we discussed the problem of nonsingularity of complex partitioned matrices, gave the conditions of complex partitioned matrices and irreducible partitioned matrices. In addition, we obtained the new criteria for a matrix to be a positive stable matrix. Therefore, we provide a new method solving differential equations in vibrating systems by these results.

II. NOTATIONS

Some basic definitions can be found in [5,6,7]. We assume $\langle m \rangle = \{1, 2, \dots, m\}$ and $|S|$ denotes the number of elements in S which is the subset of $\langle m \rangle$. And $A = (a_{ij}) \in C^{n \times n}$ is partitioned as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix} \quad (1)$$

Let $\tilde{R}_i(A) = \sum_{j \neq i} \|A_{ji}^{-1}\| \cdot \|A_{ij}\|$, $\tilde{C}_i(A) = \sum_{j \neq i} \|A_{ji}^{-1}\| \cdot \|A_{ji}\|$, and

$$\tilde{C}_{ik}(A) = \max_{|T|=k-1} \sum_{j \in T} \|A_{ji}^{-1}\| \cdot \|A_{ji}\|, i \in \langle m \rangle.$$

III. MAIN RESULTS

Theorem 1. Let $A = (a_{ij}) \in C^{n \times n}$ be partitioned as (1), in which A_{ii} is nonsingular for $i \in \langle m \rangle$. If there exists $k \in \langle m \rangle$ such that

$$|S_k| = k \text{ implies } \sum_{i \in S_k} (1 - \tilde{R}_i(A)) > 0 \text{ for all } S_k \subseteq \langle m \rangle \quad (1.1)$$

and

$$|S| > \min \left\{ \sum_{i \in S} \tilde{R}_i(A), \sum_{i \in S} \tilde{C}_{ik}(A) \right\} \text{ for all } S \subseteq \langle m \rangle. \quad (1.2)$$

Then A is nonsingular.

Proof. Assume the hypotheses of the theorem, and A is singular. Then there exists a nonzero vector Z such that $AZ = 0$, in which the partitioned vector of Z is uniform with A . Without loss of generality, we assume that $X_j = \|Z_j\|$ for $j \in \langle m \rangle$ and

$$X_1 \geq X_2 \geq \cdots \geq X_m \geq 0. \quad (1.3)$$

Let

$$P = \begin{pmatrix} A_{11}^{-1} & & & \\ & A_{22}^{-1} & & \\ & & \ddots & \\ & & & A_{mm}^{-1} \end{pmatrix}.$$

Then $(PA)Z = 0$ and $Z_i = -\sum_{j \neq i} A_{ii}^{-1} A_{ij} Z_j$ for $i \in \langle m \rangle$.

Further,

$$\|Z_i\| \leq \sum_{j \neq i} \|A_{ii}^{-1}\| \cdot \|A_{ij}\| \cdot \|Z_j\| \text{ for } i \in \langle m \rangle. \quad (1.4)$$

Define

$$\mu(PA) = \begin{pmatrix} 1 & -\|A_{11}^{-1}\| \cdot \|A_{12}\| & \cdots & -\|A_{11}^{-1}\| \cdot \|A_{1m}\| \\ -\|A_{22}^{-1}\| \cdot \|A_{21}\| & 1 & \cdots & -\|A_{22}^{-1}\| \cdot \|A_{2m}\| \\ \vdots & \vdots & \ddots & \vdots \\ -\|A_{mm}^{-1}\| \cdot \|A_{m1}\| & -\|A_{mm}^{-1}\| \cdot \|A_{m2}\| & \cdots & 1 \end{pmatrix} = K = (k_{ij})$$

Then (1.4) becomes

$$\mu(PA)X \leq 0, \quad (1.5)$$

where $X = (X_1, X_2, \dots, X_m) \in R^m \setminus \{0\}$. Our aim is to show that the hypotheses of the theorem contradict (1.5).

To do that, it is sufficient to show that, for some $l \in \langle m \rangle$,

$$\sum_{i=1}^l \sum_{j=1}^m k_{ij} X_j > 0. \quad (1.6)$$

Let

$y_1 = X_1 - X_2, y_2 = X_2 - X_3, \dots, y_{m-1} = X_{m-1} - X_m, y_m = X_m$.
By (1.3),

$$y \geq 0 \text{ (and, since } Z \neq 0, y \neq 0 \text{)}. \quad (1.7)$$

Define $B(r, s) = \sum_{i=1}^r \sum_{j=1}^s k_{ij}$. Then (1.6) becomes

$$\sum_{j=1}^m B(l, j) y_j > 0. \quad (1.8)$$

By (1.1), there exists $l \leq k$ such that

$$B(r, m) \leq 0 \text{ for } r < l, \quad (1.9)$$

But

$$B(l, m) > 0. \quad (1.10)$$

From (1.2) and (1.9), we infer $B(l, j) > 0$ for $j = 1, 2, \dots, l-1$. And from (1.10), we infer $B(l, j) > 0$ for $j = l, l+1, \dots, m$.

In fact, if $l = 1$, then $B(1, 1) \geq B(1, 2) \geq \dots \geq B(1, m) > 0$.

If $l > 1$, by (1.10), then $B(l, m) = \sum_{i=1}^l (1 - \tilde{R}_i(A)) > 0$.

Further, $B(l, l) \geq B(l, l+1) \geq \dots \geq B(l, m) > 0$.

Since

$$\begin{aligned} B(l, l-1) &= \sum_{i=1}^l \sum_{j=1}^{l-1} k_{ij} = \sum_{j=1}^{l-1} \sum_{i=1}^l k_{ij} = \sum_{j=1}^{l-1} k_{ij} + \sum_{j=1}^{l-1} \sum_{j \neq k} k_{ij} \\ &= \sum_{j=1}^{l-1} 1 - \sum_{j=1}^{l-1} \sum_{i \neq j} \|A_{ii}^{-1}\| \cdot \|A_{ij}\| \geq \sum_{j=1}^{l-1} (1 - \tilde{C}_{jk}(A)), \end{aligned} \quad (1.11)$$

By (1.9), then $B(l-1, m) = \sum_{i=1}^{l-1} \sum_{j=1}^m k_{ij} = \sum_{i=1}^{l-1} (1 - \tilde{R}_i(A)) \leq 0$.

From (1.2), then $\sum_{i=1}^{l-1} (1 - \tilde{C}_{ik}(A)) > 0$. Further,

$B(l, l-1) > 0$. Hence

$$B(l, l-2) > 0, \dots, B(l, 2) > 0, B(l, 1) > 0. \quad (1.12)$$

Further, (1.11), (1.12) and (1.7) imply (1.8), therefore (1.6), contradicting (1.5). Hence, A is nonsingular.

If we assume that

$$1 < \tilde{R}_{i_0}(A) \text{ for at least one } i, \quad (2.1)$$

then Theorem 1 can be improved.

Theorem 2. Let $A = (a_{ij}) \in C^{n \times n}$ be partitioned as (1) and satisfy (2.1), in which A_{ii} is nonsingular for $i \in \langle m \rangle$. If there exists $k \in \langle m \rangle$ such that

$$|S_k| = k \text{ implies } \sum_{i \in S_k} (1 - \tilde{R}_i(A)) \geq 0 \text{ for all } S_k \subseteq \langle m \rangle \quad (2.2)$$

and (1.6) holds. Then A is nonsingular.

Proof. We will call the index t a skip if $y_t > 0$, and SK the set of skips (which we know to be nonempty). Our object is to show that, for suitable $l \in \langle m \rangle$,

$$B(l, t) \geq 0 \text{ for all } t \in SK, \quad (2.3)$$

and

$$B(l, t) > 0 \text{ for at least one } t \in SK. \quad (2.4)$$

This will prove (1.8) and establish a contradiction of the supposition that A is singular.

Observe first that, by (2.2),

$$B(r, m) \geq 0 \text{ if } r \geq k. \quad (2.5)$$

Next, let r be the smallest index in SK . Firstly, we show the case $r \geq k$.

If $B(r, m) > 0$, then

$$B(r, r) \geq B(r, r+1) \geq \dots \geq B(r, m) > 0.$$

Let $l = r$, then $B(l, t) = B(r, t) > 0$ for all $t \in SK$.

Hence (2.3) and (2.4) hold, and we are done. So suppose $B(r, m) = 0$. Because r is the first skip, we know

$$X_1 = X_2 = \dots = X_r = \max\{X_i\}.$$

Hence $1 \leq R_i(A)$ for all $i \in \langle r \rangle$. Since (2.1) and $r \geq k$, there exists S_k such that $\sum_{i \in S_k} (1 - R_i(A)) < 0$, contradicting

(2.2).

Secondly we show the case $r < k$. Let l be the smallest skip such that $B(l, m)$ is nonnegative.

If there exists l , by (2.2), then $l < k$. Further $t \leq l$ and

$B(t, m) \leq 0$ for $t \in SK$. So $t \leq \sum_{i=1}^l R_i(A)$. By (1.2), then

$$B(l, t) = \sum_{i=1}^l \sum_{j=1}^t k_{ij} = \sum_{j=1}^t k_{jj} + \sum_{j=1}^t \sum_{i=1, i \neq j}^l k_{ij} \geq \sum_{j=1}^t (1 - \tilde{C}_{ik}(A)) > 0.$$

Hence (2.4) holds. Since

$$B(l, l) \geq B(l, l+1) \geq \dots \geq B(l, m) \geq 0,$$

(2.3) holds.

If l does not exist, then $B(t, m) < 0$ for all $t \leq k$ and

$t \in SK$. Further $t < \sum_{i=1}^t \tilde{R}_i(A)$. Let $l = k$, by (2.2),

then $B(k, m) \geq 0$. Hence $B(k, t) \geq B(k, m) \geq 0$. For all $t \geq k$ and $t \in SK$, by (1.2), then

$$B(k, t) = \sum_{i=1}^k \sum_{j=1}^t k_{ij} = \sum_{j=1}^t k_{jj} + \sum_{j=1}^t \sum_{i=1, i \neq j}^k k_{ij} \geq \sum_{j=1}^t (1 - \tilde{C}_{ik}(A)) > 0.$$

Hence (2.3) and (2.4) hold. The theorem is true.

IV. APPLICATIONS

Corollary 3. Let $A = (a_{ij}) \in C^{n \times n}$ be partitioned as (1), in which A_{ii} is nonsingular for $i \in \langle m \rangle$. If A and k satisfy (1.2) or satisfy (2.1) and (2.2), and for some α , $0 < \alpha < 1$,

$1 > \alpha \tilde{R}_i(A) + (1 - \alpha) \tilde{C}_{ik}(A)$ for all $i \in \langle m \rangle$. Then A is nonsingular.

Corollary 4. Let $A = (a_{ij}) \in C^{n \times n}$ be partitioned as (1), in which A_{ii} is nonsingular $M -$ matrix for $i \in \langle m \rangle$. Take the vector norm to be Euclidean norm. If A and k satisfy (1.1) and (1.2), then A is positive stable matrix.

Proof. For any $\lambda \in \sigma(A)$, there exists a nonzero

$$X = (X_1^T, X_2^T, \dots, X_M^T)$$

such that $AX = \lambda X$, in which the partitioned vector of X is uniform with A .

If $\operatorname{Re} \lambda \leq 0$, then $\lambda \notin \bigcup_{j \in \langle m \rangle} \sigma(A_{jj})$ (see [6]). Further,

$$\frac{\|(A_{ii} - \lambda I_{n_i})^{-1} y\|}{\|y\|} \leq \frac{\|A_{ii}^{-1}\| \cdot \|y\|}{\|y\|} \quad \text{for all } y \in C^n \setminus \{0\}$$

(see [6]).

Hence $\|(A_{ii} - \lambda I_{n_i})^{-1}\| \leq \|A_{ii}^{-1}\|$. For $A_\lambda = A - \lambda I$, then

$\tilde{R}_i(A) \geq \tilde{R}_i(A_\lambda)$, $\tilde{C}_{ik}(A) \geq \tilde{C}_{ik}(A_\lambda)$. Therefore,

$$\sum_{i \in S_k} (1 - \tilde{R}_i(A_\lambda)) \geq \sum_{i \in S_k} (1 - \tilde{R}_i(A)) > 0,$$

$$|S| > \min\{\sum_{i \in S} \tilde{R}_i(A), \sum_{i \in S} \tilde{C}_{ik}(A)\} \geq \min\{\sum_{i \in S} \tilde{R}_i(A_\lambda), \sum_{i \in S} \tilde{C}_{ik}(A_\lambda)\}.$$

By Theorem 1, $A_\lambda = A - \lambda I$ is nonsingular, contradicting A_λ to be singular. Hence the theorem is true.

Similarly, we can get the following corollary.

Corollary 5. Let $A = (a_{ij}) \in C^{n \times n}$ be partitioned as (1), in which A_{ii} is nonsingular $M -$ matrix for $i \in \langle m \rangle$. Take the vector norm to be Euclidean norm. If A and k satisfy (1.6), (2.1) and (2.2), then A is the positive stable matrix.

Clearly, the above results have generalized the corresponding results of [1]~[4].

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