

Isochronous Systems and Perturbation Theory

J-P FRANCOISE

*Laboratoire Jacques Louis Lions
Université P.-M. Curie, Paris VI and CNRS
Site Chevaleret, Bureau 8C20
165 Rue du Chevaleret, 75013 Paris
France.
E-mail: jpf@ccr.jussieu.fr*

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Abstract

This article displays examples of planar isochronous systems and discuss the new techniques found by F. Calogero with these examples. A sufficient condition is found to keep track of some periodic orbits for perturbations of isochronous systems.

1 Introduction

A large number of articles appeared recently inspired by a beautiful idea of Francesco Calogero to build new examples of isochronous systems, to such an extent to justify fully the apparently provocative title “Isochronous Systems are not rare” (cf. [5]). Before his contributions a large part of the mathematical litterature on isochronous systems was only devoted to planar systems. In this article we review briefly families of examples of planar systems and we discuss the use of the “Trick” discovered by Calogero to find new isochronous systems. Most often use of the trick provides shorter and more natural proofs. Motivated by a complex 2-dimensional family (which contains the so-called Complex Kukles Systems), we prove a sufficient condition for a persistence of periodic orbits for 1-parameter deformations of isochronous systems. This condition is expressed by the vanishing of an integral and is apparently close to “moment conditions” that were found in relation with the Poincaré-center focus problem of planar systems ([1],[2],[16]). It is a pleasure to dedicate this article to Francesco for his 70th birthday in recognition of all what I learnt from him.

2 Domains of isochrony and their boundaries, the example of holomorphic systems

These first two paragraphs review the examples previously known of isochronous systems in the theory of planar vector fields. The survey [13] and the articles [14], [15], [22] and

[21] have been quite helpful. An interesting point is that the trick found by F. Calogero provides in almost all examples a much simpler proof and that the possibility to use this trick does not seem to be known to specialists of planar systems.

The study of isochronous systems started with the cycloidal pendulum for which Huygens [20] showed that the system linearizes when the arclength is the independent variable. Next the first general class of plane systems for which isochrony was proved [17], [18], [19] is that of the complex systems:

$$\dot{z} = F(z). \quad (2.1)$$

This complex differential equation can be written as a differential system in the real plane, where $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$. For such systems it was shown that

Theorem 1.1 *A simple zero z_0 of F is a center if and only if $F'(z_0)$ is purely imaginary and in that case the center is isochronous.*

Proof. The classical proof goes like this: Consider the Cauchy integral

$$R(z) = \int^z \frac{1}{F(\zeta)d\zeta}.$$

If ζ moves once around z_0 , the value of $R(z)$ increases by $2\pi i/F'(z_0)$. Thus

$$H(z) = \exp[F'(z_0)R(z)] \quad (2.2)$$

is holomorphic at z_0 and the equation (2.2) yields

$$\frac{d^2}{dt^2}H(z) = (F'(z_0))^2H(z). \quad (2.3)$$

This shows that the system linearizes after the holomorphic change of variable $z \mapsto H(z)$ and thereby proves its isochrony. ■

A typical example is

$$\dot{z} = i\omega z + \alpha z^n, \quad (2.4)$$

where the origin is an isochronous center of period 2π . Apart from the origin there are $(n-1)$ other critical points corresponding to roots of $z^{n-1} = \frac{i}{\alpha}$. At these points $F'(z) = -(n-1)i$.

A vector field displays a domain of isochrony if there is an invariant non empty open set where all the orbits are periodic with same period. F. Calogero observed that many examples (which have some kind of homogeneity) of autonomous systems,

$$\frac{d^k z}{d\tau^k} = f\left(z, \frac{dz}{d\tau}, \dots, \frac{d^{k-1}z}{d\tau^{k-1}}\right), \quad (2.5)$$

remain autonomous when the complex time-like variable τ is changed into

$$\tau = \frac{e^{i\omega t} - 1}{i\omega} \quad (2.6)$$

and the independent variable is multiplied by an appropriate factor $e^{i\alpha t}$. Such examples can be easily constructed with $z \in C^n$ for any n . These complex differential systems can then of course be considered as plane real vector fields by representing the complex variable z via its real and imaginary parts (x, y) . Domains of initial data of equation (2.1) such that the solution exists and is holomorphic on a disc which contains the disc centered at 1 of radius $1/\omega$ yield domains such that all orbits are T -periodic, $T = 2\pi/\omega$, for the system transformed into the new real “time” variable t . This natural construction yields the isochronous domains.

Furthermore, and this last aspect was fully explored in various papers (ref.[4-12]), the singularities of the solutions of the system (2.1) as functions of the complex variables τ give some information on the boundary of the isochronous domains. Analytic extension of the solution determines other domains either isochronous with another period (algebraic branching points) or domains without any periodicity properties (logarithmic singularity, natural boundary,...). One of the most attractive prospects is perhaps to determine possible transitions from isochronicity to deterministic chaos in this framework (cf. [10,11]).

We only mention that the conclusions obtained from theorem 1 can be recovered much more easily with Calogero’s trick.

3 Examples from perturbations of centers and associated Abel equations

We consider now this type of planar system (cf [14],[15]):

$$\begin{aligned} \dot{x} &= -y + xf(x, y) \\ \dot{y} &= x + yf(x, y) \end{aligned} \tag{3.1}$$

where f is a polynomial without a linear part. This system is a perturbation of the linear center at the origin. It is usual for such systems to use polar coordinates (r, θ) . This yields

$$\begin{aligned} \dot{\theta} &= 1 \\ \dot{r} &= \frac{dr}{d\theta} = rf(r\cos(\theta), r\sin(\theta)), \end{aligned}$$

and periodic orbits (of period 2π) correspond to solutions of

$$\int_0^{2\pi} rf(r\cos(\theta), r\sin(\theta))d\theta = 0.$$

Assume furthermore that f is a homogeneous polynomial. Then we obtain in that case that (3.1) is isochronous if and only if

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = 0.$$

The general class of systems:

$$\begin{aligned} \dot{x} &= -y + f(x, y), \\ \dot{y} &= x + g(x, y) \end{aligned} \tag{3.2}$$

with homogeneous polynomials f and g of the same degree n yields

$$\begin{aligned} \dot{r} &= r^n A(\theta), \\ \dot{\theta} &= 1 + r^{n-1} B(\theta) \end{aligned} \quad (3.3)$$

in polar coordinates. The solution $r(\theta)$ of the equation

$$\frac{dr}{d\theta} = \frac{r^n A(\theta)}{1 + r^{n-1} B(\theta)} \quad (3.4)$$

can be expressed as a series in terms of θ and the initial value r_0 :

$$r(\theta) = \sum_k v_k(\theta) r_0^k.$$

If all coefficients $v_k(2\pi)$ vanish for $k > 1$ ($v_1 \equiv 1$), then all orbits of (3.2) are periodic in a neighborhood of the origin and the system is called a center.

If we apply Cherkas transform:

$$\rho = \frac{r^{n-1}}{1 + r^{n-1} B(\theta)}, \quad (3.5)$$

equation (3.5) gets transformed into the trigonometric Abel's equation:

$$\frac{d\rho}{d\theta} = p(\theta)\rho^2 + q(\theta)\rho^3 \quad (3.6)$$

with

$$p(\theta) = (n-1)A(\theta) + B'(\theta)$$

and

$$q(\theta) = -(n-1)A(\theta)B(\theta).$$

Isochronous trigonometric Abel equations correspond to centers of the planar system (3.2). The solution $\rho(\theta)$ of equation (3.5) may be expanded as a convergent series in terms of the initial data ρ_0 :

$$\rho(\theta) = \sum_k w_k(\theta) \rho_0^k.$$

There is now a nice characterisation of isochronous systems (3.2) due to Christopher and Devlin. System (3.2) is isochronous if and only if it is a center and if the following integrals vanish:

$$\int_0^{2\pi} w_k(\theta) B(\theta) d\theta = 0.$$

This is indeed a direct consequence of (cf. (3.3), (3.5)):

$$\dot{\theta} = \frac{1}{1 - \rho B(\theta)} \quad (3.7)$$

4 2-dimensional complex systems

We consider now the following 2-dimensional (complex) family of examples mentioned in [14] (which includes the Complex Kukles Systems):

$$\begin{aligned} \dot{u} &= -iu - f(u), \\ \dot{v} &= iv - g(u) \end{aligned} \tag{4.1}$$

where the polynomials f and g do not contain linear parts. The proof of the isochrony of the system (4.1) presented by Christopher and Devlin uses theorem 1 and the fact that:

$$\int_0^{2\pi} e^{-is} g(u(s)) ds = 0.$$

Note that the isochrony of (4.1) is also a consequence of the trick as example (4.1) verifies the conditions assumed for the much larger class (in any dimension) considered in [9]. More generally consider a linear isochronous system (say of period 1):

$$\begin{aligned} \dot{x} &= Ax, \quad x \in R^n, \\ \dot{y} &= By, \quad y \in R^m, \end{aligned}$$

where A and B are constant matrices, and its perturbation:

$$\dot{y} = By + g(x).$$

Then the perturbed system remains isochronous if and only if

$$\int_0^1 \exp(-sB) g(\exp(sA)x_0) ds = 0. \tag{4.2}$$

Such examples suggest to investigate the more general perturbations of isochronous systems presented below.

5 On a sufficient condition for keeping track of periodic orbits when isochronous systems are perturbed

We consider here a real n -dimensional ($x \in R^n$) isochronous system, possibly time-dependent, (of class C^2):

$$\dot{x} = f(x, t). \tag{5.1}$$

Isochrony means, of course, that all solutions of (5.1) are periodic with the same period (say $T = 1$) and f is 1-periodic in t . We consider a 1-parameter deformation:

$$\dot{x} = f(x, t) + \epsilon g(x, t, \epsilon) \tag{5.2}$$

This perturbation is possibly time-dependent, but in that case the perturbation is also assumed to be 1-periodic in t .

It is convenient to choose a parametrization of all the (periodic) orbits of system (5.1) by a n -dimensional parameter

$$\alpha = (\alpha_1, \dots, \alpha_n) \in R^n.$$

Denote as

$$x_\alpha(t)$$

the full family of 1-periodic solutions of (5.1). Linearization of system (5.1) along the particular solution $x_\alpha(t)$ yields the linear system:

$$\dot{u} = D_x f(x_\alpha(t))u, \quad u \in R^n. \quad (5.3)$$

Denote as $U(\alpha, t)$ the fundamental matrix solution of the linear system (5.3):

$$\begin{aligned} \frac{dU(\alpha, t)}{dt} &= D_x f(x_\alpha(t))U(\alpha, t), \\ U(\alpha, 0) &= Id. \end{aligned}$$

Given the deformation (4.2) it is interesting to consider the following integrals (that we propose to call the “moments” of the deformation in analogy with [1],[2],[16]):

$$\psi(\alpha) = \int_0^1 U^{-1}(\alpha, s)g(u_\alpha(s), 0)ds. \quad (5.4)$$

Theorem 1. *Assume that there exists a solution $\alpha = \alpha^0$ to the equations (vanishing of moments):*

$$\psi(\alpha) = 0$$

such that the matrix

$$\left. \frac{\partial \psi_i(\alpha)}{\partial \alpha_j} \right|_{\alpha=\alpha^0}$$

is invertible. Then there exists a unique periodic solution of (5.2) which tends to $x_{\alpha^0}(t)$ when ϵ tends to zero.

Note that these conditions depend only on the first derivative of the perturbation relative to the parameter ϵ .

A basic example (1-dimensional) is provided by the Abel equations:

$$\dot{x} = p(t)x^2 + \epsilon q(t)x^3.$$

The solution to the unperturbed system is:

$$x_\alpha(t) = \frac{\alpha}{1 - \alpha P(t)}$$

with

$$P(t) = \int_0^t p(s)ds.$$

Provided condition $P(1) = 0$ holds, the unperturbed system is isochronous. Fixing the periodic solution corresponding to $\alpha = \alpha^0$ and linearizing around this solution yields:

$$\frac{du}{dt} = 2 \frac{\alpha^0 p(t)}{1 - \alpha^0 P(t)} \cdot u$$

and

$$U(s) = [1 - \alpha^0 P(s)]^{-2}.$$

The moments read:

$$\int_0^1 U(s)^{-1} g(x_{\alpha^0}(s)) ds = (\alpha^0)^3 \int_0^1 \frac{q(s)}{1 - \alpha^0 P(s)} ds.$$

The method of proof follows the lines of Malkin (cf [23],[24],[25]) in a different context.

Proof. Consider firstly the case for which the system (5.1) is linear, namely

$$\dot{x} = f(x, t) = P(t)x + q(t). \tag{5.5}$$

Its perturbation is:

$$\frac{dx}{dt} = P(t)x + q(t) + \epsilon F(x, t, \epsilon). \tag{5.6}$$

Denote as $U(s)$ the fundamental solution of the linear homogeneous equation associated to equation (5.5). The solution to (5.5) is:

$$x(t) = U(t)[x(0) + \int_0^t U(s)^{-1} q(s) ds],$$

and we assume that both $U(1) = Id$ and $\int_0^1 U(s)^{-1} q(s) ds = 0$. Then, of course, the system (5.5) is isochronous.

Assume that the solutions $x(t, x(0), \epsilon)$ of (5.6) exist for all values of t , $0 \leq t \leq 1$, and define a differentiable function of their initial data $x(0)$ (or of α). This is for instance true for perturbations of linear systems if ϵ is small enough.

The solutions of (5.6) are:

$$x(t) = U(t)[x(0) + \int_0^t q(s) ds + \epsilon \int_0^t U(s)^{-1} F(x(s, \alpha, \epsilon), s, \epsilon) ds]. \tag{5.7}$$

Periodic solutions of (5.6) are in 1-1 correspondence with solutions of the differentiable equations:

$$\Psi(\alpha, \epsilon) = \int_0^1 U(s)^{-1} F(x(s, \alpha, \epsilon), s, \epsilon) ds = 0. \tag{5.8}$$

In this particular case the conclusion is easily obtained from the implicit function theorem.

Consider now the general situation:

$$\frac{dx}{dt} = f(x, t) + \epsilon g(x, t, \epsilon). \tag{5.9}$$

Fix once for all a solution $\alpha = \alpha^0$ which satisfies the conditions of the theorem. Consider the change of variable

$$x = x_{\alpha^0}(t) + \epsilon\xi. \quad (5.10)$$

The equation (5.9) is transformed into:

$$\frac{d\xi}{dt} = D_x f(x_{\alpha^0}(t), t)\xi + g(x_{\alpha^0}(t), t, 0) + \epsilon F(\xi, t, \epsilon). \quad (5.11)$$

Set furthermore:

$$P(t) = D_x f(x_{\alpha^0}(t), t), \quad q(t) = g(x_{\alpha^0}(t), t, 0),$$

and denote by $U(t)$ the fundamental solution of the associated homogeneous linear equation:

$$\frac{d\xi}{dt} = P(t)\xi + q(t). \quad (5.12)$$

The solution of (5.12) is:

$$\xi(t) = U(t)[\xi(0) + \int_0^t U(s)^{-1}q(s)ds]. \quad (5.13)$$

The vanishing of moments yield:

$$\int_0^1 U(s)^{-1}q(s)ds = 0 \quad (5.14)$$

and the condition of isochrony of the linearized equation is: $U(1) = Id$.

Note that the solutions of equation (5.12) can be obtained by adding to one particular solution any linear combination of solutions of the associated homogeneous equation. Thus they depend upon n arbitrary parameters, $\gamma = (\gamma_1, \dots, \gamma_n)$, such that

$$\xi(\gamma, t) = \sum_{k=1}^n \gamma_k \frac{\partial x_{\alpha^0}(t)}{\partial \alpha_k} + \bar{\xi}(t). \quad (5.15)$$

Alternatively the γ can be seen as coordinates and the mapping which takes the initial data of a solution of (5.12) to the corresponding γ as a change of coordinates on the space of periodic orbits. A solution $\xi(t, \epsilon)$ of equation (5.11) can either be seen as a differentiable function of its initial data or of the coordinates γ and, as such, we use the notation: $\xi(\gamma, t, \epsilon)$.

Periodic solutions of (5.11) (and thus of (5.2)) are in 1-1 correspondence with solutions of the differentiable equations:

$$\Psi(\gamma, \epsilon) = \int_0^1 U(s)^{-1}F(\xi(\gamma, s, \epsilon), s, \epsilon)ds = 0. \quad (5.16)$$

Observe that the quantities $F(\xi, s, 0)$ are quadratic functions of ξ :

$$\begin{aligned}
 F(\xi, s, 0) &= \frac{1}{2} \sum_{k,l} \frac{\partial^2 f}{\partial x_k \partial x_l}(x_{\alpha^0}(s), s) \xi_k \xi_l + \sum_k \frac{\partial g}{\partial x_k}(x_{\alpha^0}(s), s, 0) \xi_k \\
 &+ \frac{\partial g}{\partial \epsilon}(x_{\alpha^0}(s), s, 0).
 \end{aligned}
 \tag{5.17}$$

Then the solutions $\xi(t)$ depend linearly on γ . To simplify the notation we write z_k for the k -th component of $z = x_\alpha(s)$. We thus obtain that *a priori* $\Psi(\gamma, 0)$ are quadratic functions of γ

$$\begin{aligned}
 \Psi(\gamma, 0) &= \frac{1}{2} \sum_{qrkl} \gamma_q \gamma_r \int_0^1 U(s)^{-1} \frac{\partial^2 f}{\partial x_k \partial x_l} \frac{\partial z_k}{\partial \alpha_q} \frac{\partial z_l}{\partial \alpha_r} ds \\
 &+ \sum_{qkl} \gamma_q \int_0^1 U(s)^{-1} \left[\frac{1}{2} \frac{\partial^2 f}{\partial x_k \partial x_l} \left(\frac{\partial z_k}{\partial \alpha_q} \bar{\xi}_l + \frac{\partial z_l}{\partial \alpha_q} \bar{\xi}_k \right) + \frac{\partial g}{\partial x_k} \frac{\partial z_k}{\partial \alpha_q} \right] ds + \dots,
 \end{aligned}
 \tag{5.18}$$

where the dots represent quantities independent of γ . We use then the expression

$$\frac{d}{ds} \left(\frac{\partial^2 z}{\partial \alpha_q \partial \alpha_r} \right) = \sum_{kl} \frac{\partial^2 f}{\partial x_k \partial x_l} \frac{\partial z_k}{\partial \alpha_q} \frac{\partial z_l}{\partial \alpha_r} + \sum_k \frac{\partial f}{\partial x_k} \frac{\partial^2 z_k}{\partial \alpha_q \partial \alpha_r}.
 \tag{5.19}$$

This allows one to find the homogeneous quadratic part as:

$$\begin{aligned}
 \sum_{kl} \int_0^1 U(s)^{-1} \frac{\partial^2 f}{\partial x_k \partial x_l} \frac{\partial z_k}{\partial \alpha_q} \frac{\partial z_l}{\partial \alpha_r} ds &= \int_0^1 U(s)^{-1} \frac{d}{ds} \left(\frac{\partial^2 z}{\partial \alpha_q \partial \alpha_r} \right) ds \\
 - \sum_k \int_0^1 U(s)^{-1} \frac{\partial f}{\partial x_k} \frac{\partial^2 z_k}{\partial \alpha_q \partial \alpha_r} ds.
 \end{aligned}
 \tag{5.20}$$

Integration by parts yields:

$$\sum_{kl} \int_0^1 U(s)^{-1} \frac{\partial^2 f}{\partial x_k \partial x_l} \frac{\partial z_k}{\partial \alpha_q} \frac{\partial z_l}{\partial \alpha_r} ds = - \int_0^1 \left\{ \frac{[dU(s)^{-1}]}{ds} + U(s)^{-1} \frac{\partial f}{\partial x_k} \right\} \frac{\partial^2 z_k}{\partial \alpha_q \partial \alpha_r} ds = 0.
 \tag{5.21}$$

Consider now the coefficient of the linear part:

$$\sum_{kl} \int_0^1 U(s)^{-1} \left[\frac{\partial^2 f}{\partial x_k \partial x_l} \bar{\xi}_l + \frac{\partial g}{\partial x_k} \right] \frac{\partial z_k}{\partial \alpha_q} ds
 \tag{5.22}$$

and the moment:

$$\psi(\alpha) = \int_0^1 U(s)^{-1} g(x_\alpha(s), s, 0) ds.$$

We can write:

$$\frac{d\psi(\alpha)}{d\alpha_q} = \int_0^1 \left\{ \frac{\partial [U(s)^{-1}]}{\partial \alpha_q} g + \sum_k U(s)^{-1} \frac{\partial g}{\partial x_k} \frac{\partial x_k}{\partial \alpha_q} \right\} ds.
 \tag{5.23}$$

Note that:

$$\frac{d\bar{\xi}}{ds} = \sum_k \frac{\partial f}{\partial x_k} \bar{\xi}_k + g(x_\alpha(s), s, 0) \quad (5.24)$$

and we obtain:

$$\frac{d\psi(\alpha)}{d\alpha_q} = \int_0^1 \left\{ \frac{\partial[U(s)^{-1}]}{\partial \alpha_q} \left(\frac{d\bar{\xi}}{ds} - \sum_k \frac{\partial f}{\partial x_k} \bar{\xi}_k \right) + \sum_k U(s)^{-1} \frac{\partial g}{\partial x_k} \frac{\partial x_k}{\partial \alpha_q} \right\} ds. \quad (5.25)$$

Integration by parts yields:

$$\begin{aligned} \frac{d\psi(\alpha)}{d\alpha_q} &= - \int_0^1 \frac{d}{ds} \left\{ \frac{\partial[U(s)^{-1}]}{\partial \alpha_q} \right\} \bar{\xi} ds - \sum_k \int_0^1 \frac{\partial[U(s)^{-1}]}{\partial \alpha_q} \frac{\partial f}{\partial x_k} \bar{\xi}_k ds \\ &+ \sum_k \int_0^1 U(s)^{-1} \frac{\partial g}{\partial x_k} \frac{\partial x_k}{\partial \alpha_q} ds. \end{aligned} \quad (5.26)$$

From the equation:

$$\frac{dU(s)^{-1}}{ds} = -U(s)^{-1} \text{Jac}(f), \quad (5.27)$$

we deduce:

$$\frac{d}{ds} \left\{ \frac{\partial U(s)^{-1}}{\partial \alpha_q} \right\} \bar{\xi} = - \sum_k \frac{\partial U(s)^{-1}}{\partial \alpha_q} \frac{\partial f}{\partial x_k} \bar{\xi}_k - U(s)^{-1} \sum_{kjl} \frac{\partial^2 f}{\partial x_j \partial x_l} \frac{\partial z_k}{\partial \alpha_q} \bar{\xi}_l. \quad (5.28)$$

Hence

$$\frac{\partial \psi(\alpha)}{\partial \alpha_q} = \sum_{kl} \int_0^1 U(s)^{-1} \left[\frac{\partial^2 f}{\partial x_k \partial x_l} \bar{\xi}_l + \frac{\partial g}{\partial x_k} \right] \frac{\partial z_k}{\partial \alpha_q} ds. \quad (5.29)$$

We now consider the equation for periodic orbits of the system (5.11):

$$\Psi(\gamma, \epsilon) = 0.$$

The constant term in ϵ is

$$\int_0^1 U(s)^{-1} F(\xi(\gamma, s), s, 0) ds$$

and this term is linear in γ . The matrix of the linear part is invertible and thus there is a unique solution $\gamma = \gamma^0$ to the equation:

$$\Psi(\gamma, 0) = 0.$$

Now the matrix

$$\frac{\partial \Psi(\gamma, \epsilon)}{\partial \gamma} \Big|_{\epsilon=0}$$

is invertible. So the implicit function theorem shows that there is a unique periodic solution to equation (5.11) which corresponds to $\gamma = \gamma(\epsilon)$. If we turn back to the initial equation

(5.2), we firstly fix $\alpha = \alpha^0$, then determine γ^0 and find the conclusion of the theorem. There is a unique periodic solution to (5.2),

$$x_{\alpha^0}(t) + \sum_k \gamma_k(\epsilon) \frac{dx_{\alpha^0}(t)}{d\alpha_k}(t),$$

which tends to $x_{\alpha^0}(t)$ when ϵ tends to 0 (with the “speed” γ^0). ■

As a final remark we wish to say that for planar systems, in case of perturbation of the linear center, the moments are Abelian integrals. Such quantities should play the role of Abelian integrals for perturbations of isochronous centers which are not Hamiltonian. It is certainly quite useful to note that they depend only on the linearized isochronous system and on an explicit expression of the solutions of the isochronous system.

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