Singular Scattering Matrices

David ATKINSON

Centre for Theoretical Physics, Nijenborgh 4
9747 AG Groningen, The Netherlands
E-mail: atkinson@phys.rug.nl

Abstract

A nonlinear integrodifferential equation is solved by the methods of S-matrix theory. The technique is shown to be applicable to situations in which the effective potential is singular.

1   Introduction

Given a function, $\lambda(m)$, for $m > 0$, how can one solve the following nonlinear integrodifferential equation for $\phi(m, r)$,

$$\frac{\partial^2 \phi(m, r)}{\partial r^2} + \frac{\phi(m, r)}{\pi} \frac{\partial}{\partial r} \int_0^\infty dm' \, \lambda(m') \, \mathrm{e}^{-m'r/2} \, \phi(m', r) = \frac{m^2}{4} \phi(m, r) \,, \qquad (1)$$

subject to the boundary condition $\phi(m, r) \sim \mathrm{e}^{-mr/2}$ as $r \to \infty$? This is a problem after Francesco Calogero's heart and its solution is also quintessentially Calogerian, obtained as it is by an inverse method, beginning with the answer and working back to the question. I shall explain the technique, as Calogero and I developed it in 1969[3], and use it to derive some new results in singular Schrödinger potential theory.

First, however, let me set the scene. We are in the 1960's, the distant era of the Beatles, of flower power and the student revolution, and, in our area of expertise, the much vaunted S-matrix theory, one of the defunct attempts of particle physicists to invent a TOE, a theory of everything. The immodest claims of a dyed-in-the-wool S-matrix aficionado are lampooned in the following:

> "There is only one function that satisfies crossing and unitarity, with the analyticity required by a combination of one-particle poles with unitarity, the partial-wave projection of which is analytic in angular momentum, with no natural boundaries or delta-functions in the right-hand half of the angular momentum plane. Moreover, this function exists only for one particular single-particle mass-spectrum: this unique spectrum corresponds to the actual masses of the hadrons, and the unique function corresponds to the strong-interaction scattering matrix."

These lines are taken from one of my own papers[2], but I hasten to add that I myself was no acolyte of this 'bootstrap credo', nor was it ever part of Francesco's Weltanschauung.

Nevertheless, the process of generating an S-matrix from its left-hand cut discontinuity was a standard procedure, and one that has survived the demise of the S-matrix TOE.

In Sect. 2 the method that Calogero and I introduced in 1969[3] is explained, but a convention in the definition of the Jost solution of the Schrödinger equation is brought into line with that used in the later literature[6, 5]. In Sect. 3 a sufficient condition is given that the integral equation satisfied by the Jost solution of the Schrödinger equation have no more than one solution. This is a generalization of a technique used in [4] and moreover a new bound is given on the $\mathcal{L}^2$ norm of the kernel of the above integral equation in the case that the left-hand cut discontinuity is constant. Sect. 4 is devoted to a demonstration that the Jost function does not exist for a constant left-hand cut discontinuity and here the Mehler transform is used, as it was in[1]. Finally in Sect. 5 the Bargmann solutions are constructed from left-hand cut discontinuities that consist of a sum of delta distributions.

2   Integral Equation for Jost Solution

Consider the integral equation

$$\phi\left(m,r\right) = \mathrm{e}^{-mr/2} + \frac{1}{\pi} \int_0^\infty dm' \, \frac{\lambda(m')}{m'+m} \, \mathrm{e}^{-(m'+m)r/2} \, \phi\left(m',r\right) . \tag{2}$$

Here $\lambda(m')$ is called the left-hand cut discontinuity and we suppose it to be such that Eq.(2) is a Fredholm equation for all $r > 0$ (but not necessarily for $r = 0$).

Differentiating Eq.(2) twice with respect to r we find that

$$\phi''\left(m,r\right) = \tfrac{1}{4}m^2 \, \mathrm{e}^{-mr/2} \quad + \quad \frac{1}{\pi} \int_0^\infty dm' \, \frac{\lambda(m')}{m'+m} \, \mathrm{e}^{-(m'+m)r/2} \, \phi''\left(m',r\right)$$
$$- \quad \frac{1}{\pi} \int_0^\infty dm' \, \lambda(m') \, \mathrm{e}^{-(m'+m)r/2} \, \phi'\left(m',r\right)$$
$$+ \quad \frac{1}{4\pi} \int_0^\infty dm' \, \lambda(m')(m'+m) \, \mathrm{e}^{-(m'+m)r/2} \, \phi\left(m',r\right) ,$$

where the primes on $\phi\left(m,r\right)$ mean partial differentiations with respect to r. This expression can be cajoled into the form

$$\left[\frac{\partial^2}{\partial r^2} - \tfrac{1}{4}m^2\right] \phi\left(m,r\right) \quad = \tag{3}$$
$$V(r) \, \mathrm{e}^{-mr/2} + \frac{1}{\pi} \int_0^\infty dm' \, \frac{\lambda(m')}{m'+m} \, \mathrm{e}^{-(m'+m)r/2} \, \left[\frac{\partial^2}{\partial r^2} - \tfrac{1}{4}m'^2\right] \phi\left(m',r\right) ,$$

where

$$V(r) = -\frac{1}{\pi} \frac{\partial}{\partial r} \int_0^\infty dm' \, \lambda(m') \, \mathrm{e}^{-m'r/2} \, \phi\left(m',r\right) . \tag{4}$$

From Eq.(2) and Eq.(3) we observe that $V(r)\phi\left(m,r\right)$ and $\left(\frac{\partial^2}{\partial r^2} - \tfrac{1}{4}m^2\right) \phi\left(m,r\right)$ satisfy the same inhomogeneous Fredholm equation. Hence, on the condition that the corresponding homogeneous equation has no nontrivial solution, these functions must be equal, that is,

$$V(r)\phi\left(m,r\right) = \left[\frac{\partial^2}{\partial r^2} - \tfrac{1}{4}m^2\right] \phi\left(m,r\right) . \tag{5}$$

With $k = \frac{1}{2}im$ and $f(k,r) = \phi(m,r)$ Eq.(5) takes on the form

$$\left[-\frac{\partial^2}{\partial r^2} + V(r) \right] f(k,r) = k^2 f(k,r) \tag{6}$$

which may be interpreted as the reduced radial Schrödinger equation for S-wave scattering by a spherically symmetric potential, $V(r)$. Since m is real, k is in the first instance imaginary, but Eq.(6) can be continued into the complex k-plane without change of form.

Setting $m = -2ik$ in Eq.(2) but leaving m' unchanged, we find that

$$f(k,r) = e^{ikr} \left[1 + \frac{1}{\pi} \int_0^\infty dm' \frac{\lambda(m')}{m' - 2ik} e^{-m'r/2} \phi(m',r) \right] \tag{7}$$

which also may be continued into the k-plane. If the left-hand cut discontinuity, $\lambda(m')$, is suitably restricted, the integral term vanishes in the limit $r \to \infty$, so that $f(k,r) \sim e^{ikr}$ in this limit. We call $f(k,r)$ the Jost solution of the Schrödinger equation (6).

The scattering function is defined for real, positive, k by

$$S(k) = \lim_{r \to 0} \frac{f(-k,r)}{f(k,r)} = e^{2i\delta(k)}, \tag{8}$$

where $\delta(k)$ is called the S-wave phase shift. It may be that $S(k)$ and $\delta(k)$ are well-defined, even in singular cases for which $f(k,0)$ does not exist. We now explain how to calculate $S(k)$ in such singular situations. Multiplying Eq.(6) by $f(-k,r)$ we obtain

$$-f(-k,r)f''(k,r) + V(r)f(-k,r)f(k,r) = k^2 f(-k,r)f(k,r)$$

and, on subtracting the equation obtained from this by replacing k by $-k$ throughout, we find that the Wronskian of $f(k,r)$ and $f(-k,r)$ is independent of r. The Wronskian can be evaluated from the asymptotic forms

$$f(k,r) \sim e^{ikr} \qquad \text{and} \qquad f(-k,r) \sim e^{-ikr}$$

to yield $f(-k,r)f'(k,r) - f'(-k,r)f(k,r) = 2ik$ and so

$$\frac{\partial}{\partial r} \left[\frac{f(-k,r)}{f(k,r)} - e^{-2ikr} \right] = -2ik \left[\frac{1}{f^2(k,r)} - e^{-2ikr} \right] \tag{9}$$

whence, after integration,

$$S(k) = 1 + 2ik \lim_{\epsilon \to 0} \int_\epsilon^\infty dr \left[\frac{1}{f^2(k,r)} - e^{-2ikr} \right]. \tag{10}$$

In cases for which $f(k,r)$ diverges monotonically as $r \to 0$ this integral can be perfectly well defined in the limit $\epsilon \to 0$.

3   Fredholm Equation

A crucial step in the derivation of the Schrödinger equation (6) from the integral equation
for $\phi(m, r)$ was the assumption that Eq.(2) has a unique solution. A necessary and suf-
ficient condition for the inhomogeneous equation to have not more than one independent
solution is that the corresponding homogeneous equation,

$$\psi(m, r) = \frac{1}{\pi} \int_0^\infty dm' \frac{\lambda(m')}{m' + m} \, \mathrm{e}^{-(m'+m)r/2} \, \psi\left(m', r\right) \,, \tag{11}$$

possess only the trivial solution, $\psi(m, r) \equiv 0$. We shall now show that, if $\lambda(m) \leq 0$ for
all $m > 0$, then indeed $\psi(m, r)$ vanishes identically. This negativity condition is thus a
sufficient condition that the inhomogeneous Eq.(2) have at most one solution. Consider
the function

$$\begin{aligned} I(r) &= \int_0^\infty dm \frac{\lambda(m)}{m} \left[\psi(m, r)\right]^2 \\ &= \frac{1}{\pi} \int_0^\infty \int_0^\infty dm \, dm' \frac{\lambda(m)\lambda(m')}{m(m' + m)} \, \mathrm{e}^{-(m'+m)r/2} \, \psi(m, r) \, \psi\left(m', r\right) \,. \end{aligned} \tag{12}$$

Since

$$\frac{1}{m(m' + m)} = \left[\frac{1}{m} - \frac{1}{m' + m}\right] \frac{1}{m'} \,,$$

it follows that Eq.(12) can be written as the sum of two terms:

$$\begin{aligned} I(r) &= \frac{1}{\pi} \left[\int_0^\infty dm \frac{\lambda(m)}{m} \, \mathrm{e}^{-mr/2} \, \psi(m, r)\right]^2 \\ &\quad - \frac{1}{\pi} \int_0^\infty \int_0^\infty dm \, dm' \frac{\lambda(m)\lambda(m')}{m'(m' + m)} \, \mathrm{e}^{-(m'+m)r/2} \, \psi(m, r) \, \psi\left(m', r\right) \,. \end{aligned} \tag{13}$$

The second term on the right is clearly $I(r)$, but with a minus sign, while the first term
can be identified by setting $m = 0$ in Eq.(11) so that

$$2I(r) = \pi \left[\psi(0, r)\right]^2 \tag{14}$$

on condition that $\psi(0, r)$ exists. The right side here is nonnegative and, if $\lambda(m) \leq 0$ for
all $m > 0$, the left side is nonpositive, as we see from the first line of Eq.(12). Hence
$\psi(m, r)$ is identically zero and under these conditions the inhomogeneous Eq.(2) can have
no more than one solution. The existence of a unique solution is then guaranteed if $\lambda(m)$
is such that the equation is Fredholm. Note that this is only required for $r > 0$: it does
not matter if the equation goes wild at the point $r = 0$.

Consider next the constant left-hand cut discontinuity, $\lambda(m') = \pi\gamma\theta(m' - 1)$, such that
Eq.(2) takes the form

$$\phi(m, r) = \mathrm{e}^{-mr/2} + \gamma \int_1^\infty dm' \frac{\mathrm{e}^{-(m'+m)r/2}}{m' + m} \, \phi\left(m', r\right) \,. \tag{15}$$

The square of the $\mathcal{L}^2$ norm of this kernel is

$$L(r) = \int_1^\infty dm \int_1^\infty dm' \frac{\mathrm{e}^{-(m'+m)r}}{(m' + m)^2} = (1 + 2r)\Gamma(0, 2r) - \mathrm{e}^{-2r} \,,$$

where $\Gamma(0, z)$ is the incomplete Gamma function of order zero,

$$\Gamma(0, z) = \int_z^\infty \frac{dm}{m} \, e^{-m} \le \frac{e^{-z}}{z} \, ,$$

from which it follows that

$$L(r) \le \frac{e^{-2r}}{2r} \, .$$

Hence Eq.(15) is indeed a Fredholm equation for $r > 0$ so that the Fredholm alternative applies. In particular the equation has a unique $\mathcal{L}^2$ solution if and only if the corresponding homogeneous equation has no nontrivial solution. We have shown this condition to be satisfied if $\gamma < 0$.

4   Jost Function

In the previous section we saw that there are no special difficulties as long as r does not vanish. Here we show that Eq.(15) has no solutions, for any γ, if we do set $r = 0$. Consider the integral equation

$$\phi(m) = 1 + \gamma \int_1^\infty \frac{dm'}{m' + m} \, \phi(m') \, , \tag{16}$$

where we have written $\phi(m)$ for $\phi(m, 0)$. To show that there are no solutions we suppose *per impossibile* that there is a nontrivial solution, $\phi(m)$, and proceed to obtain a contradiction.

Subtracting Eq.(16) from its value at $m = 0$ and substituting $\psi(m) = \phi(m)/m$, we obtain

$$\psi(m) = \frac{\phi(0)}{m} - \gamma \int_1^\infty \frac{dm'}{m' + m} \, \psi(m') \, . \tag{17}$$

We do not know if $\phi(0)$ vanishes or not, so we consider both possibilities. If $\phi(0)$ is zero, the resulting homogeneous version of Eq.(17) is known to have no solution if $\gamma \ge 0$ and for $\gamma < 0$ to have the unique solution (up to normalization), $\psi(m) = P_\ell(m)$, the Legendre function of the first kind, with

$$\ell = -\frac{i}{\pi} \log \left[\sqrt{1 - \pi^2 \gamma^2} + i \pi \gamma \right] \, ,$$

the branch of the logarithm being chosen such that $-\frac{1}{2} \le \operatorname{Re} \ell < 0$ (see App. C of ref.[4]). However, in this case $\phi(m) = m \, P_\ell(m)$ and, since that diverges as $m \to \infty$, it cannot satisfy Eq.(16) for then the m'-integral would diverge. This puts paid to the possibility that Eq.(16) might have a solution for which $\phi(0) = 0$.

Suppose next that $\phi(0)$ does not vanish. We can set $\phi(0) = 1$ without loss of generality since Eq.(17) is linear. The equation can be diagonalized by a Mehler transformation[1, 4] and a particular solution may be written

$$\psi(m) = \frac{1}{m} + \frac{\gamma}{2} \int_{-\infty}^\infty ds \, \frac{s \tanh \pi s}{[\pi \gamma + \cosh \pi s] \cosh \pi s} \, P_{-\frac{1}{2}+is}(m) \, P_{-\frac{1}{2}+is}(0) \, . \tag{18}$$

If $\pi\gamma > -1$, the contour may be drawn along the real axis of s, but, if $\pi\gamma \leq -1$, the contour must be deformed to avoid the poles. In any case, as $m \to \infty$, this solution does not tend to zero more quickly than $m^{-\frac{1}{2}}$. Hence $\phi(m) = m\psi(m)$ diverges as $m^{\frac{1}{2}}$ and this is inconsistent with Eq.(16) for once more the m'-integral would diverge. For positive γ Eq.(18) is the general solution of Eq.(17), but for negative γ the general solution is obtained by adding a multiple of the solution of the homogeneous equation, $P_\ell(m)$. However, this does not help matters for, as we saw above, such a contribution to ϕ also results in divergence. This concludes the demonstration that Eq.(16) has no solution.

5   Bargmann Potentials

To conclude this paper we consider cases in which the Jost solution is meromorphic in the k-plane. Consider first a left-hand cut discontinuity that is just a single delta distribution:

$$\lambda(m') = \pi\gamma\,\delta(m' - 2b)\,. \tag{19}$$

On inserting this into Eq.(2) and setting $m = 2b$ we obtain a linear equation for $\phi(2b, r)$ with the solution

$$\phi(2b, r) = \frac{4b\,e^{br}}{4b\,e^{2br} - \gamma}\,.$$

The Jost function follows from Eq.(7):

$$f(k, r) = e^{ikr}\left[1 + \frac{2i\gamma b}{k + ib}\,\frac{1}{4b\,e^{2br} - \gamma}\right] \tag{20}$$

and Eq.(4) then yields the potential

$$V(r) = 32\gamma b^3\,\frac{e^{2br}}{\left(4b\,e^{2br} - \gamma\right)^2}\,. \tag{21}$$

Evidently we must impose $\gamma < 4b$ since for $\gamma > 4b$ the potential would have an unacceptable double pole at $r = (1/2b)\log(\gamma/4b)$. This form is called the Bargmann potential and the Jost function, i.e. the Jost solution at $k = 0$, is very simple:

$$f(k, 0) = \frac{k + ia}{k + ib} \qquad \text{with} \qquad a = \frac{4b + \gamma}{4b - \gamma}\,. \tag{22}$$

With $\gamma < 0$ the potential is negative (attractive), but, for $0 < \gamma < 4b$, it is positive (repulsive). For any $\gamma < 4b$ it is finite for all $r \geq 0$ and it tends exponentially to zero in the limit $r \to \infty$: $V(r) \sim 2\gamma\,b\,e^{-2br}$.

To generalize this result it is convenient firstly to define $\psi(m, r) = e^{mr/2}\phi(m, r)$ so that Eq.(2) takes the form

$$\psi(m, r) = 1 + \frac{1}{\pi}\int_0^\infty dm'\,\frac{\lambda(m')}{m' + m}\,e^{-m'r}\,\psi\left(m', r\right)\,. \tag{23}$$

Suppose now that

$$\lambda(m') = \pi\sum_{n=1}^N \gamma_n\,\delta(m' - b_n)$$

so that, from Eq.(23), we obtain

$$\psi(b_m, r) = 1 + \sum_{n=1}^{N} \frac{\gamma_n}{b_m + b_n} \, e^{-b_n r} \psi(b_n, r)$$

or in compact form

$$\sum_{n=1}^{N} M_{mn}(r) \, \psi(b_n, r) = 1 \,, \tag{24}$$

where

$$M_{mn}(r) = \delta_{mn} - \frac{\gamma_n}{b_m + b_n} \, e^{-b_n r} \,.$$

The necessary and sufficient condition for the existence of a solution of the matrix equation (24) is that $||M(r)||$ does not vanish. If this determinant is nonzero for all $r > 0$, then the unique solution is given in terms of the inverse matrix by

$$\psi(m, r) = \sum_{n=1}^{N} M_{mn}^{-1}(r) \,.$$

Finally this yields the Jost solution itself:

$$f(k, r) = e^{ikr} \left[1 + \sum_{n=1}^{N} \frac{\gamma_n}{b_n - 2ik} \, e^{-b_n r} \sum_{q=1}^{N} M_{nq}^{-1}(r) \right] \,.$$

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