Journal of Nonlinear Mathematical Physics Volume 12, Supplement 1 (2005), 625–632 Birthday Issue

RG Solutions for α_s at large N_c in $d = 3 + 1$ QCD

Yu A SIMONOV

ITEP, B.Cheremushkinskaya 25, Moscow, 117218, Russia
E-mail: simonov@heron.itep.ru

This article is part of the special issue published in honour of Francesco Calogero on the occasion of his 70th birthday

Abstract

Solutions of RG equations for $\beta(\alpha)$ and $\alpha(Q)$ are found in the class of meromorphic functions satisfying asymptotic conditions at large Q (resp. small α), and analyticity properties in the Q^2 plane. The resulting $\alpha_R(Q)$ is finite in the Euclidean Q^2 region and agrees well at $Q \geq 1$ GeV with the $\overline{MS}$ $\alpha_s(Q)$.

1 Introduction

QCD is known to simplify at large N_c: i) the perturbation theory dictates that only planar diagrams survive in the leading order [1]; ii) assuming that confinement exists also in the limit $N_c \to \infty$, the spectrum of mesons and glueballs consists of bound-state poles, and the decay width vanishes at large N_c[1, 2, 3].

In the $1+1$ QCD this property was proved both analytically and numerically [3] (for a review see [4]) since confinement in this case is induced by the perturbative gluon exchange.

On the lattice numerical calculations have confirmed that the large N_c limit is achieved with few percent accuracy already at $N_c = 3$ for several tested observables [5]. It is thus likely that the large N_c QCD is a good first approximation for the realistic $N_c = 3$ case, which has an accuracy of 10% or better.

From theoretical point of view the large N_c limit of QCD is very useful since it drastically simplifies the analytic structure of amplitudes, e.g. one has sums of simple poles in the two-point function, and for the four-point function one expects formulas of the Veneziano type.

Thus one expects the large N_c amplitudes as functions of external momenta to be meromorphic. On the other hand the perturbation series yields the typical logarithmic dependencies and the RG equation prescribes for $\alpha_s(Q)$ the structure with unphysical poles and cuts which are incompatible with unitarity and analyticity. E.g. the one-loop expression for $\alpha_s(Q)$ has the form

$$\alpha_s(Q) = \frac{4\pi}{\beta_0 \ln Q^2/\Lambda^2}, \quad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f \tag{1.1}$$

with the Landau ghost pole at $Q^2 = \Lambda^2$ and the cut with the branch point $Q^2 = 0$. This property being true for all N_c, strikingly violates the expected meromorphic structure of amplitudes for $N_c \to \infty$.

One might argue that $\alpha_s(Q)$ by itself is not yet the physical amplitude, and in the latter the unphysical features of α_s may be somehow compensated. This is however contradicted by examples of amplitudes, e.g. for the process $e^+e^- \to$ hadrons, where $\frac{\alpha_s(Q)}{\pi}$ enters directly into the hadronic ratio $R(s)$.

$$R(s) = R_{parton} + \frac{\alpha_s(s)}{\pi} + O\left(\frac{\alpha_s}{\pi}\right)^2. \tag{1.2}$$

Moreover one can define the "effective coupling" $\alpha_R(Q)$ for the process R, which enters the physical amplitude Ω_R directly [6]

$$\Omega_R = \Omega_R^{(0)} + \omega_R \alpha_R(Q) \tag{1.3}$$

and this "process dependent" α_R satisfies the standard RG equation

$$\frac{d\alpha_R(\mu)}{d\ln\mu} = \beta^{(R)}(\alpha) = -\frac{\beta_0^{(R)}\alpha^2}{2\pi} - \frac{\beta_1^{(R)}\alpha^3}{4\pi^2} + O(\alpha^4) \tag{1.4}$$

where $\beta_0^{(R)}$ and $\beta_1^{(R)}$ are standard scheme-independent coefficients, while $\beta_n^{(R)}, n > 1$ depend on the process. One can easily see, that the solution of (1.4) has the dominant asymptotic term of the form (1.1) with the unphysical features discussed above.

One might still argue that perturbative series and $\alpha_s(Q)$ itself should be considered only in the asymptotic regime when Q is large and therefore the logarithmic singularities and Landau ghost pole are far away.

However the analytic structure of Riemann surfaces with cuts is different everywhere from that of meromorphic function and this argument of asymptotics can be turned around to imply that the logarithmic asymptotics of $\alpha_s(Q)$ is an asymptotic approximation of the true analytic function which is meromorphic at large N_c and is in agreement with all expected properties of large N_c physical amplitudes.

It is the purpose of the present paper to exemplify the solutions of the RG equations which have the desired meromorphic properties. These solutions will have the standard logarithmic asymptotics in good numerical and analytic similarity with the standard perturbation theory.

Moreover we show that this meromorphic-logarithmic duality has deeper roots in the quark-hadron duality and provide explicit example of this connection.

The paper is organized as follows. In section 2 the RG equation for $\alpha_s(Q)$ is considered and the general form of solution is written down, containing an arbitrary function with known asymptotics. In section 3 the meromorphic solution is suggested and its properties are studies. In section 4 connection with the standard perturbation theory is investigated. The concluding section is devoted to general discussion of the meromorphic – logarithmic duality – in QCD and QED.

2 General solutions of RG equations

In this section we follow the line of reasoning which was given in [7]. We write the RG equation as in (1.4) suppressing the subscript R everywhere and express $\beta(\alpha)$ through another unknown function $\varphi\left(\frac{1}{\alpha}\right)$:

$$\frac{d\alpha(\mu)}{d\ln\mu}=\beta(\alpha),\quad \beta(\alpha)=-\frac{\beta_0}{2\pi}\frac{\alpha^2}{\left[1-\frac{\beta_1}{2\pi\beta_0}\varphi'\left(\frac{1}{\alpha}\right)\right]}. \tag{2.1}$$

Here $\varphi'(x)$ means the derivative of function $\varphi(x)$ in the argument x, and μ is the RG scale, which will be later traded as usual for external parameters (momenta) of the given process P_i^2 since α can depend only on the ratio P_i^2/μ.

In terms of the function φ the solution of the RG equation for $\alpha(\mu)$ can be immediately written

$$\alpha=\frac{4\pi}{\beta_0\left[\ln\mu^2+\chi+\frac{2\beta_1}{\beta_0^2}\varphi\left(\frac{1}{\alpha}\right)\right]}. \tag{2.2}$$

Here χ is an arbitrary function of P_i such that $\ln\mu^2+\chi$ is some function of $P_i^2/\mu^2, P_i^2/P_j^2$ the form of which depends on the process and will be found below for concrete examples.

The Standard Perturbation Theory (SPT) which is assumed to be valid at large Q^2, provides some limitations on the properties of functions χ and φ, which one must impose. Namely, the SPT Taylor expansion of $\beta(\alpha)$ is known for the first four terms

$$\beta(\alpha)=-\frac{\beta_0}{2\pi}\alpha^2-\frac{\beta'}{4\pi^2}\alpha^3-\frac{\beta_2}{64\pi^3}\alpha^4-\frac{\beta_3}{(4\pi)^4}\alpha^5+O(\alpha^6). \tag{2.3}$$

Here β_0,β_1 are scheme-independent and for $n_f=0$ are equal to $\beta_0=\frac{11}{3}N_c-\frac{2}{3}n_f$ and $\beta_1=\frac{17}{3}N_c^2-\frac{5}{3}N_cn_f-N_cn_f$, while β_2 and β_3 have been calculated in the $\overline{MS}$ scheme. Comparing (2.3) and (2.1) one obtains the following condition on the asymptotic behaviour of $\varphi(x)$ at large $x (x\equiv\frac{1}{\alpha})$:

$$\varphi(x)|_{x\to\infty}\cong\ln x+O\left(\frac{1}{x}\right). \tag{2.4}$$

The first term on the r.h.s. of (2.4) reproduces the scheme-independent coefficients β_0,β_1 in the expansion (2.3), while the term $O(\frac{1}{x})$ contributes to the higher order coefficients $\beta_2,\beta_3,...$ Conditions on the function $\chi(P^2)$ are more subtle and in general depend on the process in question. Here one should distinguish in P_i^2 the external parameters which can be made arbitrarily large, as the momentum Q^2 in the two-point function $\Pi(Q^2)$ or in the formfactor, and other renorm-invariant parameters, which define the scale of confinement, e.g. the string tension σ, or the RG invariant gluonic condensate $\frac{\beta(\alpha)}{16\alpha}\langle F_{\mu\nu}(0)F_{\mu\nu}(0)\rangle$. In the framework of the Background Perturbation Theory (BPT) [8, 9] both appear as vacuum expectation values of operators made of RG invariant combinations $gF_{\mu\nu}$, where $F_{\mu\nu}$ refers to the background field. Here we do not use BPT, but only consider all RG invariant parameters like σ on the same ground as the true external parameters – the external momenta like Q^2. In what follows we shall use the generic mass parameter $m^2\equiv 4\pi\sigma$ as the confinement scale and disregard for simplicity other vacuum field characteristics, like the gluonic correlation length λ, which can be computed in principle and in practice [10] through m^2.

As a result keeping only one external momentum Q^2, χ can be written as $\chi \equiv \chi(Q^2/m^2)$. Then the large Q^2 behaviour of $\alpha(Q^2)$,

$$\alpha(Q^2)(Q^2 \to \infty) \sim \frac{4\pi}{\beta_0 \ln Q^2/\Lambda^2} + O\left(\frac{\ln\ln Q^2}{(\ln Q^2)^2}\right) \tag{2.5}$$

dictates the following behaviour of $\chi + \ln \mu^2$, where we go over to the standard Λ_{QCD} parametrization instead of μ parametrization

$$(\chi + \ln \mu^2) \equiv \ln \frac{m^2}{\Lambda^2} + \Psi(Q^2, m^2), \quad \Psi|_{Q\to\infty} \cong \ln \frac{Q^2}{m^2}. \tag{2.6}$$

3 Meromorphic realization of RG solutions

As was discussed in Introduction, at large N_c the analytic structure of physical amplitudes and of effective charge $\alpha(Q)$ is simplified and reduced to the meromorphic functions with simple poles at $Q^2 = -M_n^2$, corresponding to the bound states of quarks and gluons – mesons, glueballs and hybrids.

Therefore the function $\Psi(Q^2, m^2)$ in (4.4) can be represented in the spectral form

$$\Psi(Q^2, m^2) = -\sum_{n=0}^{\infty} \frac{c_n}{Q^2 + M_n^2} + const \tag{3.1}$$

where the coefficients c_n are independent of Q^2. The spectrum M_n^2 was found for large N_c for mesons, glueballs and hybrids in the limit of no mixing between them [11]. In the large N_c limit the meson-glueball mixing vanishes, while the the meson-hybrid mixing is $O(\alpha)$ [12], and following [11] we write the lowest order spectrum in N_c and α independent, for glueballs and mesons as

$$M_n^{(0)2} = m^2(n + L/2) + M_0^2 + O(1/n) \tag{3.2}$$

where $m^2 = 4\pi\sigma$, and for glueballs $\sigma(glue) = \frac{9}{4}\sigma(quark)$. In what follows we shall be interested only in the asymptotic part of the spectrum at large n and $L = 0, 1$, (the correction to (3.2) for $n = 1, 2$ come from spin splittings and is of the order of 20%). In general one can rewrite the effective coupling for the given process R with external parameters denoted as Q^2 as

$$\alpha_R = \frac{4\pi}{\beta_0 \left[\ln \frac{m_R^2}{\Lambda^2} + \Psi_R(Q^2, m_R^2) + \frac{2\beta_1}{\beta_0^2}\varphi_R\left(\frac{1}{\alpha_R}\right)\right]}. \tag{3.3}$$

Here m_R^2, Ψ_R and φ_R depend on the process and Ψ_R contains the poles of the spectrum in the lowest (one loop) approximation. Conditions on Ψ_R, φ_R, Eqs. (2.6) and (2.4) respectively, impose restrictions on coefficients $c_n, M_n^{(0)}$ in the spectral representation (3.1). As an example one can choose $\Psi_R(Q^2, m_R^2)$ in the following form

$$\Psi_R \equiv \psi\left(\frac{Q^2 + (M_R^{(0)})^2}{m_R^2}\right), \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{3.4}$$

which implies that the spectrum responsible for the one-loop RG evolution of $\alpha_R(Q)$ is

$$M^2_{RN}(1-\text{loop}) = m^2_R n + (M^{(0)}_R)^2. \tag{3.5}$$

The asymptotics of $\psi(x)$ at large x is

$$\psi(x) = \ln x - \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}} \tag{3.6}$$

where B_{2k} are Bernoulli numbers behaving at large k as $B_{2k} \sim (-)^k(2k)!$.

Therefore the asymptotic condition (2.6) is satisfied by the choice (3.4). Note that one can also satisfy this condition by the realistic spectrum which asymptotically has the form (3.5) but differs from it for the first N_0 terms, provided the coefficients c_n tend to constant for large n.

At this point it is important to define the physical system which provides the one-loop spectrum (3.5). To this end one can compare the one-loop expansion in SPT, $\alpha_s(1-\text{loop}) = \alpha_s^{(0)} - \frac{\beta_0}{4\pi}\ln(Q^2/\mu_0^2)(\alpha_s^{(0)})^2 + ...$ with the equivalent expansion of α_R (3.3):

$$\alpha_R(\text{"}1-\text{loop"}) = \alpha_R^{(0)} - \frac{\beta_0}{4\pi}\Psi_R(Q^2, m_R^2)(\alpha_R^{(0)})^2 + ...$$

where $\alpha_R^{(0)} = \frac{4\pi}{\beta_0 \ln\frac{m_R^2}{\Lambda^2}}$. One can see that $\Psi_R(Q^2, m_R^2)$ plays the role of gluon loop, or better, two-gluon intermediate states imbedded in the framework of the process R. In particular, when R is the e^+e^- annihilation into hadrons, then Ψ_R is responsible for bound states of quark, antiquark and two gluons, i.e. in the large N_c limit this is the two-gluon hybrid state.

4 Meromorphic solution and standard perturbation theory

In previous section it was argued that the general solution of RG equation (2.1) which correctly reproduces the one-loop result of SPT and scheme-independent coefficients β_0, β_1 in the expansion of $\beta(\alpha)$ is given by

$$\alpha_R = \frac{4\pi}{\beta_0\left[\ln\frac{m_R^2}{\Lambda^2} + \psi\left(\frac{Q^2+(M_r^{(0)})^2}{m_R^2}\right) + \frac{2\beta_1}{\beta_0^2}\varphi_R\left(\frac{1}{\alpha_R}\right)\right]}. \tag{4.1}$$

where $\varphi_R(x) \to \ln x, x \to \infty$. The analytic properties of $\alpha_R(Q^2)$ depend on the form of $\varphi_R\left(\frac{1}{\alpha}\right)$. Neglecting the latter (one-loop approximation) one obtains for α_R the meromorphic function of Q^2 with simple poles at the negative values of Q^2 (for $m_R^2 > \Lambda^2$). If one adopts $\varphi_R(x) = \ln x$, then one can reproduce the two-loop result of SPT, but then α_R acquires the logarithmic branch points, in the Q^2 plane which are unacceptable for $N_c \to \infty$. To keep the correct meromorphic properties of α_R one can choose $\varphi_R\left(\frac{1}{\alpha_R}\right)$ as a meromorphic function ot its argument. A particular choice was made in [7], namely

$$\varphi_R\left(\frac{1}{\alpha_R}\right) = \psi\left(\frac{4\pi}{\beta_0\alpha_R} + \Delta\right) \tag{4.2}$$

where $\psi(x)$ is the same as in (3.4) with the asymptotics (3.6) which satisfies asymptotic condition on $\varphi_R(x)$ given above. The resulting expression for $\beta(\alpha)$ is

$$\beta(\alpha) = -\frac{\beta_0}{2\pi}\frac{\alpha^2}{1-\frac{2\beta_1}{\beta_0}\psi'\left(\frac{4\pi}{\beta_0\alpha}+\Delta\right)}. \tag{4.3}$$

One can persuade oneself that for $\Delta > \Delta_0 = 1.255$, $\beta(\alpha)$ is analytic for $\alpha > 0$, and has infinite number of zeros for negative α condensing on the negative side of the point $\alpha = 0$. For positive α the function $\beta(\alpha)$ is always negative and monotonically decreasing, and thus has no IR fixed point. Also the coefficients β_2, β_3 computed from (4.3) are several times smaller than computed in $\overline{MS}$ scheme.

To compare α_R in Eqs. (4.1), (4.2) with the loop expansion of SPT one can consider Q^2 large enough so that one can use the asymptotics (3.6) for $\psi(x)$ and keep the leading (logarithmic) term. With the notation

$$L = \ln\frac{Q^2+(M_R^{(0)})^2}{\Lambda^2}, \alpha_R^{(0)} \equiv \frac{4\pi}{\beta_0 L}, \tag{4.4}$$

one obtains the expansion in $1/L$ and $\frac{\ln L}{L}$, which looks like

$$\alpha_R = \alpha_R^{(0)}\left\{1-\frac{2\beta_1}{\beta_0^2}\frac{\ln L}{L}+\frac{4\beta_1^2}{\beta_0^4 L^2}\left[\left(\ln L-\frac{1}{2}\right)^2+b\right]+O\left(\frac{\ln L}{L}\right)^3\right\}. \tag{4.5}$$

Here $b = -\frac{\beta_0^2}{2\beta_1}\left(\Delta-\frac{1}{2}\right)-\frac{1}{4}$, and for $\Delta = 1.5$ one has $b = -1.436$, while in the $\overline{MS}$ $b = 0.26$ for $n_f = 0$. Thus the difference with SPT occurs only in the 3-loop result and is not large. Typically at $Q^2 = 3$ GeV^2 the exact expression (4.1) yields $\alpha_R = 0.26$ while the 2-loop $\overline{MS}$ result is $\alpha_R = 0.256(n_f = 0)$.

5 Conclusions

We conclude this paper with the discussion of the OPE for the case of e^+e^- annihilation which will stress the new features of the $\alpha_R(Q^2)$ behaviour (3.3). Expanding the polarization function $\Pi(Q)$ in powers of α_s one has

$$\Pi(Q) = \Pi^{(0)}(Q) + \alpha_s\Pi^{(1)}(Q) + \alpha_s^2\Pi^{(2)}(Q) + ... \tag{5.1}$$

where $\Pi^{(0)}(Q)$ can be computed as the spectral sum [13] as follows

$$\Pi^{(0)}(Q) = \frac{1}{12\pi^2}\sum\frac{c_n}{(M_n^{(0)})^2+Q^2} = -\frac{N_c}{12\pi^2}\psi\left(\frac{Q^2+(M_n^{(0)})^2}{m^2}\right) \tag{5.2}$$

and similarly or the hybrid sums $\Pi^{(1)}, \Pi^{(2)}, ...$

This should be compared with the standard OPE expansion [14]

$$Pi(Q^2) = -\frac{N_c}{12\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{Q^2}{\mu^2}+\frac{6m_q^2}{Q^2}+\frac{2m_q\langle\bar{q}q\rangle}{Q^4}+\frac{\alpha_s\langle FF\rangle}{12\pi Q^4}+... \tag{5.3}$$

where $m_q = \frac{m_n+m_d}{2}$. As was discussed in [1, 3, 9], the leading asymptotic term in (5.2) reproduces the partonic answer $\frac{N_c}{12\pi^2} \ln \frac{Q^2}{\mu^2}$, supporting the idea of Quark-Hadron Duality (QHD).

The next asymptotic terms of (5.2) produce the OPE terms $O\left(\frac{1}{Q^2}\right)$, $O\left(\frac{1}{Q^4}\right)$, etc. One usually imposes the QHD requirement, trying to reproduce the OPE coefficients in (5.3), and in particular the absence of $O\left(\frac{1}{Q^2}\right)$term for $m_q = 0$ [13]. Here we notice, that the form (4.1) produces additional power terms of somewhat different structure. Namely, taking the subleading asymptotic term in (3.6) for α_R one has

$$\alpha_R = \alpha_R^{(0)} + \frac{m^2}{2(Q^2+m^2)} \frac{\beta_0}{4\pi} (\alpha_R^{(0)})^2 + ... \tag{5.4}$$

Assuming that $\Pi^{(1)}(Q)$ has the same asymptotics as $\Pi^{(0)}$ (to reproduce the term $\frac{\alpha_s}{\pi} \ln \frac{Q2}{\mu^2}$ in (5.3)) one obtains a new power term

$$\Delta\Pi(Q^2) = -\frac{N_c}{12\pi^2} \frac{\alpha_R^{(0)}}{\pi} \frac{m^2}{2Q^2} \tag{5.5}$$

which is $O\left(\frac{1}{Q^2 \ln Q^2}\right)$ and has a negative sign.

Thus the meromorphic α_R, Eq. (4.1) produces the power-like terms of new kind. Numerically these terms are smaller than those originating from asymptotic expansion of (5.2), and phenomenologically the appearance of $O(1/Q^2)$ terms is not forbidden by experiment [15].

As a summary, a new type of solutions of RG equations is suggested where $\alpha_R(Q)$ is a meromorphic function of Q with poles corresponding to the physical poles of two-gluon hybrid states. The resulting $\alpha_R(Q)$ is finite for $Q^2 > 0$ and agrees well with $\alpha_{\overline{MS}}(Q)$ for $Q > 1$ GeV and is phenomenologicaly acceptable for all positive Q^2.

The author is grateful to I.Ya.Arefieva, A.M.Badalian, A.L.Kataev, N.V.Krasnikov, A.A.Pivovarov, D.V.Shirkov, for useful remarks and discussions. The work is supported by the Federal Program of the Russian Ministry of Industry, Science, and Technology No 40.052.1.1.1112.

References

[1] 'tHooft G,*Nucl. Phys.* **B72** (1974), 461; *Phys. Lett.* **B109** (1982), 474.

[2] Witten E, *Nucl. Phys.* **B156** (1979), 269.

[3] 'tHooft G, *Nucl. Phys.* **B75** (1974), 461.

[4] Kalashnikova Yu S, Nefediev A V, *Phys. Usp.* **45** (2002), 347.

[5] Lucini B, Teper M, Wenger U, hep-lat/0309170;
Teper M, hep-ph/0203203.

[6] Bogolubov N N, Shirkov P V, Introduction to the Theory of Quantum Fields, 1959, (1980), Interscience–Wiley, N.Y.;
Shirkov P V, *Theor. Math. Phys.* **132** (2002), 1307;
Grunberg G, *Phys. Rev.* **D29** (1984), 2315;
Krasnikov N V, *Nucl. Phys.* **B192** (1981), 497;
Kataev A L, Krasnikov N V, Pivovarov A A, *Phys. Lett.* **B107** (1981), 115, *Nucl. Phys.* **B198** (1982), 508;
Pivovarov A A, *Phys. At. Nucl.* **66** (2003), 724;
Binger M, Brodsky S J, hep-ph/0310322.

[7] Simonov Yu A, *Phys. At. Nucl.* **65** (2002), 135; ibid **66** (2003), 764.

[8] De Witt B S, *Phys. Rev.* **162**(1967), 1195, 1239
Honerkamp J, *Nucl. Phys.* **B48** (1972), 269;
'tHooft G, *Nucl. Phys.* **B62** (1973), 444; Lectures at Karpacz, in : Acta Univ. Wratislaviensis **368** (1976), 345;
Abbot L F, *Nucl. Phys.* **B185** (1981),189.

[9] Simonov Yu A, in: Lecture Notes in Physics, v.479, p.144, Springer, 1996;
Simonov Yu A, *Phys. At. Nucl.* **58** (1995), 107 ; (*Yad. Fiz.* **58** (1995), 113).

[10] Di Giacomo A, Dosch H G,Shevchenko V I, Simonov Yu A, *Phys.Rept.* **372 (2002)**, 319; hep-ph/0007223.

[11] Simonov Yu A, *Phys. At. Nucl.* **66** (2003), 2038 .

[12] Simonov Yu A, *Phys. At. Nucl.* **67** (2004), 553.

[13] Badalian A M, Simonov Y A, *Phys. At. Nucl.* **60** (1997), 630.

[14] Shifman M A, Vainshtein AI, Zakharov V I, *Nucl. Phys.* **B147** (1979), 385.

[15] Chetyrkin K G, Narison S, Zakharov V I, *Nucl. Phys.* **B550** (1999), 353 .