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Factorization of the Loop Algebras and Compatible Lie Brackets

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Abstract

It is shown that any decomposition of the loop algebra over a simple Lie algebra into a direct sum of the Taylor series and a complementary subalgebra is defined by a pair of compatible Lie brackets.

1 Introduction

Let $\mathcal{G}$ be a finite-dimensional Lie algebra. Denote by $\mathcal{G}((\lambda))$ the loop algebra over $\mathcal{G}$, i.e. the Lie algebra of all Laurant series with respect to a parameter λ with coefficients being elements of $\mathcal{G}$. Suppose that the loop algebra is decomposed into a direct sum of *vector spaces*:

$$\mathcal{G}((\lambda)) = \mathcal{A} \oplus \mathcal{G}[[\lambda]], \tag{1.1}$$

where $\mathcal{G}[[\lambda]]$ is the subalgebra of all Taylor series and $\mathcal{A}$ is a complementary Lie subalgebra. Using the terminology by I. Cherednik [1], we will call $\mathcal{A}$ *a factoring* subalgebra. There exist deep relationships between solutions of the classical Yang-Baxter equation [2] and factoring subalgebras that are isotropic with respect to the invariant bilinear form

$$< X(\lambda),\, Y(\lambda) >= \operatorname{res}\Big(X(\lambda),\, Y(\lambda)\Big), \qquad X(\lambda), Y(\lambda) \in \mathcal{G}((\lambda)), \tag{1.2}$$

where $(\cdot, \cdot)$ is the Killing form on $\mathcal{G}$. Non-isotropic factoring subalgebras have been studied in [4, 5].

One of important objects related to a factoring subalgebra $\mathcal{A}$ is the (associative) algebra of multiplicands. A Laurant series

$$\mathbf{m} = \sum_{i=-n}^{\infty} c_i \lambda^i, \qquad c_i \in C$$

is called a *multiplicand* if $\mathbf{m}\,\mathcal{A} \subset \mathcal{A}$. The number n is called *the order* of the multiplicand $\mathbf{m}$.

Theorem 1. *(see [3]) Let $\mathcal{G}$ be a simple Lie algebra. Then for any factoring subalgebra the following statements are fulfilled:*

- *i) Do not exist multiplicands of negative orders;*
- *ii) The complement of the set of orders of all multiplicands to the set of natural numbers is finite.*

We don't know whether the Theorem 1 holds for semi-simple Lie algebras $\mathcal{G}$.
Since the algebra $\mathcal{A}$ is defined up to transformations of the form

$$\bar{\lambda} = k_1\lambda + k_2\lambda^2 + k_3\lambda^3 + \cdots, \qquad k_1 \neq 0, \tag{1.3}$$

without loss of generality we can assume that $\mathbf{m} = \lambda^{-n}$.

The case when $\mathcal{A}$ admits a multiplicand $\mathbf{m}$ of order $n = 1$ have been considered in [6]. Such factoring subalgebras are called *homogeneous*. It follows from item i) of Theorem 1 that any multiplicand for a homogeneous factoring subalgebra is a polynomial in $\mathbf{m}$ with constant coefficients.

An one-to-one correspondence between homogeneous factoring subalgebras and compatible Lie brackets on $\mathcal{G}$ have been established in [6]. Two Lie brackets are called *compatible* if any linear combination of these brackets is a Lie bracket as well (see [7, 6, 8]). We recall the main result of the paper [6].

Let $A(\lambda)$ be a formal series of the form

$$A = E + R\,\lambda + S\,\lambda^2 + \cdots, \tag{1.4}$$

where the coefficients $R, S, \ldots$ are linear operators from $\mathcal{G}$ to $\mathcal{G}$ and E is the identical operator.

Theorem 2.

- *i) Any homogeneous factoring subalgebra $\mathcal{A}$ can be represented as*

$$\mathcal{A} = \{\sum_{i=1}^{k} \lambda^{-i}\, A(g_i), \mid g_i \in \mathcal{G}, \quad k \in \mathbb{N}\}, \tag{1.5}$$

where $A(\lambda)$ is a formal series of the form (1.4).

- *ii) The vector space (1.5) is a Lie subalgebra iff the following identity is fulfilled for any $X, Y \in \mathcal{G}$ and some Lie bracket $[\cdot,\cdot]_1$ compatible with $[\cdot,\cdot]$:*

$$[A(X),\ A(Y)] = A\Big([X,\ Y] + \lambda\,[X,\ Y]_1\Big). \tag{1.6}$$

- *iii) For any homogeneous subalgebra the bracket $[\cdot,\cdot]_1$ is given by the formula*

$$[X, Y]_1 = [R(X), Y] + [X, R(Y)] - R([X, Y]), \tag{1.7}$$

where R is the coefficient of λ in (1.4).

In this paper we generalize Theorem 2 to the case $n > 1$. Using our construction and results from [2], one can derive new interesting examples of compatible Lie brackets on $\mathcal{G} \oplus \mathcal{G}$.

2 Compatible Lie brackets for factoring subalgebras.

Let $\mathcal{A}$ be a factoring subalgebra having the multiplicand $\mathbf{m} = \lambda^{-n}$. Denote by $\mathbf{V}$ the vector space

$$\mathbf{V} = \lambda^{-n}\mathcal{G}[[\lambda]] \cap \mathcal{A}. \tag{2.1}$$

It follows from (1.1) that $\dim\mathbf{V} = n \dim \mathcal{G}$. It is easy to see that

$$[\mathbf{V},\, \mathbf{V}] \subset \lambda^{-n}\mathbf{V} \oplus \mathbf{V} \tag{2.2}$$

and

$$[\mathbf{V},\, [\mathbf{V},\, \mathbf{V}]] \subset \lambda^{-2n}\mathbf{V} \oplus \lambda^{-n}\mathbf{V} \oplus \mathbf{V}. \tag{2.3}$$

Using (2.2), we define two brackets $[\cdot\,\cdot]_1$ and $[\cdot,\,\cdot]_2$ on $\mathbf{V}$ by the formula

$$[a,\, b] = \lambda^{-n}[a,\, b]_1 + [a,\, b]_2. \tag{2.4}$$

Proposition 1. *The bilinear operations $[\cdot\,\cdot]_1$ and $[\cdot,\,\cdot]_2$ are compatible Lie brackets.*

Proof. Consider the Jacobi identity in the loop algebra for arbitrary three elements of $\mathbf{V}$ and calculate it's projections on $\mathbf{V}$, $\lambda^{-n}\mathbf{V}$, $\lambda^{-2n}\mathbf{V}$ with respect to (2.3). It is easy to check that the projection on $\mathbf{V}$ coincides with the Jacobi identity for $[\cdot,\,\cdot]_2$, projection on $\lambda^{-2n}\mathbf{V}$ leads to the Jacobi identity for $[\cdot,\,\cdot]_1$. Thus $[\cdot,\,\cdot]_i$ are Lie brackets. The projection on $\lambda^{-n}\mathbf{V}$ gives rise to an identity, which means just the compatibility of these brackets. ■

The standard example of factoring subalgebra is given by

$$\mathcal{A}^{st} = \{\sum_{i=1}^{k} g_i \lambda^{-i} \quad | \quad g_i \in \mathcal{G}, \quad k \in \mathbb{N}\}.$$

For this case, we have

$$\mathbf{V}^{st} = \lambda^{-n}\mathcal{G}[[\lambda]] \cap \mathcal{A}^{st}$$

and the condition

$$[\mathbf{V}^{st},\, \mathbf{V}^{st}] \subset \lambda^{-n}\mathbf{V}^{st} \oplus \mathbf{V}^{st}$$

yields two compatible brackets $[\cdot\,\cdot]_1^{st}$ and $[\cdot,\,\cdot]_2^{st}$. It is easy to check that these brackets are given by

$$[\lambda^{-n+i}\, p,\, \lambda^{-n+j}\, q]_1^{st} = \lambda^{-n+i+j}\, [p,\, q] \qquad \text{for} \quad i+j < n,$$
$$[\lambda^{-n+i}\, p,\, \lambda^{-n+j}\, q]_1^{st} = 0 \qquad \text{for} \quad i+j \geq n$$

and

$$[\lambda^{-n+i}\, p,\, \lambda^{-n+j}\, q]_2^{st} = \lambda^{-2n+i+j}\, [p,\, q] \qquad \text{for} \quad i+j \geq n,$$
$$[\lambda^{-n+i}\, p,\, \lambda^{-n+j}\, q]_2^{st} = 0 \qquad \text{for} \quad i+j < n,$$

where $0 \leq i, j < n, \quad p, q \in \mathcal{G}$.

Let $\mathcal{A}$ be a factoring subalgebra such that $\lambda^{-n}\mathcal{A} \subset \mathcal{A}$ and $\mathbf{V}$ defined by (2.1). For any $b \in \mathbf{V}$, we denote by $\pi(b)$ the principle part of b, i.e. the projection of b on $\mathbf{V}^{st}$ parallel to the Taylor series vector space. It follows from (1.1) that the map $\pi : \mathbf{V} \to \mathbf{V}^{st}$ is invertible. The formula

$$b = \pi(b) + \lambda^n R\Big(\pi(b)\Big) + \lambda^{2n} S\Big(\pi(b)\Big) + \cdots \tag{2.5}$$

uniquely defines the linear operators $R, S, \dots$ on $\mathbf{V}^{st}$.

The brackets (2.4) defined by (1.1) induce the following brackets on $\mathbf{V}^{st}$:

$$[a,\, b]_i = \pi\Big([\pi^{-1}(a),\, \pi^{-1}(b)]_i\Big), \qquad a, b \in \mathbf{V}^{st}, \qquad i = 1, 2$$

The following statement is a direct generalization of Lemma 2 from [6]:

Proposition 2. *Let c, d be arbitrary elements of $\mathbf{V}^{st}$; then*

- *i)* $$[c,\, d]_1 = [c,\, d]_1^{st};$$
- *ii)*
$$[c,\, d]_2 = [c,\, d]_2^{st} + [R(c),\, d]_1^{st} + [c,\, R(d)]_1^{st} - R([c,\, d]_1^{st}), \tag{2.6}$$
where R is the operator from (2.5);
- *iii) Suppose that the operation (2.6) is a Lie bracket for some operator R; then (2.6) is compatible with $[\cdot, \cdot]_1$;*
- *iv) The operators R and S - from (2.5) satisfy the following identity:*

$$R([c,\, d]_2) = [R(c),\, R(d)]_1^{st} + [S(c),\, d]_1^{st} + [c,\, S(d)]_1^{st} - S([c,\, d]_1^{st}) + [R(c),\, d]_2^{st} + [c,\, R(d)]_2^{st}; \tag{2.7}$$

- *v) Suppose some operators R and S satisfy (2.7); then (2.6) is a Lie bracket.*

Remark 1. It is easy to see that the Lie algebra defined by the bracket $[\cdot, \cdot]_1$ is isomorphic to $\mathcal{G}[\mu]/(\mu^n)$. When $n > 1$ this algebra is not semi-simple. If $R = \mathbf{Id}$ then the second bracket $[\cdot, \cdot]_2$ gives rise to the Lie algebra isomorphic to $\mathcal{G}[\mu]/(\mu^n - 1)$, which is in its turn isomorphic to the direct sum of n copies of $\mathcal{G}$. The same is true if $|R - \mathbf{Id}|$ is sufficiently small.

Remark 2. The existence of the multiplicand $\mathbf{m} = \lambda^{-n}$ is not destroyed under transformations $\lambda^{-n} = k_1 \bar{\lambda}^{-n} + k_2$. Such transformations also preserve the first bracket. However it follows from (2.5) that the operator R and the bracket $[\cdot, \cdot]_2$ are deformed. In particular, if $k_2 = 1$ and $k_1 \to \infty$, then $R \to \mathbf{Id}$ and according to Remark 1 the Lie algebra with the bracket $[\cdot, \cdot]_2$ becomes isomorphic to n copies of $\mathcal{G}$.

Thus starting with any factoring subalgebra, one can construct a pair of compatible brackets on the direct sum of n copies of $\mathcal{G}$, where n is the order of arbitrary multiplicand (see Theorem 1).

Example. Let $r : \mathcal{G} \to \mathcal{G}$ be a constant solution of the modified Yang-Baxter equation (see [9])

$$[r(a),\, r(b)] = r\Big([r(a), b] - [r(b), a]\Big) - [a,\, b]. \tag{2.8}$$

In [10] the following factoring subalgebra has been introduced:

$$\mathcal{A} = \{ \sum_{i=-m}^{-1} \lambda^i q_i + r_i(q_i) \quad | \quad m > 0, \quad q_i \in \mathcal{G}\},$$

where by definition $r_{2k} = -\mathbf{Id}$, $r_{2k-1} = r$. It is clear that $\lambda^{-2}\mathcal{A} \subset \mathcal{A}$. The vector spaces corresponding to this multiplicand have the following form

$$\mathbf{V} = \{\lambda^{-2}p + \lambda^{-1}q + r(q) - p \,|\, p, q \in \mathcal{G}\}, \qquad \mathbf{V}^{st} = \{\lambda^{-2}p + \lambda^{-1}q \,|\, p, q \in \mathcal{G}\}.$$

The operator $R : \mathbf{V}^{st} \to \mathbf{V}^{st}$ from (2.7) is given by

$$R\Big(\lambda^{-2}p + \lambda^{-1}q\Big) = \lambda^{-2}\Big(r(q) - p\Big)$$

and the operator S is identically zero. The pair of compatible brackets on $\mathbf{V}^{st}$ is defined by Proposition 2. According to Remarks 1,2 the Lie algebra defined by the second bracket is isomorphic to $\mathcal{G} \oplus \mathcal{G}$ after admissible change of λ.

3 Generalized loop algebras generated by compatible brackets.

Suppose that we have two compatible Lie brackets $[\cdot\,\cdot]_1$ and $[\cdot,\,\cdot]_2$ on a k-dimensional vector space $\mathbf{V}$. We assume that the bracket $[\cdot\,\cdot]_1$ has the trivial center. Let us consider the following vector subspace

$$\mathcal{L} = \{ \sum_{i=-m}^{\infty} \lambda^i (ad_1 q_i + \lambda\, ad_2 q_i) \quad | \quad m \in \mathbb{Z},\, q_i \in \mathbf{V}\} \tag{3.1}$$

in the loop algebra over linear operators on $\mathbf{V}$. Here $ad_i q$ are linear operators defined by

$$ad_i q(p) = [q,\, p]_i, \qquad i = 1, 2.$$

It follows from compatibility of $[\cdot\,\cdot]_1$ and $[\cdot,\,\cdot]_2$ that

$$[ad_1 p + \lambda\, ad_2 p,\, ad_1 q + \lambda\, ad_2 q] = ad_1[p, q]_1 + \lambda\, ad_2[p, q]_1 + \lambda(ad_1[p, q]_2 + \lambda\, ad_2[p, q]_2). \tag{3.2}$$

The latter formula implies that $\mathcal{L}$ is a Lie subalgebra.

It can be easily checked that $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$, where $\mathcal{L}_+$ and $\mathcal{L}_-$ are defined as the intersections of $\mathcal{L}$ with the Laurent polynomials and Taylor series:

$$\mathcal{L}_+ = \{ \sum_{i=-m}^{-1} \lambda^i (ad_1 q_i + \lambda\, ad_2 q_i) \quad | \quad m > 0,\, q_i \in \mathbf{V}\},$$

$$\mathcal{L}_- = \{\sum_{i=0}^{\infty} \lambda^i (ad_1 q_i + \lambda\, ad_2 q_i) \quad | \quad q_i \in \mathbf{V}\}.$$

It follows from the triviality of the common center for $[\cdot\,\cdot]_1$ and $[\cdot,\,\cdot]_2$ that $\mathcal{L}_+ \cap \mathcal{L}_- = \{0\}$.

Suppose now that $\mathbf{V}$, $[\cdot\,\cdot]_1$ and $[\cdot,\,\cdot]_2$ are generated by a factoring subalgebra $\mathcal{A}$ by formulas (2.1), (2.4). It is clear that

$$\mathcal{G}((\lambda)) = \{\sum_{i=-m}^{\infty} \lambda^{ni} q_i \,|\, m > 0,\, q_i \in \mathbf{V}\}. \tag{3.3}$$

Define a mapping $\Phi : \mathcal{G}((\lambda)) \to \mathcal{L}$ by the following formula:

$$\Phi\Big(\sum_{i=-m}^{\infty} \lambda^{ni} q_i\Big) = \sum_{i=-m}^{\infty} \lambda^{i-1} (ad_1 q_i + \lambda\, ad_2 q_i) \tag{3.4}$$

Proposition 3. • *i). The mapping Φ is an isomorphism of Lie algebras;*

- *ii).* $\Phi(\mathcal{A}) = \mathcal{L}_+, \quad \Phi(\mathcal{G}[[\lambda]]) = \mathcal{L}_-$.

Proof. To verify that Φ is a homomorphism it is suffices to compare formulas (2.4) and (3.2). Since

$$\Phi(q) = \lambda^{-1} ad_1\, q + ad_2\, q,$$

it follows from the triviality of the common center for $[\cdot\,\cdot]_1$ and $[\cdot,\,\cdot]_2$ that Φ is bijective. The statement ii) follows from (3.3) and from the definitions of the corresponding subalgebras. ■

Remark 3. Theorem 1 and Propositions 2, 3 show that any factoring subalgebra in the loop algebra over *simple* Lie algebra $\mathcal{G}$ is defined by a pair of compatible brackets generated by identity (2.7). Thus the classification problem for factoring subalgebras is embedded into a finite-dimensional problem of the description of all solutions for operator equation (2.7).

4 Integrable nonlinear hyperbolic systems and the generalized loop algebras.

In [6] the hyperbolic systems

$$q_y = [q,\, p]_1, \qquad p_x = [p,\, q]_2, \qquad p, q \in \mathbf{V} \tag{4.1}$$

associated with two compatible Lie brackets $[\cdot,\,\cdot]_i$ on $\mathbf{V}$ have been considered. If one of the bracket is semi-simple, a Lax representation for (4.1) have been constructed. The next statement provides a different Lax representation for the same system.

Proposition 4. *Let $[\cdot\,\cdot]_1$ and $[\cdot,\,\cdot]_2$ be a pair of compatible brackets defined on a vector space $\mathbf{V}$. Suppose that the center of $[\cdot\,\cdot]_1$ is trivial. Then (4.1) is equivalent to the condition*

$$\left[\frac{d}{dx} + \lambda^{-1} ad_1\, q + ad_2\, q, \quad \frac{d}{dy} + ad_1\, p + \lambda ad_2\, p\right] = 0. \tag{4.2}$$

The proof follows from (3.2).

Remark 4. The Lax operators from Proposition 4 are rational functions in λ in contrast to operators constructed in [6].

Proposition 5. *Suppose* $\mathbf{V}$ *is given by (2.1) and the compatible Lie brackets are defined by (2.4); then (4.1) is equivalent to*

$$\left[\frac{d}{dx} + q, \quad \frac{d}{dy} + \lambda^n p\right] = 0. \tag{4.3}$$

The statement follows from (2.4).

It is evident that (4.1) admits the reduction $p \in I_1$, $q \in I_2$, where I_k is an ideal for the bracket $[\cdot,\cdot]_k$. In particular, suppose that the algebra defined by $[\cdot,\cdot]_1$ is isomorphic to $\mathcal{G}[\mu]/(\mu^n)$ and the algebra with $[\cdot,\cdot]_1$ is isomorphic to $\mathcal{G}[\mu]/(\mu^n - 1)$ or, in other words, is the direct sum of n copies of $\mathcal{G}$ (see. Remarks 1,2). Then taking $I_1 = \mu^{n-1}\mathcal{G}$ for I_1 and one of the copies of $\mathcal{G}$ for I_2, we get an integrable hyperbolic system with respect to two elements of the algebra $\mathcal{G}$.

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