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Extended Prelle-Singer Method and Integrability/Solvability of a Class of Nonlinear nth Order Ordinary Differential Equations

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Abstract

We discuss a method of solving n^{th} order scalar ordinary differential equations by extending the ideas based on the Prelle-Singer (PS) procedure for second order ordinary differential equations. We also introduce a novel way of generating additional integrals of motion from a single integral. We illustrate the theory for both second and third order equations with suitable examples. Further, we extend the method to two coupled second order equations and apply the theory to two-dimensional Kepler problem and deduce the constants of motion including Runge-Lenz integral.

1 Introduction

During the past two decades or so immense interest has been shown in identifying integrable/solvable nonlinear dynamical systems. Different ideas have been developed or employed in isolating such nonlinear ordinary/partial differential equations [1]. Focussing our attention on ordinary differential equations (ODEs), one finds that using certain novel ideas, Calogero and his coworkers have generated a wide class of integrable/solvable nonlinear systems and explored their underlying features [2–4]. In this paper we introduce another straightforward method of identifying such solvable and/or integrable equations. To do so we employ the extended Prelle-Singer (PS) procedure [5].

Sometime ago Prelle and Singer [5] have proposed a procedure for solving first order ODEs that presents the solution in terms of elementary functions if such a solution exists. The attractiveness of the PS method is that if the given system of first order ODEs has a solution in terms of elementary functions then the method guarantees that this solution will be found. Very recently Duarte et al [6] have modified the technique developed by Prelle and Singer [5] and applied it to second order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second order ODE then there exists at least one elementary first integral $I(t, x, \dot{x})$ whose derivatives are all rational functions of t, x and $\dot{x}$. For a class of systems these authors have deduced first integrals and in some cases for the first time through their procedure [6]. Recently the present authors have generalized the theory given in [6] and pointed out a procedure to obtain all the integrals of motion/general solution and solved a class of nonlinear equations [7–9].

The PS procedure has many attractive features. For a large class of integrable systems, this procedure provides the integrals of motion/general solution in a straightforward way. In fact this is true for any order. The PS method not only gives the first integrals but also the underlying integrating factors. Further, like Lie-symmetry analysis and Noether's theorem the PS method can also used to solve linear as well as nonlinear ODEs. In addition to the above, the PS procedure is applicable to deal with both Hamiltonian and non-Hamiltonian systems.

In this paper we extend the above theory [6–8,10] to n^{th} order scalar ODEs and derive a relation which connects integrals of motion with the integrating factors. We demonstrate the method to second and third order ODEs. We also introduce a novel way of generating all the integrals of motion from a single integral, *which is applicable to a class of ODEs*, and demonstrate our ideas by considering the previous examples. Finally, we prolong the method to two coupled second order ODEs.

We note here that several works have been undertaken earlier to study and classify systematically the nth order ODEs based on different ideas. To name a few, besides the works of Calogero [2,3], we cite, Lie symmetry approach [11–14], the method of Bluman and Anco [15,16], Painlevé analysis [1,17] and so on.

The organization of the material is as follows. In the following section we extend the theory of modified PS method to n^{th} order ODEs. In Sec.3, we describe a procedure to generate second, third and higher integrals of motion from a single integral, provided the first integral can be written in certain specific form. In Secs. 4 and 5, respectively, we demonstrate the theory to second and third order ODEs with suitable examples. In Sec. 6, we briefly discuss the application of PS procedure to higher order ODEs. In Sec. 7 we prolong the theory to two-coupled second order ODEs and illustrate the theory with an example, namely the two-dimensional Kepler problem. We give our final remarks in Sec. 8.

2 Prelle-Singer procedure

Let us consider a class of n^{th} order ODEs of the form

$$x^{(n)} = \frac{P}{Q}, \qquad P, Q \in C[t, x, x^{(1)}, x^{(2)}, ..., x^{(n-1)}], \tag{2.1}$$

where $x^{(1)} = \frac{dx}{dt}, x^{(2)} = \frac{d^2x}{dt^2}$ and $x^{(n)} = \frac{d^nx}{dt^n}$ and P and Q are polynomials in $t, x, x^{(1)}, x^{(2)}$, $..., x^{(n-2)}$ and $x^{(n-1)}$ with coefficients in the field of complex numbers. Let us assume that the ODE (2.1) admits a first integral $I(t, x, x^{(1)}, x^{(2)}, ..., x^{(n-1)}) = C$, with C constant on the solutions, so that the total differential gives

$$dI = I_t dt + I_x dx + I_{x^{(1)}} dx^{(1)} + \ldots + I_{x^{(n-1)}} dx^{(n-1)} = 0, \tag{2.2}$$

where subscript denotes partial differentiation with respect to that variable. Rewriting equation (2.1) of the form $\frac{P}{Q}dt - dx^{(n-1)} = 0$ and adding null terms $S_1(t, x, x^{(1)}, ..., x^{(n-1)})$

$x^{(1)}dt - S_1(t,x,x^{(1)},...,x^{(n-1)})dx$ and $S_i(t,x,x^{(1)},...,x^{(n-1)})x^{(i)}dt - S_i(t,x,x^{(1)},...,x^{(n-1)})$ $dx^{(i-1)}$, $i = 2,3,...n-1$, to it we obtain that on the solutions the 1-form

$$\left(\frac{P}{Q} + \sum_{i=1}^{n-1} S_i x^{(i)}\right)dt - S_1 dx - \sum_{i=2}^{n-1} S_i dx^{(i-1)} - dx^{(n-1)} = 0. \tag{2.3}$$

Hence, on the solutions, the 1-forms (2.2) and (2.3) must be proportional. Multiplying (2.3) by the factor $R(t,x,x^{(1)},...,x^{(n-1)})$ which acts as the integrating factor for (2.3), we have on the solutions that

$$dI = R\left[\left(\phi + \sum_{i=1}^{n-1} S_i x^{(i)}\right)dt - S_1 dx - \sum_{i=2}^{n-1} S_i dx^{(i-1)} - dx^{(n-1)}\right] = 0, \tag{2.4}$$

where $\phi \equiv P/Q$. Comparing equations (2.2) with (2.4) we have, on the solutions, the relations

$$\begin{aligned}
I_t &= R(\phi + \sum_{i=1}^{n-1} S_i x^{(i)}),\\
I_x &= -RS_1,\\
I_{x^{(i)}} &= -RS_{i+1}, \;\; i = 1,2,\ldots,n-2,\\
I_{x^{(n-1)}} &= -R.
\end{aligned} \tag{2.5}$$

The compatibility conditions, $I_{tx} = I_{xt}$, $I_{tx^{(n-1)}} = I_{x^{(n-1)}t}$, $I_{xx^{(n-1)}} = I_{x^{(n-1)}x}$, $I_{tx^{(i)}} = I_{x^{(i)}t}$, $I_{xx^{(i)}} = I_{x^{(i)}x}$, $I_{x^{(i)}x^{(n-1)}} = I_{x^{(n-1)}x^{(i)}}$, $i = 2,3,\ldots,n-2$, between the equations (2.5), provide us the conditions

$$D[S_1] = -\phi_x + S_1\phi_{x^{(n-1)}} + S_1 S_{n-1}, \tag{2.6}$$
$$D[S_i] = -\phi_{x^{(i-1)}} + S_i\phi_{x^{(n-1)}} + S_i S_{i+1} - S_{i-1}, \; i = 2,3,\ldots,n-2, \tag{2.7}$$
$$D[S_{n-1}] = -\phi_{x^{(n-2)}} + S_{n-1}\phi_{x^{(n-1)}} + S_{n-1}^2 - S_{n-2}, \tag{2.8}$$
$$D[R] = -R(S_{n-1} + \phi_{x^{(n-1)}}), \tag{2.9}$$
$$R_{x^{(i)}} S_1 = -RS_{1x^i} + R_x S_{i+1} + RS_{i+1x}, \;\; i = 1,2,\ldots,n-2, \tag{2.10}$$
$$R_{x^{(i)}} S_{j+1} = -RS_{j+1x^{(i)}} + R_{x^j} S_{i+1} + RS_{i+1x^j}, \;\; i,j = 1,2,\ldots,n-2, \tag{2.11}$$
$$R_{x^{(i)}} = R_{x^{(n-1)}} S_{i+1} + RS_{i+1x^{(n-1)}}, \;\; i = 1,2,\ldots,n-2, \tag{2.12}$$
$$R_x = R_{x^{(n-1)}} S_1 + RS_{1x^{(n-1)}}, \tag{2.13}$$

where the total differential operator D is defined by

$$D = \frac{\partial}{\partial t} + x^{(1)}\frac{\partial}{\partial x} + \sum_{i=2}^{n} x^{(i)}\frac{\partial}{\partial x^{(i-1)}}.$$

We note that (2.6)-(2.13) form an overdetermined system for the unknowns, S_i, $i = 1,\ldots,n-1$, and R. For example, one can check that for a second order ODE, $n = 2$, one gets three equations for two unknowns, say, S and R, whereas for a third order ODE one gets six equations for three unknowns, say, S_1, S_2 and R. Thus in this procedure, for a given n^{th} order ODE, one gets $\frac{n(n+1)}{2}$ number of equations, for n unknowns, out of which $\frac{n(n-1)}{2}$ equations are just extra constraints.

The crux of the problem lies in solving the determining equations and identifying sufficient number of integrating factors and null forms. But the point is that any particular solution will suffice for the purpose. We solve equations (2.6)-(2.13) in the following way. Substituting the expression for $\phi = \frac{P}{Q}$, obtained from equation (2.1), into (2.6)-(2.8) we get a system of differential equations for the unknowns $S_i,\ i = 1, \ldots, n-1$. Solving them we can obtain expressions for the null forms S_i's. Once S_i's are known then equation (2.9) becomes the determining equation for the function R. Solving the latter we can get an explicit form for R. Now the functions R and $S_i,\ i = 1, \ldots, n-1$, have to satisfy the extra constraints (2.10)-(2.13). However, once a compatible solution satisfying all the equations have been found then the functions R and $S_i,\ i = 1, 2, \ldots, n-1$, fix the first integral $I(t, x, x^{(1)}, ..., x^{(n-1)})$ by the relation

$$I(t, x, x^{(1)}, ..., x^{(n-1)}) = \sum_{i=1}^{n} r_i - \int \left[R + \frac{d}{dx^{(n-1)}} \left(\sum_{i=1}^{n} r_i \right) \right] dx^{(n-1)}, \tag{2.14}$$

where

$$\begin{aligned}
r_1 \quad &= \textstyle\int R \bigg(\phi + \sum_{i=1}^{n-1} S_i x^{(i)} \bigg) dt, \\
r_2 \quad &= -\textstyle\int \bigg(RS_1 + \frac{d}{dx} r_1 \bigg) dx, \\
r_j \quad &= -\textstyle\int \Big[RS_{j-1} + \frac{d}{dx^{(j-1)}} \Big(\sum_{k=1}^{j-1} r_k \Big) \Big] dx^{(j-1)}, \quad j = 3, \ldots, n.
\end{aligned}$$

Equation (2.14) can be derived straightforwardly by integrating the equations (2.5). Now substituting the expressions for ϕ, R and $S_i, i = 1, 2, \ldots, n$, into (2.14) and evaluating the integrals one can get the associated integrals of motion. However, we have to point out that we have not examined the question of existence of consistent solutions to equations (2.6)-(2.13) at present.

3 Method of generating integrals of motion

In the above, we derived the integrals of motion, $I_i,\ i = 1, 2, \ldots, n$, by constructing sufficient number of integrating factors. Interestingly, for a *class of equations*, one can also generate the required number of integrals of motion from one of the integrals, say I_1, if it is known, provided its form can be written in a specific form. For example, for the n^{th} order equation (2.1), one can generate $I_2, I_3, \ldots, I_{n-1}$ and I_n from I_1 itself. In the following we illustrate this possibility.

Let us assume that there exists a first integral for the n^{th} order equation (2.1) of the form, $I_1(t, x, x^{(1)}, x^{(2)}, ..., x^{(n-1)}) = C$. Now let us split the functional form of the first integral I_1 into two terms such that one involves all the variables $(t, x, x^{(1)}, ..., x^{(n-1)})$ while the other excludes $x^{(n-1)}$, that is,

$$I_1 = F_1(t, x, x^{(1)}, ..., x^{(n-1)}) + F_2(t, x, x^{(1)}, ..., x^{(n-2)}). \tag{3.1}$$

Of course, such a splitting is not unique, which can in fact be used profitably further to identify new integrals.

Now let us split the function F_1 further in terms of two functions such that F_1 itself is a function of the product of the two functions, say, a perfect differentiable function $\frac{d}{dt}G_1(t,x,x^{(1)},...,x^{(n-2)})$ and another function $G_2(t,x,x^{(1)},...,x^{(n-1)})$, that is,

$$I_1 = F_1\left(\frac{1}{G_2(t,x,x^{(1)},...,x^{(n-1)})}\frac{d}{dt}G_1(t,x,x^{(1)},...,x^{(n-2)})\right) \\ +F_2\left(G_1(t,x,x^{(1)},...,x^{(n-2)})\right). \qquad (3.2)$$

We note that while rewriting equation (3.1) in the form (3.2), we demand the function $F_2(t,x,x^{(1)},...,x^{(n-2)})$ in (3.1) automatically to be a function of $G_1(t,x,x^{(1)},...,x^{(n-2)})$. Now identifying the function G_1 as the new dependent variable and the integral of G_2 over time as the new independent variable, that is,

$$w = G_1(t,x,x^{(1)},...,x^{(n-2)}), \quad z = \int_o^t G_2(t',x,x^{(1)},...,x^{(n-1)})dt', \qquad (3.3)$$

one can rewrite equation (3.2) of the form

$$I = F_1\left(\frac{dw}{dz}\right) + F_2(w). \qquad (3.4)$$

In other words

$$F_1\left(\frac{dw}{dz}\right) = I - F_2(w). \qquad (3.5)$$

Now rewriting equation (3.5) one obtains a separable equation

$$\frac{dw}{dz} = f(w), \qquad (3.6)$$

which can be integrated by quadrature, and I_2 can be obtained.

The procedure given above is easy to follow and can be used to solve *a class of problems* straightforwardly. In fact, for the linearizable second order ODEs, we find that the function F_2 turns out to be zero and as a consequence one gets $\frac{dw}{dz} = I_1$ (from (3.6)) which can be integrated to obtain the second integration constant trivially. On the other hand, if F_2 is not zero then the second integration constant can be deduced after the integration of (3.6), which can be done for a number of examples. As far as third order ODEs are concerned the first order equation (3.6) provides us the second integral whereas a different choice $\hat{G}_1$ and $\hat{G}_2$, where $\hat{G}_1$ and $\hat{G}_2$ are different from G_1 and G_2, directly leads us to the third integral (see Sec. 5). The procedure can in principle be extended to higher order ODEs.

4 Second order ODEs

The theory developed in Secs. 2 and 3, in principle, can be used to solve *a class of equations*. To illustrate the underlying ideas let us first consider a second order ODE. Since we are going to discuss in detail only second and third order ODEs, hereafter, we use the notation $\dot{x}$, $\ddot{x}$ and $\dddot{x}$ instead of $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ for $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$ and $\frac{d^3x}{dt^3}$, respectively.

Fixing, $n = 2$, the determining equations (2.6)-(2.13) get simplified to

$$S_t + \dot{x}S_x + \phi S_{\dot{x}} = -\phi_x + S\phi_{\dot{x}} + S^2, \tag{4.1}$$

$$R_t + \dot{x}R_x + \phi R_{\dot{x}} = -R(S + \phi_{\dot{x}}), \tag{4.2}$$

$$R_x = R_{\dot{x}}S + RS_{\dot{x}}, \tag{4.3}$$

The integral of motion, (2.14), is fixed by the relation

$$I(t, x, \dot{x}) = r_1 - r_2 - \int \left[R + \frac{d}{d\dot{x}}(r_1 - r_2)\right] d\dot{x}, \tag{4.4}$$

where

$$r_1 = \int R(\phi + \dot{x}S)dt \quad \text{and} \quad r_2 = \int (RS + \frac{d}{dx}r_1)dx.$$

The method of solving the determining equations (4.1)-(4.3) is described in detail in Ref. [6,7]. However, in order to be self-contained, we briefly summarize the main ideas in the following.

Let us first solve the equation (4.1) with the given ϕ and obtain expression for S. Once S is known then equation (4.2) becomes the determining equation for the function R. Solving the latter one can get an explicit form for R. Now the functions R and S have to satisfy an extra constraint, that is, equation (4.3). We note at this point that all solutions which satisfy equations (4.1)-(4.2) need not satisfy the constraint (4.3) since equations (4.1)-(4.3) constitute an overdetermined system for the unknowns R and S. For example, let us consider a set (S, R) be a solution of equations (4.1)-(4.2) and not of the constraint equation (4.3). After examining several examples we observe that one can make the set (S, R) compatible by modifying the form of R as

$$\hat{R} = F(t, x, \dot{x})R, \tag{4.5}$$

where $\hat{R}$ satisfies equation (4.2), so that we have

$$(F_t + \dot{x}F_x + \phi F_{\dot{x}})R + FD[R] = -FR(S + \phi_{\dot{x}}). \tag{4.6}$$

Further, if F is a constant of motion (or a function of it), then the first term on the left hand side vanishes and one gets the same equation (4.2) for R provided F is non-zero. In other words, whenever F is a non-zero constant or a function of the integral of motion then the solution of equation (4.2) may provide only a factor of the complete solution $\hat{R}$ without the factor F in equations (4.5). This general form of $\hat{R}$ with S will form a complete solution to the equations (4.1)-(4.3). In a nutshell we describe the procedure as follows. First we determine S and R from equations (4.1)-(4.2). If the set (S, R) satisfies equation (4.3) then we take it as a compatible solution. On the other hand if it does not satisfy (4.3) then we assume the modified form $\hat{R} = F(I)R$, and find the explicit form of $F(I)$ from equation (4.3), which in turn fixes the compatible solution $(S, \hat{R})$. Once I_1 is derived then the second integration constant can be deduced either utilizing our procedure described in Sec. 3, or finding another set of solutions (S_2, R_2) for equations (4.1)-(4.3). The method has been applied to several interesting nonlinear dynamical systems and interesting results have been obtained [7]. In the following example we illustrate both the above ideas.

4.1 Example

Let us consider an equation of the following form for illustrative purpose:

$$\ddot{x} = \frac{(2x-1)}{(1+x^2)}\dot{x}^2 \tag{4.7}$$

so that the equations (4.1)-(4.3) become

$$S_t + \dot{x}S_x + \frac{(2x-1)\dot{x}^2}{(1+x^2)}S_{\dot{x}} = 2\bigg(\frac{x(1+x)-1}{(1+x^2)^2}\bigg)\dot{x}^2 + \frac{2(2x-1)\dot{x}}{(1+x^2)}S + S^2, \tag{4.8}$$

$$R_t + \dot{x}R_x + \frac{(2x-1)\dot{x}^2}{(1+x^2)}R_{\dot{x}} = -\bigg(S + \frac{2(2x-1)\dot{x}}{(1+x^2)}\bigg)R, \tag{4.9}$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \tag{4.10}$$

As mentioned in Sec. 2, any particular solution satisfying equations (4.8)-(4.10) is sufficient to derive an integral of motion. We solve equations (4.8)-(4.10) in the following way. Equation (4.8) is a first order partial differential equation in S with variable coefficients. To seek a particular solution for S we consider a simple ansatz for S of the form

$$S = a(t,x) + b(t,x)\dot{x}, \tag{4.11}$$

where a and b are arbitrary functions of t and x (in other examples one may need to take rational forms in $\dot{x}$). Substituting (4.11) into (4.8) and equating the coefficients of different powers of $\dot{x}$ to zero we get a set of partial differential equations for the variables a and b. Solving them we arrive at

$$S_1 = \frac{(2x-1)}{(1+x^2)}\dot{x}, \quad S_2 = -\frac{1}{t} + \frac{(1-2x)}{(1+x^2)}\dot{x}. \tag{4.12}$$

Substituting the forms of S_1 and S_2 separately into (4.9) and solving the resultant equations one can obtain the corresponding forms of R. Let us first consider S_1. Substituting the latter into (4.9) we obtain an equation for R of the form

$$R_t + \dot{x}R_x + \frac{(2x-1)\dot{x}^2}{(1+x^2)}R_{\dot{x}} \quad = -\bigg(\tfrac{(2x-1)\dot{x}}{(1+x^2)}\bigg)R. \tag{4.13}$$

One can immediately identify a particular solution

$$R_1 = -\frac{e^{tan^{-1}x}}{(1+x^2)}, \tag{4.14}$$

to this equation with a polynamial ansatz in $\dot{x}$. One can easily check that S_1 and R_1 satisfy equation (4.10) also. As a consequence one can deduce the first integral, using the relation (4.4), of the form

$$I_1 = \frac{\dot{x}e^{tan^{-1}x}}{(1+x^2)}. \tag{4.15}$$

Now substituting the expression S_2 into (4.9) we obtain an equation for R. As in the previous case, one can easily fix a particular solution of the form

$$R_2 = -\frac{t}{\dot{x}}. \tag{4.16}$$

However, this set (S_2, R_2) does not satisfy the extra constraint (4.10) and so to deduce the correct form of R_2 we assume that

$$\hat{R}_2 = F(I_1)R_2 = -F(I_1)\frac{t}{\dot{x}}, \tag{4.17}$$

where F is an arbitrary function of the first integral I_1. Substituting (4.17) into equation (4.10) we obtain $F = \frac{1}{I_1}$, which fixes the form of $\hat{R}$ as

$$\hat{R}_2 = -\frac{te^{tan^{-1}x}}{(1+x^2)}. \tag{4.18}$$

Now one can easily check that this set $(S_2, \hat{R}_2)$ is a compatible solution for the set (4.8)-(4.10) which in turn provides I_2 through the relation (4.4) in the form

$$I_2 = e^{tan^{-1}x}\bigg(1 - \frac{t\dot{x}}{(1+x^2)}\bigg). \tag{4.19}$$

Using the explicit forms of the first integrals I_1 and I_2, the solution of Eq. (4.1) can be deduced directly as

$$x(t) = \tan\bigg[\log\left(I_1 t + I_2\right)\bigg]. \tag{4.20}$$

However, as shown in the Sec. 3, one can also deduce the second integral from the first integral itself. By using the procedure indicated in Sec. 3, by demanding that I_1 be put in the form

$$I_1 = F_1\left(\frac{1}{G_2(t,x,\dot{x})}\frac{d}{dt}G_1(t,x)\right) + F_2\left(G_1(t,x)\right), \tag{4.21}$$

so that in the transformed variables become $w = G_1(t,x)$ and $z = \int_o^t G_2(t', x, \dot{x})dt'$ and a first order ODE results which can be solved by quadrature. For example, in the present case, it is easy to rewrite the first integral (4.15) in the form (4.21), by inspection, namely,

$$I_1 = \frac{d}{dt}\left(e^{tan^{-1}x}\right), \tag{4.22}$$

and identifying (4.22) with (3.2), we get

$$G_1 = e^{tan^{-1}x}, \qquad G_2 = 1, \qquad F_2 = 0. \tag{4.23}$$

With the above choices, equation (3.3) furnishes the transformation variables,

$$w = e^{tan^{-1}x}, \qquad z = t. \tag{4.24}$$

Substituting (4.24) into (4.22) we get

$$\frac{dw}{dz} = I_1, \tag{4.25}$$

which in turn gives the free particle equation by differentiation or leads to the solution (4.20) by an integration. On the other hand vanishing of the function F_2 in this analysis is precisely the condition for the system to be transformed into the free particle equation.

5 Third order ODEs

Now we focus our attention on third order ODEs, $n = 3$ in equation (2.1). Fixing $n = 3$ in the determining equations (2.6)-(2.13), we get

$$\begin{aligned} D[S_1] &= -\phi_x + S_1\phi_{\ddot{x}} + S_1S_2, && (5.1)\\ D[S_2] &= -\phi_{\dot{x}} + S_2\phi_{\ddot{x}} + S_2^2 - S_1, && (5.2)\\ D[R] &= -R(S_2 + \phi_{\ddot{x}}), && (5.3)\\ R_{\dot{x}}S_1 &= -RS_{1\dot{x}} + R_xS_2 + RS_{2x}, && (5.4)\\ R_{\dot{x}} &= R_{\ddot{x}}S_2 + RS_{2\ddot{x}}, && (5.5)\\ R_x &= R_{\ddot{x}}S_1 + RS_{1\ddot{x}}, && (5.6)\end{aligned}$$

where the total differential operator D is defined by

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \phi\frac{\partial}{\partial \ddot{x}}.$$

The associated integral of motion is fixed by the relation

$$I = r_1 - r_2 - r_3 - \int \left(R + \frac{d}{d\ddot{x}}(r_1 - r_2 - r_3)\right) d\ddot{x}, \tag{5.7}$$

where

$$r_1 = \int R(\phi + S_1\dot{x} + S_2\ddot{x})dt, \qquad r_2 = \int (RS_1 + \frac{d}{dx}r_1)dx,$$
$$r_3 = \int \left(RS_2 + \tfrac{d}{d\dot{x}}(r_1 - r_2)\right) d\dot{x}.$$

As mentioned earlier, (5.1)-(5.6) form an overdetermined system for the unknowns, S_1, S_2 and R. We solve the (5.1)-(5.6) in the following way. Substituting the expression for ϕ into (5.1)-(5.2) we get a system of differential equations for the unknowns S_1 and S_2. Solving them we can obtain expressions for the null forms (S_1, S_2). Once S_2 is known then equation (5.3) becomes the determining equation for the function R. Solving the latter we can get an explicit form for R. Now the functions R, S_1 and S_2 have to satisfy the extra constraints (5.4)-(5.6).

5.1 Example: 1

Let us consider an equation discussed by Bluman and Anco from the symmentres point of view [16],

$$\dddot{x} = \frac{6t\ddot{x}^3}{\dot{x}^2} + \frac{6\ddot{x}^2}{\dot{x}}. \tag{5.8}$$

Substituting $\phi = \frac{6t\ddot{x}^3}{\dot{x}^2} + \frac{6\ddot{x}^2}{\dot{x}}$ into (5.1)-(5.3), we get

$$\begin{aligned} S_{1t} + \dot{x}S_{1x} + \ddot{x}S_{1\dot{x}} + \left(\frac{6t\ddot{x}^3}{\dot{x}^2} + \frac{6\ddot{x}^2}{\dot{x}}\right)S_{1\ddot{x}} &= S_1\left(\tfrac{18t\ddot{x}^2}{\dot{x}^2} + \tfrac{12\ddot{x}}{\dot{x}} + S_2\right),\\ S_{2t} + \dot{x}S_{2x} + \ddot{x}S_{2\dot{x}} + \left(\frac{6t\ddot{x}^3}{\dot{x}^2} + \frac{6\ddot{x}^2}{\dot{x}}\right)S_{2\ddot{x}} &= \tfrac{12t\ddot{x}}{\dot{x}^3} + \tfrac{6\ddot{x}^2}{\dot{x}^2} + S_2\left(\tfrac{18t\ddot{x}^2}{\dot{x}^2} + \tfrac{12\ddot{x}}{\dot{x}}\right) + S_2{}^2 - S_1,\\ R_t + \dot{x}R_x + \ddot{x}R_{\dot{x}} + \left(\frac{6t\ddot{x}^3}{\dot{x}^2} + \frac{6\ddot{x}^2}{\dot{x}}\right)R_{\ddot{x}} &= -R\left(S_2 + \tfrac{18t\ddot{x}^2}{\dot{x}^2} + \tfrac{12\ddot{x}}{\dot{x}}\right). \end{aligned} \tag{5.9}$$

One can easily verify that equation (5.9) admits the following solutions,

$$S_1 = 0, \quad S_2 = -\frac{6t\ddot{x}^2 + 3\dot{x}\ddot{x}}{\dot{x}^2}, \quad R = \frac{\dot{x}^3}{\ddot{x}^2}, \tag{5.10}$$

$$\bar{S}_1 = 0, \quad \bar{S}_2 = -\frac{6t\ddot{x}^2 + 4\dot{x}\ddot{x}}{\dot{x}^2}, \quad \bar{R} = \frac{\dot{x}^4}{\ddot{x}^2}, \tag{5.11}$$

$$\hat{S}_1 = \frac{2\ddot{x}^2}{\dot{x}^2}, \quad \hat{S}_2 = -\frac{6t\ddot{x}^2 + 2\dot{x}\ddot{x}}{\dot{x}^2}, \quad \hat{R} = \frac{\dot{x}^2}{\ddot{x}^2}. \tag{5.12}$$

Having determined the functions S_i's and R, $i = 1, 2$, one can proceed to determine the associated integrals of motion. Substituting the expressions into (5.7) separately and evaluating the integrals, one obtains

$$I_1 = 3t\dot{x}^2 + \tfrac{\dot{x}^3}{\ddot{x}}, \tag{5.13}$$

$$I_2 = 2t\dot{x}^3 + \tfrac{\dot{x}^4}{\ddot{x}}, \tag{5.14}$$

$$I_3 = 2x - 6t\dot{x} - \tfrac{\dot{x}^2}{\ddot{x}}, \tag{5.15}$$

respectively. One can easily check that $I_i's, i = 1, 2, 3$, are constants on the solutions, that is, $\frac{dI_i}{dt} = 0, i = 1, 2, 3$. From the integrals, I_1, I_2 and I_3, we can deduce the general solution for the equation (5.8) straightforwardly.

In the following we generate second and third integrals, say, (5.14) and (5.15), from the first integral, that is, (5.13), using our procedure described in Sec. 3.

Rewriting (5.13) in the form (3.2) we get

$$I_1 = -\frac{1}{\ddot{x}}\frac{d}{dt}(-t\dot{x}^3) = \frac{dt}{dz}\frac{dw}{dt} = \frac{dw}{dz}, \tag{5.16}$$

so that

$$w = -t\dot{x}^3, \quad z = -\dot{x}. \tag{5.17}$$

Integrating (5.16) and rewriting the latter in terms of the old variables we get

$$I_2 = 2t\dot{x}^3 + \frac{\dot{x}^4}{\ddot{x}}, \tag{5.18}$$

which is exactly the same as the one we derived (vide equation (5.14)) earlier through the PS procedure.

To generate I_3 from I_1 we rewrite the latter in the form (3.2) but with different functions $\hat{w}$ and $\hat{z}$, namely,

$$I_1 = -\frac{\dot{x}^2}{\ddot{x}}\frac{d}{dt}(2x - 3t\dot{x}) = \frac{dt}{d\hat{z}}\frac{d\hat{w}}{dt} = \frac{d\hat{w}}{d\hat{z}}, \tag{5.19}$$

so that

$$\hat{w} = 2x - 3t\dot{x}, \quad \hat{z} = \dot{x}. \tag{5.20}$$

Integrating (5.19) we get

$$\hat{w} = I_1\hat{z} + I_3 \Rightarrow I_3 = \hat{w} - I_1\hat{z}. \tag{5.21}$$

Substituting (5.20) and (5.13) into (5.21) we get

$$I_3 = 2x - 6t\dot{x} - \frac{\dot{x}^2}{\ddot{x}}, \tag{5.22}$$

which exactly coincides with (5.15).

In a similar way one can derive I_1 and I_2 from I_3 and I_1 and I_3 from I_2.

5.2 Example: 2

Let us consider another nontrivial example which was discussed by Steeb [18] in the context of invertible point transformations, namely,

$$\dddot{x} + \frac{3\dot{x}\ddot{x}}{x} - 3\ddot{x} - \frac{3\dot{x}^2}{x} + 2\dot{x} = 0. \tag{5.23}$$

Substituting the form $\phi = -\frac{3}{x}(\dot{x}\ddot{x} - x\ddot{x} - \dot{x}^2 + \frac{2}{3}x\dot{x})$ into (5.1)-(5.3) and solving them, we find the following three particular solutions which satisfy the determining equations

$$\begin{aligned}
(S_1, F_1, R_1) &= (\tfrac{\ddot{x}-\dot{x}}{x}, \tfrac{2\dot{x}-x}{x}, xe^{-2t}), &(5.24)\\
(S_2, F_2, R_2) &= (\tfrac{\ddot{x}-2\dot{x}}{x}, \tfrac{2\dot{x}-2x}{x}, xe^{-t}), &(5.25)\\
(S_3, F_3, R_3) &= (\tfrac{\ddot{x}-3\dot{x}+2x}{x}, \tfrac{2\dot{x}^2-3x}{x}, x). &(5.26)
\end{aligned}$$

Substituting the expressions (5.24)-(5.26) separately into the expression (5.7) and evaluating the integrals we obtain

$$\begin{aligned}
I_1 &= (\dot{x}^2 + x\ddot{x} - x\dot{x})e^{-2t}, &(5.27)\\
I_2 &= (\dot{x}^2 + x\ddot{x} - 2x\dot{x})e^{-t}, &(5.28)\\
I_3 &= (\dot{x}^2 + x\ddot{x} - 3x\dot{x} + x^2), &(5.29)
\end{aligned}$$

from which the general solution can be written of the form

$$x(t) = \left(\frac{I_1}{2}e^{2t} + I_2e^t + I_3\right)^{\frac{1}{2}}. \tag{5.30}$$

From the integrals I_1, I_2 and I_3, one can also obtain the linearizing transformation.

6 Extension to higher order ODEs

In the previous two sections we discussed the PS procedure applicable for second and third order ODEs. Following the same steps one can derive the determining equations for the fourth order ODE also from the equations (2.6)-(2.13). This can be done by restricting to $n = 4$ in these equations. For example, for the present case one obtains ten equations for four unknowns, namely, S_1, S_2, S_3, and R. Solving them consistently one can obtain explicit expressions for the null forms, S_i's, $i = 1, 2, 3$, and the integrating factor R. Once the S_i's and R are known then the associated integral of motion can be constructed using the relation (2.14) with $n = 4$. Here also, if one is able to find four sets of particular solutions, say, (S_{ij}, R_i), $i = 1, 2, 3, 4$ and $j = 1, 2, 3$, one can straightforwardly construct four integrals of motion. On the other hand if one has less number of integrals of motion then using the procedure described in Sec. 3 one can generate the remaining integrals of motion from the known ones and establish the integrability. The extension to higher order ODEs follows along similar lines.

7 Extension to coupled second order ODEs

So far we discussed the applicability of the PS procedure to scalar differential equations. Interstingly, the procedure can also be extended to coupled ODEs. In the following we describe the procedure to two coupled second order ODEs and the application to higher dimensional equations will be discussed elsewhere.

Let us consider a class of second order ODEs of the form

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{P_1}{Q_1}, \quad \ddot{y} = \frac{d^2y}{dt^2} = \frac{P_2}{Q_2}, \quad P_i, Q_i \in C[t, x, y, \dot{x}, \dot{y}],\ i = 1, 2. \tag{7.1}$$

Let us suppose that the system (7.1) admits a first integral of the form $I(t, x, y, \dot{x}, \dot{y}) = C$ with C constant on the solution, so that the total differential gives

$$dI = I_t dt + I_x dx + I_y dy + I_{\dot{x}} d\dot{x} + I_{\dot{y}} d\dot{y} = 0, \tag{7.2}$$

where subscript denotes partial differentiation with respect to that variable. Rewriting (7.1) in the form

$$\frac{P_1}{Q_1}dt - d\dot{x} = 0, \qquad \frac{P_2}{Q_2}dt - d\dot{y} = 0 \tag{7.3}$$

and adding null terms $S_1(t, x, y, \dot{x}, \dot{y})\dot{x}dt - S_1(t, x, y, \dot{x}, \dot{y})dx$ and $S_2(t, x, y, \dot{x}, \dot{y})\dot{y}dt$ $-S_2(t, x, y, \dot{x}, \dot{y})dy$ suitably, we obtain that, on the solutions, the 1-forms

$$(\frac{P_1}{Q_1} + S_1\dot{x})dt - S_1 dx - d\dot{x} = 0, \tag{7.4a}$$

$$(\frac{P_2}{Q_2} + S_2\dot{y})dt - S_2 dy - d\dot{y} = 0. \tag{7.4b}$$

Hence, on the solutions, the 1-forms (7.2) and (7.4) must be proportional. Multiplying (7.4a) by the factor $R_1(t, x, y, \dot{x}, \dot{y})$ and (7.4b) by the factor $R_2(t, x, y, \dot{x}, \dot{y})$, which act as the integrating factors for (7.4a) and (7.4b), respectively, we have on the solutions that

$$dI = R_1(\phi_1 + S_1\dot{x})dt + R_2(\phi_2 + S_2\dot{y})dt - R_1S_1dx - R_2S_2dy - R_1d\dot{x} - R_2d\dot{y} = 0, \tag{7.5}$$

where $\phi_i \equiv P_i/Q_i,\ i = 1, 2$. Comparing equations (7.5) and (7.2) we have, on the solutions, the relations

$$I_t = R_1(\phi_1 + S_1\dot{x}) + R_2(\phi_2 + S_2\dot{y}), \tag{7.6a}$$

$$I_x = -R_1S_1, \tag{7.6b}$$

$$I_y = -R_2S_2, \tag{7.6c}$$

$$I_{\dot{x}} = -R_1, \tag{7.6d}$$

$$I_{\dot{y}} = -R_2. \tag{7.6e}$$

The compatibility conditions between the equations (7.6) provide us the conditions,

$$D[S_1] = -\phi_{1x} - \frac{R_2}{R_1}\phi_{2x} + \frac{R_2}{R_1}S_1\phi_{2\dot{x}} + S_1\phi_{1\dot{x}} + S_1^2, \tag{7.7}$$
$$D[S_2] = -\phi_{2y} - \frac{R_1}{R_2}\phi_{1y} + \frac{R_1}{R_2}S_2\phi_{1\dot{y}} + S_2\phi_{2\dot{y}} + S_2^2, \tag{7.8}$$
$$D[R_1] = -(R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}} + R_1S_1), \tag{7.9}$$
$$D[R_2] = -(R_2\phi_{2\dot{y}} + R_1\phi_{1\dot{y}} + R_2S_2), \tag{7.10}$$
$$S_1R_{1y} = -R_1S_{1y} + S_2R_{2x} + R_2S_{2x}, \tag{7.11}$$
$$R_{1x} = S_1R_{1\dot{x}} + R_1S_{1\dot{x}}, \tag{7.12}$$
$$R_{2y} = S_2R_{2\dot{y}} + R_2S_{2\dot{y}}, \tag{7.13}$$
$$R_{1y} = S_2R_{2\dot{x}} + R_2S_{2\dot{x}}, \tag{7.14}$$
$$R_{2x} = S_1R_{1\dot{y}} + R_1S_{1\dot{y}}, \tag{7.15}$$
$$R_{1\dot{y}} = R_{2\dot{x}}, \tag{7.16}$$

where the total differential operator is now defined by

$$D = \tfrac{\partial}{\partial t} + \dot{x}\tfrac{\partial}{\partial x} + \dot{y}\tfrac{\partial}{\partial y} + \phi_1\tfrac{\partial}{\partial \dot{x}} + \phi_2\tfrac{\partial}{\partial \dot{y}}. \tag{7.17}$$

Integrating equations (7.6), we obtain the integral of motion,

$$I = r_1 + r_2 + r_3 + r_4 - \int \left[R_2 + \frac{d}{d\dot{y}}\left(r_1 + r_2 + r_3 + r_4\right)\right]d\dot{y}, \tag{7.18}$$

where

$$r_1 = \int \left(R_1(\phi_1 + S_1\dot{x}) + R_2(\phi_2 + S_2\dot{y})\right)dt, \qquad r_2 = -\int \left(R_1S_1 + \tfrac{d}{dx}(r_1)\right)dx,$$
$$r_3 = -\int \left(R_2S_2 + \tfrac{d}{dy}(r_1 + r_2)\right)dy, \qquad r_4 = -\int \left[R_1 + \tfrac{d}{d\dot{x}}\left(r_1 + r_2 + r_3\right)\right]d\dot{x}.$$

As we did earlier, solving the determining equations (7.7)-(7.16) consistently we can obtain expressions for the function S_i's and R_i's, $i = 1, 2$. Substituting them into (7.18) and evaluating the integrals we can deduce the associated integrals of motion. However, unlike the scalar case, the determining equations in the present case are highly coupled and pose difficulties to approach them directly. To overcome this problem we adopt the following technique. We rewrite equations (7.7)-(7.16) for two variables, namely, R_1 and R_2, by eliminating S_1 and S_2, and solve the resultant equations and obtain expressions for R_1 and R_2. From the latter we deduce the forms of S_1 and S_2 by using the relations (7.9) and (7.10). To implement this algorithm we use all the equations (7.7)-(7.16) effectively such that the functions S_i's $i = 1, 2$, can be eliminated and the functions R_1 and R_2 can be deduced from an optimal set of equations.

To begin with we deduce the following two identities,

$$D[R_1S_1] = -(R_1\phi_{1x} + R_2\phi_{2x}), \tag{7.19}$$
$$D[R_2S_2] = -(R_1\phi_{1y} + R_2\phi_{2y}), \tag{7.20}$$

which can be obtained by combining (7.7)-(7.10). Now we have explicit forms for the total derivatives R_1S_1 and R_2S_2. Let us now take total derivative of equations (7.9) and (7.10) and substitute (7.19) and (7.20) in the resultant equations to obtain

$$\begin{aligned}
R_{1tt} &+ 2\dot{x}R_{1tx} + 2\dot{y}R_{1ty} + 2\phi_1 R_{1t\dot{x}} + 2\phi_2 R_{1t\dot{y}} + \dot{x}^2 R_{1xx} + 2\dot{x}\dot{y}R_{1xy} + \dot{y}^2 R_{1yy} + \phi_{1t}R_{1\dot{x}} \\
&+\phi_{2t}R_{1\dot{y}} + \dot{x}\phi_{1x}R_{1\dot{x}} + \dot{y}\phi_{1y}R_{1\dot{x}} + 2\dot{x}\phi_1 R_{1x\dot{x}} + 2\dot{y}\phi_1 R_{1y\dot{x}} + 2\dot{x}\phi_2 R_{1x\dot{y}} + 2\dot{y}\phi_2 R_{1y\dot{y}} \\
&+\dot{x}\phi_{2x}R_{1\dot{y}} + \dot{y}\phi_{2y}R_{1\dot{y}} + \phi_1 R_{1x} + \phi_2 R_{1y} + \phi_1\phi_{1\dot{x}}R_{1\dot{x}} + \phi_1^2 R_{1\dot{x}\dot{x}} + \phi_2\phi_{1\dot{y}}R_{1\dot{x}} \\
&+\phi_1\phi_{2\dot{x}}R_{1\dot{y}} + \phi_2\phi_{2\dot{y}}R_{1\dot{y}} + \phi_{1\dot{x}}(R_{1t} + \dot{x}R_{1x} + \dot{y}R_{1y} + \phi_1 R_{1\dot{x}} + \phi_2 R_{1\dot{y}}) \\
&+\phi_2^2 R_{1\dot{y}\dot{y}} + 2\phi_1\phi_2 R_{1\dot{x}\dot{y}} + \phi_{2\dot{x}}(R_{2t} + \dot{x}R_{2x} + \dot{y}R_{2y} + \phi_1 R_{2\dot{x}} + \phi_2 R_{2\dot{y}}) \\
&-R_1\phi_{1x} - R_2\phi_{2x} + R_1(\phi_{1t\dot{x}} + \dot{x}\phi_{1x\dot{x}} + \dot{y}\phi_{1y\dot{x}} + \phi_1\phi_{1\dot{x}\dot{x}} + \phi_2\phi_{1\dot{x}\dot{y}}) \\
&+R_2(\phi_{2t\dot{x}} + \dot{x}\phi_{2x\dot{x}} + \dot{y}\phi_{2y\dot{x}} + \phi_1\phi_{2\dot{x}\dot{x}} + \phi_2\phi_{2\dot{x}\dot{y}}) = 0,
\end{aligned} \tag{7.21}$$

$$\begin{aligned}
R_{2tt} &+ 2\dot{x}R_{2tx} + 2\dot{y}R_{2ty} + 2\phi_1 R_{2t\dot{x}} + 2\phi_2 R_{2t\dot{y}} + \dot{x}^2 R_{2xx} + 2\dot{x}\dot{y}R_{2xy} + \dot{y}^2 R_{2yy} + \phi_{1t}R_{2\dot{x}} \\
&+\phi_{2t}R_{2\dot{y}} + \dot{x}\phi_{1x}R_{2\dot{x}} + \dot{y}\phi_{1y}R_{2\dot{x}} + 2\dot{x}\phi_1 R_{2x\dot{x}} + 2\dot{y}\phi_1 R_{2y\dot{x}} + 2\dot{x}\phi_2 R_{2x\dot{y}} + 2\dot{y}\phi_2 R_{2y\dot{y}} \\
&+\dot{x}\phi_{2x}R_{2\dot{y}} + \dot{y}\phi_{2y}R_{2\dot{y}} + \phi_1 R_{2x} + \phi_2 R_{2y} + \phi_1\phi_{1\dot{x}}R_{2\dot{x}} + \phi_1^2 R_{2\dot{x}\dot{x}} + \phi_2\phi_{1\dot{y}}R_{2\dot{x}} \\
&+\phi_1\phi_{2\dot{x}}R_{2\dot{y}} + \phi_2\phi_{2\dot{y}}R_{2\dot{y}} + \phi_{1\dot{y}}(R_{1t} + \dot{x}R_{1x} + \dot{y}R_{1y} + \phi_1 R_{1\dot{x}} + \phi_2 R_{1\dot{y}}) \\
&+\phi_2^2 R_{2\dot{y}\dot{y}} + 2\phi_1\phi_2 R_{2\dot{x}\dot{y}} + \phi_{2\dot{y}}(R_{2t} + \dot{x}R_{2x} + \dot{y}R_{2y} + \phi_1 R_{2\dot{x}} + \phi_2 R_{2\dot{y}}) \\
&-R_1\phi_{1y} - R_2\phi_{2y} + R_1(\phi_{1t\dot{y}} + \dot{x}\phi_{1x\dot{y}} + \dot{y}\phi_{1y\dot{y}} + \phi_1\phi_{1\dot{x}\dot{y}} + \phi_2\phi_{1\dot{y}\dot{y}}) \\
&+R_2(\phi_{2t\dot{y}} + \dot{x}\phi_{2x\dot{y}} + \dot{y}\phi_{2y\dot{y}} + \phi_1\phi_{2\dot{x}\dot{y}} + \phi_2\phi_{2\dot{y}\dot{y}}) = 0.
\end{aligned} \tag{7.22}$$

equations (7.12)-(7.16) can also be written of the form

$$R_{1x} = \frac{\partial}{\partial \dot{x}}(R_1S_1), \quad R_{2y} = \frac{\partial}{\partial \dot{y}}(R_2S_2), \quad R_{1y} = \frac{\partial}{\partial \dot{x}}(R_2S_2), \quad R_{2x} = \frac{\partial}{\partial \dot{y}}(R_1S_1). \tag{7.23}$$

These identities help us to obtain some additional equations for the variables R_1 and R_2. For example differentiating (7.9) with respect to $\dot{x}$ and (7.10) with respect to $\dot{y}$ and using the identities (7.23) in the resulting equations, we get

$$\begin{aligned}
R_{1t\dot{x}} + \dot{x}R_{1x\dot{x}} + \dot{y}R_{1y\dot{x}} + \phi_1 R_{1\dot{x}\dot{x}} + \phi_2 R_{2\dot{x}\dot{x}} + 2R_{1x} + 2\phi_{2\dot{x}}R_{2\dot{x}} + 2\phi_{1\dot{x}}R_{1\dot{x}} \\
+R_2\phi_{2\dot{x}\dot{x}} + R_1\phi_{1\dot{x}\dot{x}} = 0,
\end{aligned} \tag{7.24}$$

$$\begin{aligned}
R_{2t\dot{y}} + \dot{x}R_{2x\dot{y}} + \dot{y}R_{2y\dot{y}} + \phi_1 R_{1\dot{y}\dot{y}} + \phi_2 R_{2\dot{y}\dot{y}} + 2R_{2y} + 2\phi_{2\dot{y}}R_{2\dot{y}} + 2\phi_{1\dot{y}}R_{1\dot{y}} \\
+R_2\phi_{2\dot{y}\dot{y}} + R_1\phi_{1\dot{y}\dot{y}} = 0.
\end{aligned} \tag{7.25}$$

On the other hand, differentiating (7.9) with respect to $\dot{y}$ and using (7.23) we get

$$\begin{aligned}
R_{1t\dot{y}} + \dot{x}R_{1x\dot{y}} + \dot{y}R_{1y\dot{y}} + \phi_1 R_{1\dot{x}\dot{y}} + \phi_2 R_{1\dot{y}\dot{y}} + R_{1y} + R_{2x} + \phi_{2\dot{x}}R_{2\dot{y}} \\
+\phi_{1\dot{x}}R_{1\dot{y}} + R_2\phi_{2\dot{x}\dot{y}} + R_1\phi_{1\dot{x}\dot{y}} + \phi_{1\dot{y}}R_{1\dot{x}} + \phi_{2\dot{y}}R_{1\dot{y}} = 0.
\end{aligned} \tag{7.26}$$

The remaining possibility, that is, differentiation of (7.10) with respect to $\dot{x}$, leads us to the same equation (7.26) and so we can discard it.

We can further simplify equations (7.21) and (7.22) by utilizing the equations (7.24)-(7.26). The final form of the equations (7.21) and (7.22) reads

$$\begin{aligned}&R_{1tt}+2\dot{x}R_{1tx}+2\dot{y}R_{1ty}+\phi_1R_{1t\dot{x}}+\phi_2R_{1t\dot{y}}+\dot{x}^2R_{1xx}+2\dot{x}\dot{y}R_{1xy}-R_1\phi_{1x}\\&-R_2\phi_{2x}+\dot{y}^2R_{1yy}+\phi_{1t}R_{1\dot{x}}+\phi_{2t}R_{1\dot{y}}+\dot{x}\phi_{1x}R_{1\dot{x}}+\dot{y}\phi_{1y}R_{1\dot{x}}+\dot{x}\phi_1R_{1x\dot{x}}\\&+\dot{y}\phi_1R_{1y\dot{x}}+\dot{x}\phi_2R_{1x\dot{y}}+\dot{y}\phi_2R_{1y\dot{y}}+\dot{x}\phi_{2x}R_{1\dot{y}}+\dot{y}\phi_{2y}R_{1\dot{y}}-\phi_1R_{1x}\\&-\phi_2R_{2x}+\phi_{1\dot{x}}(R_{1t}+\dot{x}R_{1x}+\dot{y}R_{1y})+R_1(\phi_{1t\dot{x}}+\dot{x}\phi_{1x\dot{x}}+\dot{y}\phi_{1y\dot{x}})\\&+\phi_{2\dot{x}}(R_{2t}+\dot{x}R_{2x}+\dot{y}R_{2y})+R_2(\phi_{2t\dot{x}}+\dot{x}\phi_{2x\dot{x}}+\dot{y}\phi_{2y\dot{x}})=0,\end{aligned}\tag{7.27}$$

$$\begin{aligned}&R_{2tt}+2\dot{x}R_{2tx}+2\dot{y}R_{2ty}+\phi_1R_{2t\dot{x}}+\phi_2R_{2t\dot{y}}+\dot{x}^2R_{2xx}+2\dot{x}\dot{y}R_{2xy}-R_1\phi_{1y}\\&-R_2\phi_{2y}+\phi_{1t}R_{2\dot{x}}+\phi_{2t}R_{2\dot{y}}+\dot{y}^2R_{2yy}+\dot{x}\phi_{1x}R_{2\dot{x}}+\dot{y}\phi_{1y}R_{2\dot{x}}+\dot{x}\phi_1R_{2x\dot{x}}\\&+\dot{y}\phi_1R_{2y\dot{x}}+\dot{x}\phi_2R_{2x\dot{y}}+\dot{y}\phi_2R_{2y\dot{y}}+\dot{x}\phi_{2x}R_{2\dot{y}}+\dot{y}\phi_{2y}R_{2\dot{y}}-\phi_1R_{1y}\\&-\phi_2R_{2y}+\phi_{1\dot{y}}(R_{1t}+\dot{x}R_{1x}+\dot{y}R_{1y})+R_1(\phi_{1t\dot{y}}+\dot{x}\phi_{1x\dot{y}}+\dot{y}\phi_{1y\dot{y}})\\&+\phi_{2\dot{y}}(R_{2t}+\dot{x}R_{2x}+\dot{y}R_{2y})+R_2(\phi_{2t\dot{y}}+\dot{x}\phi_{2x\dot{y}}+\dot{y}\phi_{2y\dot{y}})=0.\end{aligned}\tag{7.28}$$

As a result now we have a system of five equations (of course in second order) for the unknowns R_1 and R_2, namely equations (7.24)-(7.28). Substituting the expressions for ϕ_1 and ϕ_2 into (7.24)-(7.28) and solving them one gets the integrating factors $R_1$and R_2. Once R_i's, $i=1,2$, are known the null forms S_i's, $i=1,2$, can be fixed through the relation (7.9)-(7.10).

7.1 Example: Two-dimensional Kepler problem

Here we consider a simple, but physically important example, namely, two dimensional Kepler problem and illustrate the method developed in the previous section.

Let us consider the Kepler problem in the $x-y$ plane, that is,

$$\ddot{\hat{\mathbf{r}}}+\frac{\hat{\mathbf{r}}}{r^3}=0,\tag{7.29}$$

where $\hat{\mathbf{r}}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}$ and $r=|\hat{\mathbf{r}}|$. The respective equations of motions are

$$\ddot{x}=-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}=\phi_1(x,y),\tag{7.30}$$

$$\ddot{y}=-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}=\phi_2(x,y).\tag{7.31}$$

In this case the determining equations (7.24)-(7.28) simplify to

$$R_{1t\dot{x}}+\dot{x}R_{1x\dot{x}}+\dot{y}R_{1y\dot{x}}+\phi_1R_{1\dot{x}\dot{x}}+\phi_2R_{2\dot{x}\dot{x}}+2R_{1x}=0,\tag{7.32}$$

$$R_{2t\dot{y}}+\dot{x}R_{2x\dot{y}}+\dot{y}R_{2y\dot{y}}+\phi_1R_{1\dot{y}\dot{y}}+\phi_2R_{2\dot{y}\dot{y}}+2R_{2y}=0,\tag{7.33}$$

$$R_{1t\dot{y}}+\dot{x}R_{1x\dot{y}}+\dot{y}R_{1yvy}+\phi_1R_{1\dot{x}\dot{y}}+\phi_2R_{1\dot{y}\dot{y}}+R_{1y}+R_{2x}=0,\tag{7.34}$$

$$\begin{aligned}
&R_{1tt}+2\dot{x}R_{1tx}+2\dot{y}R_{1ty}+\dot{x}^2R_{1xx}+2\dot{x}\dot{y}R_{1xy}+\dot{y}^2R_{1yy}+\dot{x}\phi_{1x}R_{1\dot{x}}+\dot{y}\phi_{1y}R_{1\dot{x}}\\
&\quad+\dot{x}\phi_1R_{1x\dot{x}}+\dot{y}\phi_1R_{1y\dot{x}}+\dot{x}\phi_2R_{1x\dot{y}}+\dot{y}\phi_2R_{1y\dot{y}}+\dot{x}\phi_{2x}R_{1\dot{y}}+\dot{y}\phi_{2y}R_{1\dot{y}}\\
&\quad+\phi_1R_{1t\dot{x}}+\phi_2R_{1t\dot{y}}-R_1\phi_{1x}-R_2\phi_{2x}-\phi_1R_{1x}-\phi_2R_{2x}=0, \qquad (7.35)\\
&R_{2tt}+2\dot{x}R_{2tx}+2\dot{y}R_{2ty}+\dot{x}^2R_{2xx}+2\dot{x}\dot{y}R_{2xy}+\dot{y}^2R_{2yy}+\dot{x}\phi_{1x}R_{2\dot{x}}+\dot{y}\phi_{1y}R_{2\dot{x}}\\
&\quad+\dot{x}\phi_1R_{2x\dot{x}}+\dot{y}\phi_1R_{2y\dot{x}}+\dot{x}\phi_2R_{2x\dot{y}}+\dot{y}\phi_2R_{2y\dot{y}}+\dot{x}\phi_{2x}R_{2\dot{y}}+\dot{y}\phi_{2y}R_{2\dot{y}}\\
&\quad+\phi_1R_{2t\dot{x}}+\phi_2R_{2t\dot{y}}-R_1\phi_{1y}-R_2\phi_{2y}-\phi_1R_{1y}-\phi_2R_{2y}=0. \qquad (7.36)
\end{aligned}$$

To solve equations (7.32)-(7.36) we seek an ansatz

$$R_1=a_1(x,y)+a_2(x,y)\dot{x}+a_3(x,y)\dot{y}, \qquad (7.37)$$

$$R_2=b_1(x,y)+b_2(x,y)\dot{x}+b_3(x,y)\dot{y}, \qquad (7.38)$$

where a_i's and b_i's, $i=1,2,3$, are arbitrary functions of x and y. Substituting (7.37) and (7.38) into (7.32)-(7.36) and solving the resultant equations we can obtain at least the following three solutions.

$$(i) \quad R_1=\dot{x}, \qquad R_2=\dot{y}, \qquad (7.39)$$

$$(ii) \quad \bar{R}_1=y, \qquad \bar{R}_2=x, \qquad (7.40)$$

$$(iii) \quad \hat{R}_1=2y\dot{x}-x\dot{y}, \qquad \hat{R}_2=-x\dot{x}. \qquad (7.41)$$

We are also now searching for other possible forms with a modified ansatz. Substituting (7.39) into (7.9) and (7.10) we get

$$(i) \quad S_1=\frac{x}{\dot{x}(x^2+y^2)^{\frac{3}{2}}}, \quad S_2=\frac{y}{\dot{y}(x^2+y^2)^{\frac{3}{2}}}. \qquad (7.42)$$

In similar way equations (7.40) and (7.41) provide us

$$(ii) \quad \bar{S}_1=-\frac{\dot{y}}{y}, \qquad \bar{S}_2=-\frac{\dot{x}}{x}, \qquad (7.43)$$

$$(iii) \quad \hat{S}_1=-\frac{(-\dot{x}\dot{y}+\frac{xy}{r^3})}{2y\dot{x}-x\dot{y}}, \quad \hat{S}_2=\frac{(\dot{x}^2-\frac{x^2}{r^3})}{x\dot{x}}. \qquad (7.44)$$

Substituting the expressions R_i's and S_i's, $i=1,2$, into (7.18) and evaluating the integrals we get,

$$I_1=\frac{1}{2}(\dot{x}^2+\dot{y}^2)-\frac{1}{\sqrt{x^2+y^2}}, \qquad (7.45)$$

which is of course the Hamiltonian of the system. The forms $\bar{R}_i$'s and $\bar{S}_i$'s, $i=1,2$, provide us

$$I_2=y\dot{x}-x\dot{y}, \qquad (7.46)$$

the second integral, namely, the angular momentum. The integrating factors $\hat{R}_i$'s and null form $\hat{S}_i$'s, $i=1,2$, lead us to

$$I_3=\dot{x}(y\dot{x}-x\dot{y})-\frac{y}{\sqrt{x^2+y^2}}, \qquad (7.47)$$

namely, the Runge-Lenz constant.

8 Final remarks

In this paper, we have discussed the method of solving a class of ODEs through the modified PS method. The method is applicable to both scalar and multicomponent equations of any order. We also demonstrated the theory with examples. Apart from the above, in the scalar case, we introduced a novel way of generating integral of motion from a single integral and illustrated our ideas with the same example considered previously. The application of this method to multicomponent systems and their integrability and linearization properties will be published elsewhere.

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