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Gauge Theory Approach Towards an Explicit Solution of the (Classical) Elliptic Calogero-Moser System

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Abstract

We discuss the relation of the trigonometric Calogero-Moser (CM) system to Yang-Mills gauge theories and its generalization to the elliptic case. This yields a linearization of the time evolution of the elliptic CM system and suggests two promising strategies for finding a fully explicit solution of this model. We also present a large class of integrable spin-particle systems generalizing the elliptic CM system.

Prologue. *Integrable systems associated with the name of Calogero [1, 2, 3, 4] seem to have a strong attraction to me: on two different occasions I started to work on seemingly unrelated problems, made progress, but at some point realized that the natural application of our results was, unexpectedly for me, in the context of Calogero-type systems. One such occasion was a project which was initially aiming at a better understanding of certain aspects of the fractional quantum Hall effect [5] but eventually led to a method for solving the quantum elliptic Calogero-Sutherland system [6]. The present paper is based on the other such occasion which started as a project on two dimensional quantum chromodynamics [7, 8, 9] but eventually led to an alternative solution method for classical trigonometric Calogero-Moser-type systems [10, 11] (this somewhat curious story is told in more detail in the Epilogue of this paper). I will shortly review the solution method thus obtained for the trigonometric Calogero-Moser system, explaining in particular its particle physics motivation. I then show that this method has a natural generalization to the elliptic case, and this leads to an explicit linearization of the time evolution of the (classical) elliptic Calogero-Moser (eCM) model. I finally discuss possible strategies to turn this result into a fully explicit solution of the eCM model. I should mention various closely related papers [12, 13, 14, 15, 16, 17, 18, 19] and in particular [20, 21] whose relation to our work will be also mentioned. However, I have tried to keep my discussion self-contained.*

This paper is based on notes which I wrote in 1999. They have remained unpublished up to now since I have felt I was just one step away from a fully explicit solution of the eCM system. However, I did not succeed with the final step since 1999, and I hope it could be helpful to make available what I have: somebody else might be interested and able to complete this story.

Francesco Calogero is an outstanding example for me, not only as a scientist but also as a person. It is a pleasure for me to dedicate this paper to him.

1 Introduction

We begin by formulating the problem, mainly to fix notation. The Calogero-Moser (CM) systems are classical many-body models which can be defined by the following system of ODEs,

$$\ddot{q}^\alpha(t) = -\lambda^2 \sum_{\substack{\beta=1 \\ \alpha\neq\beta}}^{N} V'\big(q^\alpha(t) - q^\beta(t)\big) \tag{1.1}$$

for $\alpha = 1, 2, \ldots, N$, which are the Newton's equations for $N \geq 2$ identical particles moving in one dimension and interacting via a 2-body potential V which, in the general elliptic case, is given by the Weierstrass elliptic $\wp$-functions, i.e.,

$$V'(q) = \frac{\partial}{\partial q}\wp(q) \equiv -\sum_{m,n\in\mathbb{Z}} \frac{2}{(ma + \mathrm{i}nb + q)^3} \qquad \text{(elliptic case)}; \tag{1.2}$$

the dots mean differentiation with respect to the time variable t, λ is a real coupling constant, and we write the periods of the elliptic function as $2\omega_1 = a$, $2\omega_2 = \mathrm{i}b$ with $a, b > 0$; see e.g. [22]. The problem is to find the particle positions $q^\alpha(t)$ for all times $t > 0$ subject to the usual initial conditions

$$q^\alpha(0) = q_0^\alpha, \quad \dot{q}^\alpha(0) = p_0^\alpha. \tag{1.3}$$

The integrability of these equations was shown by Calogero [3]; for review see [23, 24] and references therein. An explicit solutions for the limiting cases where both periods are infinite,

$$V'(q) = V'_{\mathrm{rat}}(q) \equiv \frac{\partial}{\partial q}\frac{1}{q^2} \qquad \text{(rational case)}, \tag{1.4}$$

is know since a long time: the particle positions $q^\alpha(t)$ in this case are given by the eigenvalues of the $N \times N$ self-adjoint matrix with the matrix elements

$$D^{\alpha\beta}(t) = \delta_{\alpha\beta}\big(q_0^\alpha + p_0^\alpha t\big) - \frac{\mathrm{i}\lambda t\big(1 - \delta_{\alpha\beta}\big)}{q_0^\alpha - q_0^\beta}; \tag{1.5}$$

see [23, 24]. A similarly explicit solution is known in the limiting case where only one period is infinite

$$V'(q) = V'_{\mathrm{trig}}(q) \equiv \frac{\partial}{\partial q}\frac{(\pi/a)^2}{\sin(\pi q/a)^2} \qquad \text{(trigonometric case)}; \tag{1.6}$$

see [23, 24]. Our aim is to find a similarly explicit solution also for the elliptic case. Despite of various remarkable results in this direction [20, 16, 17], this problem is still open, to our knowledge.

The next section shortly reviews a solution method for the trigonometric CM (tCM) model [10] which is alternative to the standard one [23, 24]. It is based on the well-known relation of Yang-Mills gauge theories on a cylinder to the tCM model [13, 8, 25, 15]. In Section 3 we show how to generalize this to the elliptic case by using a particular Yang-Mills gauge theory in 2+1 dimensions.[1] This provides an alternative derivation and interpretation of Krichever's Lax pair and linearization of the time evolution for the eCM system [20] and a promising route to a fully explicit solution of this model. In the final Section 4 we shortly discuss a natural generalization of our construction which leads to integrable spin generalizations of the eCM system. As we argue, the quantum versions of these systems seem to give generalizations of the Inozemtsev spin models [26] which recently have received attention in the context of $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theories in 4 dimensions; see [27] and references therein.

2 Gauge theories and the trigonometric CM system

This section is mainly to motivate our construction, and to explain some particle physics terminology which we use.

2.1 Yang-Mills theory on a cylinder

We recall the definition of classical Yang-Mills theory in 1+1 spacetime dimensions with gauge group $\mathrm{U}(N)$ in the fundamental representation. We consider the Yang-Mills fields $A_\mu = A_\mu(t,x)$ and matter currents $J_\mu = J_\mu(t,x)$ which are functions of the time variable $t \in \mathbb{R}$ and space variable $x \in [-\pi, \pi]$ and with values in the self-adjoint $N \times N$ matrices, with $\mu = 0, 1$ a spacetime index. We assume that space is a circle, i.e., all functions above are periodic. We are interested in equations of motion which are invariant under gauge transformations

$$A_\mu \to A_\mu^U \equiv U^{-1} A_\mu U - \mathrm{i} U^{-1} \partial_\mu U, \quad J_\mu \to J_\mu^U \equiv U^{-1} J_\mu U \tag{2.1}$$

for unitary matrix valued functions functions $U = U(t,x)$, with $\mathrm{i} = \sqrt{-1}$, $\partial_0 \equiv \partial_t$ and $\partial_1 \equiv \partial_x$.[2] The natural such equations are the non-Abelian analog of Maxwell's equations,[3]

$$[D_\mu, F^{\mu\nu}] = J^\nu \tag{2.2}$$

with the covariant derivatives $D_\mu = \partial_\mu + \mathrm{i} A_\mu$, where

$$F_{\mu\nu} = -\mathrm{i}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu]; \tag{2.3}$$

we use the metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1)$ to rise and lower spacetime indices μ, ν. By standard computations one convinces oneself that the equations above indeed are gauge invariant. It is important to note that gauge transformations are periodic, $U(t,x) = U(t, x + 2\pi)$.

[1]This construction is similar to one in [21]; see also [18, 19].
[2]We write ∂_t for $\partial/\partial t$, etc.
[3]It is only in the following equation that we use the summation convention in this paper.

2.2 Relation to the trigonometric CM system

We now explain that this *Yang-Mills theory on a cylinder restricted to matter fields with $J^1 = 0$ can be solved exactly and, for particular initial conditions, is gauge equivalent to the time evolution of the tCM system defined by Eqs. (1.1), (1.3) and (1.6).*[4]

For that we find it convenient to rewrite the equations of motion above as follows, $E = -\mathrm{i}[D_0, D_1]$, $[D_0, E] = 0$, $[D_1, E] = J^0$, or more explicitly,

$$\partial_t A_1 = E + \partial_x A_0 + \mathrm{i}[A_1, A_0] \tag{2.4a}$$
$$\partial_t E + \mathrm{i}[A_0, E] = 0 \tag{2.4b}$$
$$\partial_x E + \mathrm{i}[A_1, E] = \rho \tag{2.4c}$$

where we introduce the notation $E \equiv F_{01}$ and $\rho \equiv J^0$. We will refer to Eq. (2.4c) as *Gauss' law* in the following.[5] It is important to note that these equations also determine the time evolution of the matter fields as follows,[6]

$$\partial_t \rho + \mathrm{i}[A_0, \rho] = 0. \tag{2.4d}$$

We now use the gauge invariance of these equations to impose additional gauge conditions: the first such condition is usually called *axial gauge* in the particle physics literature,

$$A_0 = 0; \tag{2.5}$$

we will show below that this condition is indeed consistent. If we impose this condition the time evolution equations above become trivial, $\partial_t E = \partial_t \rho = 0$ and $\partial_t A_1 = E$, and thus can be solved explicitly as follows, $E(t,x) = E(0,x)$, $\rho(t,x) = \rho(0,x)$, and

$$A_1(t,x) = A_1(0,x) + E(0,x)t. \tag{2.6}$$

It is important to note that Gauss' law is then fulfilled for all times t if it is fulfilled at $t = 0$: Gauss' law gives a constraint on possible initial conditions.

To obtain non-trivial time evolution equations we impose another gauge conditions, namely the so-called *diagonal Coulomb gauge*[7]

$$A_1^{\alpha\beta}(t,x) = \delta_{\alpha\beta} q^\alpha(t) \quad \text{independent of } x \tag{2.7}$$

where $A_1^{\alpha\beta}$, $\alpha, \beta = 1, 2, \ldots, N$, are the matrix elements of A_1, etc.; we will show below that this condition is consistent as well. A straightforward computation shows that *the $q^\alpha(t)$ obey precisely the equations of the tCM system provided we choose the initial conditions for the Yang-Mills equation as follows,*

$$A_1^{\alpha\beta}(0,x) = \delta_{\alpha\beta} q_0^\alpha, \quad \int_{-\pi}^{\pi} \frac{\mathrm{d}x}{2\pi} E^{\alpha\alpha}(0,x) = p_0^\alpha, \quad \rho^{\alpha\beta}(0,x) = \lambda(1 - \delta_{\alpha\beta})\delta(x); \tag{2.8}$$

see [11] for details.

[4] We will set $a = 1$ in the rest of this section, to simplify notation.
[5] Since this is the standard name of the corresponding Maxwell equation $\nabla \cdot \vec{E} = \rho$.
[6] Since $[D_0, \rho] = [D_0, [D_1, E]] = [[D_0, D_1], E] + [D_1, [D_0, E]] = -\mathrm{i}[E, E] = 0$, due to the Jacobi identity.
[7] This name is motivated by the Coulomb gauge $\nabla \cdot \vec{A} = 0$ in electrodynamics.

2.3 Gauge fixing

In our discussion above we used two different gauge conditions. It is important to convince oneself that these conditions are consistent. Consistency of a gauge condition means that, *for any solution $A_\mu(t,x)$ of our Yang-Mills equations (2.4) one can find a gauge transformation U such that the gauge transformed A_μ^U obeys this condition.* One then can impose this condition and will not loose anything.

Axial gauge. To show consistency of the axial gauge in (2.5) we need to find a gauge transformation U such that $A_0^U = 0$, or equivalently, $[\partial_t + \mathrm{i}A_0(t,x)]U(t,x) = 0$. This latter first order ODE has always a solution which we write symbolically as follows,

$$U(t,x) = \mathcal{P}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}s A_0(s,x)\right) \tag{2.9}$$

where $\mathcal{P}\exp$ is the path ordered exponential.[8]

Diagonal Coulomb gauge. This gauge is the closest one can get to the condition $A_1 = 0$ which is *not* consistent on a cylinder: Superficially space and time seem symmetric, and our discussion above therefore suggests that it should be possible to impose $A_1 = 0$. However, this is not possible in general since the solution

$$S(t,x) = \mathcal{P}\exp\left(-\mathrm{i}\int_{-\pi}^x \mathrm{d}y A_1(t,y)\right) \tag{2.10}$$

of $A_1^S = 0$ is not a gauge transformation: a gauge transformation must be periodic, but $S(t,\pi) \neq I = S(t,-\pi)$ in general. However, one can correct for this as follows,

$$U(t,x) = S(t,x)V(t)\mathrm{e}^{\mathrm{i}(x+\pi)Q(t)} \tag{2.11}$$

where V is an invertible matrix and Q a diagonal matrix independent of x. Indeed, $U(t,-\pi) = U(t,\pi)$ leads to the following equation

$$V(t)^{-1}S(t,\pi)V(t) = \mathrm{e}^{-2\pi\mathrm{i}Q(t)} \tag{2.12}$$

which always has a solution: $V(t)$ has to be chosen as the unitary matrix diagonalizing $S(t,x)$, and $\exp\big(-2\pi\mathrm{i}Q(t)\big)$ is the diagonal matrix of eigenvalues. It is easy to see that, with that gauge transformation U, $A_1^U = Q$, which implies that the diagonal Coulomb gauge in Eq. (2.7) is consistent.

This discussion implies a simple recipe to transform from the axial gauge to the diagonal Coulomb gauge: *compute $S(t,\pi)$, and the eigenvalues of this matrix are then equal to* $\exp\big(-2\pi\mathrm{i}q^\alpha(t)\big)$. This recipe provides a useful tool for finding explicit solutions for a large class of integrable systems generalizing the tCM model [11].

[8] The r.h.s. of Eq. (2.9) can be defined as limit $K \to \infty$ of

$$[I - \mathrm{i}(t/K)A_0(t_K,x)][I - \mathrm{i}(t/K)A_0(t_{K-1},x)]\cdots[I - \mathrm{i}(t/K)A_0(t_1,x)]$$

where $t_j = jt/K$ for $j = 1,2,\ldots,K$ and I is the $N\times N$ unit matrix.

2.4 Implications for the trigonometric CM system

The facts summarized above imply various interesting results for the tCM model [11]: Firstly, Eq. (2.4b) has the form of a Lax equation, and to get the Lax pair for the tCM model one only needs to compute $L \equiv E$ and $M \equiv \mathrm{i}A_0$ in the diagonal Coulomb gauge which is a straightforward computation. Secondly, Eq. (2.6) gives a linear time evolution which is gauge equivalent to the time evolution of the tCM system, and this yields a strategy for constructing the explicit solution of the tCM system: compute $E(0,x)$ by solving Gauss' law, and then transform the explicit solution in the axial gauge to the diagonal Coulomb gauge. As shown in Section 4.2 of Ref. [11], this reproduces the well-known explicit solution of the tCM model.

3 Generalization to the elliptic case

We now discuss the 2+1 dimensional gauge theory equivalent to the eCM system and various implications of this relations. We start with the heuristic argument which led us to this results.

3.1 Heuristic argument

The key to the result discussed in the previous section is the following well-known identity, $\sum_{n\in\mathbb{Z}}(n+q)^{-2} = \pi^2\sin^{-2}(\pi q)$, and it is essentially through this that the interaction potential of the tCM model appears. This sum results from the Fourier transform of the operator D_1 in the diagonal Coulomb gauge, which is a diagonal matrix with elements $\mathrm{i}(n+q^\alpha)\delta_{\alpha\beta}$ [11]. Since the Weierstrass elliptic function $\wp$ can be represented by a similar sum, $\wp(q) = \sum_\Omega(\Omega+q)^{-2} + const.$ with $\Omega = ma + \mathrm{i}nb$ and m, n integers, this suggested to us that one should be able to generalize this construction to the elliptic case if one replaces the Fourier modes n of the circle by $\Omega = ma + \mathrm{i}nb$, i.e., use a gauge theory where the circle is replaced by a torus. This idea led us to the construction below.

3.2 2+1 dimensional gauge theory equivalent to the eCM system

We consider functions $A_0 = A_0(t,x,y)$, $A = A(t,x,y)$ and $\rho = \rho(t,x,y)$ of the variables $t \in \mathbb{R}$ (=time) and $(x,y) \in [-\pi,\pi]^2$ (=space) and with values in the complex $N\times N$ matrices (which no longer need to be self-adjoint). We define covariant derivatives

$$D_0 \equiv \partial_t + \mathrm{i}A_0 \quad \text{and} \quad D \equiv a\partial_x + \mathrm{i}b\partial_y + \mathrm{i}A \tag{3.1}$$

and the function $E \equiv -\mathrm{i}[D_0, D]$, and we consider the following system of partial differential equations, $[D_0, E] = 0$ and $[D_1, E] = \rho$. These equations imply $[D_0, \rho] = 0$. More explicitly, this system of matrix valued PDEs can be written as follows,

$$\partial_t A = E + (a\partial_x + \mathrm{i}b\partial_y)A_0 + \mathrm{i}[A, A_0] \tag{3.2a}$$
$$\partial_t E + \mathrm{i}[A_0, E] = 0 \tag{3.2b}$$
$$(a\partial_x + \mathrm{i}b\partial_y)E + \mathrm{i}[A, E] = \rho \tag{3.2c}$$
$$\partial_t \rho + \mathrm{i}[A_0, \rho] = 0\,. \tag{3.2d}$$

It is interesting to note that these equations can be obtained from 2+1 dimensional gauge theory of Yang-Mills fields coupled to non-dynamical matter by imposing gauge invariant reduction conditions: The Yang-Mills theory is defined as in the last section but with spacetime indices $\mu, \nu = 0, 1, 2$, and the reduction conditions are $F_{01}/2a - F_{02}/2\mathrm{i}b = 0$, $F_{12} = 0$, and $J^1 = J^2 = 0$, where $A \equiv aA_1 + \mathrm{i}bA_2$ and $E \equiv F_{01}/2a + F_{02}/2\mathrm{i}b$. We stress the periodicity, i.e., invariance under $x \to x + 2\pi$ and $y \to y + 2\pi$, of all fields A_0, A and ρ, which again is crucial for making the gauge theory non-trivial.

It is obvious that Eqs. (3.2) are invariant under the following gauge transformations,

$$\begin{aligned} &A_0 \to A_0^U \equiv U^{-1}A_0U - \mathrm{i}U^{-1}\partial_t U, \quad A \to A^U \equiv U^{-1}AU - \mathrm{i}U^{-1}(a\partial_x + \mathrm{i}b\partial_y)U \\ &E \to E^U \equiv U^{-1}EU, \quad \rho \to \rho^U \equiv U^{-1}\rho U \end{aligned} \tag{3.3}$$

where $U = U(t, x, y)$ is an arbitrary differentiable, periodic function with values in the invertible $N \times N$-matrices. Again it is possible to impose the Weyl gauge $A_0 = 0$, and as in the previous section this allows to solve these equations explicitly as follows, $E(t, x, y) = E(0, x, y)$, $\rho(t, x, y) = \rho(0, x, y)$, and

$$A(t, x, y) = A(0, x, y) + E(0, x, y)t \tag{3.4}$$

for arbitrary initial conditions $E(0, x, y)$, $A(0, x, y)$ and $\rho(0, x, y)$ satisfying Gauss' law in Eq. (3.2c). Note that our solution satisfies Eq. (3.2c) for all t if it satisfies it for $t = 0$.

We now state the precise relation of this gauge theory to the eCM model. As before we denote the matrix elements of ρ as $\rho^{\alpha\beta}$ with $\alpha, \beta = 1, 2, \ldots, N$, and similarly for A, A_0 and E.

Proposition. *The gauge theory equations in (3.2) with the initial conditions*

$$\begin{aligned} &A^{\alpha\beta}(0, x, y) = \delta_{\alpha\beta}\, q_0^\alpha, \quad \int_{-\pi}^{\pi} \frac{\mathrm{d}x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}y}{2\pi}\, E^{\alpha\alpha}(0, x, y) = p_0^\alpha, \\ &\rho^{\alpha\beta}(t, x, y) = (2\pi)^2 \big(1 - \delta_{\alpha\beta}\big)\, \lambda\delta(x)\delta(y) \end{aligned} \tag{3.5}$$

are gauge equivalent to the equations (1.1)–(1.3) defining the eCM system. More specifically, it is then possible to impose the gauge conditions

$$A(t, x, y) = Q(t) = \mathrm{diag}\left(q^1(t), q^2(t), \ldots, q^N(t)\right) \tag{3.6a}$$

and

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}y}{2\pi}\, A_0^{\alpha\alpha}(t, x, y) = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \lambda\, \wp\big(q^\alpha(t) - q^\beta(t)\big) \tag{3.6b}$$

with the Weierstrass elliptic $\wp$-function defined as usual [22], and the equations (3.2) and (3.5) in this gauge imply Eqs. (1.1)–(1.3).

Proof. We make the following ansatz for the time evolution of the matter charges,

$$\rho^{\alpha\beta}(t, x, y) = (2\pi)^2 \lambda^{\alpha\beta}(t)\delta(x)\delta(y); \tag{3.7}$$

we will show below that this indeed is consistent. It is interesting to note that this has the physical interpretation of matter charges localized at the point $(x,y) = (0,0)$. We now write Eqs. (3.2) in the diagonal Coulomb gauge (3.6a) using Fourier transformation,

$$\hat{E}^{\alpha\beta}(m,n) = \int_{-\pi}^{\pi} \mathrm{d}x \int_{-\pi}^{\pi} \mathrm{d}y \, \mathrm{e}^{-imx-iny} E^{\alpha\beta}(x,y), \quad m,n \in \mathbb{Z}, \tag{3.8}$$

and similarly for A_0 and ρ, suppressing the t-dependence. Eq. (3.2a) then reads

$$\delta^{\alpha\beta}_{mn}(2\pi)^2 \partial_t q^\alpha = \hat{E}^{\alpha\beta}(m,n) + \mathrm{i}\left(am + \mathrm{i}bn + [q^\alpha - q^\beta]\right) \hat{A}_0^{\alpha\beta}(m,n) \tag{3.9}$$

where we use the shorthand notation $\delta^{\alpha\beta}_{mn} \equiv \delta_{\alpha\beta}\delta_{m,0}\delta_{n,0}$. Taking $\alpha = \beta$ and $m = n = 0$ we obtain

$$\partial_t q^\alpha = \frac{\hat{E}^{\alpha\alpha}(0,0)}{(2\pi)^2} \equiv p^\alpha \tag{3.10}$$

where the last equality introduces a suggestive notation. The time evolution of the p^α follows from Eq. (3.2b),

$$\partial_t p^\alpha = -\frac{\mathrm{i}}{(2\pi)^4} \sum_{m,n,\beta\neq\alpha} \left(\hat{A}_0^{\alpha\beta}(m,n)\hat{E}^{\beta\alpha}(-m,-n) - \hat{E}^{\alpha\beta}(m,n)\hat{A}_0^{\beta\alpha}(-m,-n) \right).$$

The r.h.s. of this equation can be evaluated by taking the Fourier transform of Eq. (3.2c),

$$\mathrm{i}\left(am + \mathrm{i}bn + [q^\alpha - q^\beta]\right) \hat{E}^{\alpha\beta}(m,n) = (2\pi)^2 \lambda^{\alpha\beta}$$

where we used Eq. (3.7), and this yields

$$\hat{E}^{\alpha\beta}(m,n) = \delta^{\alpha\beta}_{mn}(2\pi)^2\, p^\alpha + \frac{(1-\delta^{\alpha\beta}_{mn})(2\pi)^2\lambda^{\alpha\beta}}{\mathrm{i}\,(am + \mathrm{i}bn + [q^\alpha - q^\beta])}. \tag{3.11}$$

Similarly Eq. (3.9) determines all components of $\hat{A}_0$ except for the zero modes $\hat{A}_0^{\alpha\alpha}(0,0) \equiv (2\pi)^2\, h^\alpha$ which are arbitrary at this point,

$$\hat{A}_0^{\alpha\beta}(m,n) = \delta^{\alpha\beta}_{mn}(2\pi)^2\, h^\alpha + \frac{(1-\delta^{\alpha\beta}_{mn})(2\pi)^2\lambda^{\alpha\beta}}{(am + \mathrm{i}bn + [q^\alpha - q^\beta])^2}. \tag{3.12}$$

Inserting this we get

$$\partial_t p^\alpha = 2 \sum_{m,n,\beta\neq\alpha} \frac{\lambda^{\alpha\beta}\lambda^{\beta\alpha}}{(am + \mathrm{i}bn + [q^\alpha - q^\beta])^3} = -\sum_{\beta\neq\alpha} \lambda^{\alpha\beta}\lambda^{\beta\alpha}\wp'(q^\alpha - q^\beta). \tag{3.13}$$

We are left to determine the time evolution of the $\lambda^{\alpha\beta}$. From Eqs. (3.2d) and (3.7) we get

$$\partial_t \lambda^{\alpha\beta} = -\mathrm{i}\sum_{\gamma=1}^{N} \left(M^{\alpha\gamma}\lambda^{\gamma\beta} - \lambda^{\alpha\gamma}M^{\gamma\beta} \right)$$

where $M^{\alpha\beta} \equiv A_0^{\alpha\beta}(0,0) = (2\pi)^{-2} \sum_{m,n} \hat{A}_0^{\alpha\beta}(m,n)$. Using Eqs. (3.11) and (3.7) yields, for $\alpha \neq \beta$, $M^{\alpha\beta} = \wp_0(q^\alpha - q^\beta)\lambda^{\alpha\beta}$ where $\wp_0(q) \equiv \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}}(ma + \mathrm{i}nb + q)^{-2}\right)$ is equal to the Weierstrass function $\wp(q)$ up to an additive constant. Using $M^{\alpha\alpha} - M^{\beta\beta} = h^\alpha - h^\beta$ we obtain

$$\partial_t \lambda^{\alpha\beta} = -\mathrm{i}\left(h^\alpha - h^\beta\right)\lambda^{\alpha\beta} - \mathrm{i}\sum_{\gamma\neq\alpha,\beta} \lambda^{\alpha\gamma}\lambda^{\gamma\beta}\left(\wp(q^\alpha - q^\gamma) - \wp(q^\gamma - q^\beta)\right) \tag{3.14}$$

where we could replace the function $\wp_0$ by $\wp$. Thus our gauge theory in the diagonal Coulomb gauge is equivalent to the Eqs. (3.10), (3.13) and (3.14). This corresponds to a dynamical system of particles coupled to a spin degrees of freedom $\lambda^{\alpha\beta}$. The functions h^α in Eq. (3.14) can be regarded as external time-dependent fields which can be chosen arbitrarily. In particular, we can choose them such that the spin degrees of freedom are time independent: for $h^\alpha = \sum_{\beta\neq\alpha} \lambda\, \wp(q^\alpha - q^\beta)$, which is equivalent to (3.6b), one finds that the unique solution of Eq. (3.14) with the initial condition in Eq. (3.5) is $\lambda^{\alpha\beta}(t) = (1 - \delta_{\alpha\beta})\,\lambda$, independent of t. With that, Eqs. (3.13) and (3.10) imply Eq. (1.1). It is straightforward to check that the other components of Eq. (3.2b) are then obeyed.

Since the equations (1.1)–(1.3) have a unique solution $q^\alpha(t)$ we also have shown that the diagonal Coulomb gauge in (3.6a) is indeed consistent. This completes our proof. ∎

It interesting to note that our construction above actually gives the well-known spin-generalization of the eCM system [28, 29].

3.3 Lax pairs

We now show that the construction above yields, in a simple manner, a family of Lax pairs which is essentially Krichever's [20].

Eq. (3.2b) is of the Lax form: E and A_0 define a family of Lax pairs (depending on x and y) for the elliptic CM system. They can be computed by taking the inverse Fourier transform of $\hat{E}$ in (3.11) and $\hat{A}_0$ in (3.12) and (3.6b). We obtain

$$E^{\alpha\beta}(t,x,y) = \delta_{\alpha\beta}\, p^\alpha(t) - (1 - \delta_{\alpha\beta})\,\mathrm{i}\lambda\, G(x,y;q^\alpha(t) - q^\beta(t)) \tag{3.15}$$

and

$$A_0^{\alpha\beta}(t,x,y) = \delta_{\alpha\beta}\sum_{\gamma\neq\alpha} \lambda\,\wp(q^\alpha(t) - q^\gamma(t)) + \left(1 - \delta_{\alpha\beta}\right)\mathrm{i}\lambda\, G'(x,y;q^\alpha(t) - q^\beta(t)) \tag{3.16}$$

where $G'(x,y;q) \equiv \partial_q G(x,y;q)$, with

$$G(x,y;q) \equiv \sum_{m,n\in\mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}mx + \mathrm{i}ny}}{(am + \mathrm{i}bn + q)} \tag{3.17}$$

the periodic solution of $(a\partial_x + \mathrm{i}b\partial_y + \mathrm{i}q)G(x,y;q) = \mathrm{i}(2\pi)^2\delta(x)\delta(y)$. To compute this function G it is convenient to change from x and y to complex variables,

$$z = \frac{ay - \mathrm{i}bx}{2\pi}, \quad \bar{z} = \frac{ay + \mathrm{i}bx}{2\pi}. \tag{3.18}$$

Note that z lives on the same spectral curve as q, i.e., $y \to y + 2\pi$ and $x \to x - 2\pi$ correspond to $z \to z + a$ and $z \to z + \mathrm{i}b$, respectively. Noting that

$$a\partial_x + \mathrm{i}b\partial_y = \frac{\mathrm{i}}{c}\partial_{\bar{z}}, \quad c = \frac{\pi}{ab}, \tag{3.19}$$

one can compute G and obtain (we will give some details of this computation below)

$$G(x, y; q) = \mathrm{e}^{-c\bar{z}q}\, \phi(z; q) \tag{3.20}$$

where

$$\phi(z; q) = \mathrm{e}^{\eta z q}\, \frac{\sigma(z + q)}{\sigma(z)\, \sigma(q)}, \quad \eta = \frac{2\eta_1}{a} - c, \tag{3.21}$$

and the constant $2\eta_1 = \xi(z+a) - \xi(z)$, with σ and ξ the usual Weierstrass elliptic function [22].

It is easy to see that, up to a trivial transformation, this family of Lax pairs $(E, \mathrm{i}A_0) \equiv (L, M)$ coincides with Krichever's [20].

In the rest of this subsection we sketch the computation of G in (3.17). This function obviously satisfies $\mathrm{i}(c\partial_{\bar{z}} + \mathrm{i}q)\, G = 0$ for all $z \notin a\mathbb{Z} + \mathrm{i}b\mathbb{Z}$, which implies Eq. (3.20) for some function ϕ depending only on z. The definition of G implies the following quasi-periodicity properties of the function ϕ,

$$\phi(z + \Omega; q) = \mathrm{e}^{c\bar{\Omega}q}\, \phi(z; q) = \phi(z; q + \Omega) \tag{3.22}$$

where

$$\Omega = ma + \mathrm{i}nb, \quad \bar{\Omega} = ma - \mathrm{i}nb, \quad m, n \in \mathbb{Z}, \tag{3.23}$$

are the periods for z and $\bar{z}$. Moreover, from Eq. (3.17) it is obvious that, for fixed $z \notin a\mathbb{Z} + \mathrm{i}b\mathbb{Z}$, $\phi(z; q)$ is meromorphic function of $q \in \mathbb{C}$ with first order poles in $q \in a\mathbb{Z} + \mathrm{i}b\mathbb{Z}$ and $\lim_{q\to 0} q\phi(z; q) = 1$. The same is true with q and z interchanged, as follows from (3.22) and the fact that, close to $(x, y) = (0, 0)$, G behaves like the solution of $(a\partial_x + \mathrm{i}b\partial_y)G = \mathrm{i}(2\pi)^2\delta(x)\delta(y)$ (with periodicity ignored), i.e., $G \sim 2\pi/(by - \mathrm{i}ax) = 1/z$. Using the well-known properties of the Weierstrass function σ [22] one then shows that the function

$$f(z; q) \equiv \mathrm{e}^{-\eta z q} \frac{\sigma(z)\sigma(q)}{\sigma(z + q)} \phi(z, q)$$

is holomorphic and periodic in both arguments q and z, which implies $f(z; q) = 1$ and yields Eq. (3.21).

3.4 Towards an explicit solution of the eCM system

The results above give the following characterization of the solution of the eCM system.

Corollary: *The solution $q^\alpha(t)$ of the eCM system is given by the eigenvalues of the following 'chiral Dirac operator' depending linearly on time t,*

$$D(t) = a\partial_x + \mathrm{i}b\partial_y + \mathrm{i}A(t, x, y), \quad A(t, x, y) = Q_0 + E(0, x, y)t \tag{3.24}$$

with $Q_0 = \mathrm{diag}(q_0^1, \dots, q_0^N)$ *and* $E(0,x,y)$ *in Eqs. (3.15) and (3.20)–(3.21) a known matrix valued function on the torus determined by the initial conditions* $q^\alpha(0) = q_0^\alpha$ *and* $p^\alpha(0) = p_0^\alpha$*: There is some invertible-matrix valued functions* $U(t,x,y)$ *on the torus such that* $U(t,x,y)^{-1}D(t)U(t,x,y) = a\partial_x + \mathrm{i}b\partial_y + Q(t)$ *where* $Q(t) = \mathrm{diag}(q^1(t), \dots, q^N(t))$.

This provides a linearization of the time evolution of the eCM model. However, it would be desirable to make the solution $q^\alpha(t)$ more explicit. In the following we sketch two possible strategies in this direction.

Strategy 1: *Compute the determinant of* $D(t) - \mu I$. The characterization above implies that $D(t)$ is a normal (diagonalizable) operator on the Hilbert space $L^2([-\pi,\pi]^2) \otimes \mathbb{C}^N$ with eigenvalues $\Omega + q^\alpha(t)$ where $\Omega = am + \mathrm{i}bn$, $m,n \in \mathbb{Z}$. One thus could get the hands on the solution $q^\alpha(t)$ by computing the determinant of $D(t) - \mu I$: Formally,

$$\Sigma(\mu,t) \equiv \overset{reg}{\det}(D(t) - \mu I) = \prod_{\alpha=1}^{N} \overset{reg}{\prod_{\Omega}} \big(\Omega + q^\alpha(t) - \mu\big) = \prod_{\alpha=1}^{N} \sigma(q^\alpha(t) - \mu) \tag{3.25}$$

where '*reg*' indicates some natural (gauge invariant) regularization generalizing the one used to define the Weierstrass elliptic function σ [22]. The explicit solution of the eCM system is given by the zeros of this function $\Sigma(\mu,t)$. The problem is to find a simple, explicit formula for Σ. It is interesting to note that $\Sigma(\mu,t)$ is identical with the object which Krichever obtains in his solution [20], but his characterization of this function seems less explicit than ours.

To compute this functions Σ the following explicit formula of $D(t)$ in Fourier space could be useful,

$$\hat{D}^{\alpha\beta}_{\Omega\Omega'}(t) = \delta_{\alpha\beta}\delta_{\Omega\Omega'}\left(\Omega + q^\alpha + p^\alpha t\right) - \frac{\mathrm{i}\lambda t\big(1 - \delta_{\alpha\beta}\delta_{\Omega\Omega'}\big)}{\Omega + q_0^\alpha - (\Omega' + q_0^\beta)}. \tag{3.26}$$

We have recently started to compute $\log\Sigma(\mu,t)$ as a formal power series in λ using this latter formula, and our preliminary results seem promising.

It is interesting to note the following interpretation of Eq. (3.26): comparing with Eqs. (1.5), our solution of the elliptic CM model can be interpreted as the solution of a rational CM system with and infinite number of particles $q^{\mathcal{A}} = \Omega + q^\alpha$ where $\mathcal{A} = (\alpha, \Omega)$. This interpretation is also natural if one compares the time evolution equations for the elliptic and rational CM models (this is well-known; see e.g. [4, 30]), and this provides therefore a simple way to understand our result stated in the Corollary above.

Strategy 2. *Compute a gauge transformation diagonalizing* $D(t)$. This amounts to trying to generalize the explicit construction of the gauge transformation transforming to the diagonal Coulomb gauge in Section 2.3.

Introducing the matrix

$$B^{\alpha\beta}(z) \equiv \delta_{\alpha\beta}\, p_0^\alpha - \mathrm{i}\big(1 - \delta_{\alpha\beta}\big)\,\lambda\phi(z; q_0^\alpha - q_0^\beta) \tag{3.27}$$

and using (3.20), Eq. (3.15) implies $E(0,x,y) = \exp(-c\bar{z}Q_0)\, B(z) \exp(c\bar{z}Q_0)$ and the following explicit solution of our gauge theory in the Weyl gauge,

$$A(t,x,y) = \mathrm{e}^{-c\bar{z}Q_0}\big(Q_0 + B(z)t\big)\mathrm{e}^{c\bar{z}Q_0}. \tag{3.28}$$

Note also that the periodicity of A implies

$$B(z+\Omega) = \mathrm{e}^{c\bar{\Omega}Q_0} B(z) \mathrm{e}^{-c\bar{\Omega}Q_0} \tag{3.29}$$

with Ω and $\bar{\Omega}$ defined in Eq. (3.23).

We now try to construct the gauge transformations $U(t,x,y)$ transforming from the Weyl- to the diagonal Coulomb gauge. It has to obey the equation

$$\big(a\partial_x + \mathrm{i}b\partial_y + \mathrm{i}A(t,x,y)\big)\, U(t,x,y) = U(t,x,y)\mathrm{i}Q(t) \tag{3.30}$$

where $Q(t)$ is a diagonal matrix independent of x and y which we need to determine. A straightforward computation yields

$$U(t,x,y) = \mathrm{e}^{-c\bar{z}Q_0} \mathrm{e}^{-c\bar{z}B(z)t} V(t,z) \mathrm{e}^{c\bar{z}Q(t)} \tag{3.31}$$

where $V(t,z)$ is an arbitrary function with values in the invertible $N\times N$ matrices. We now use that *U has to be periodic in both arguments x and y with period 2π.* Requiring $U(t,x-2\pi n, y+2\pi m) = U(t,x,y)$ leads to the following condition,

$$\mathrm{e}^{c\bar{\Omega}Q(t)} = V(t,z+\Omega)^{-1} \mathrm{e}^{c\bar{\Omega}Q_0} \mathrm{e}^{c\bar{\Omega}B(z)t} V(t,z). \tag{3.32}$$

This seems close to an explicit solution of the eCM system. We expect that $V(t,z)$ is determined by the condition in (3.32), but we do not know how to compute is. In fact, one does not really need $V(t,z)$ to determine $Q(t)$, all one needs is its periodicity properties which might be easier to compute. The **final missing step** in an explicit solution of the eCM system can thus be described as follows: *Determine $V(t,z)V(t,z+\Omega)^{-1}$ where $V(t,z)$ is some matrix valued functions which is holomorphic (except perhaps in $z=0$) and such that the eigenvalues $s^\alpha(t)$ of the matrix on the r.h.s. in Eq. (3.32) are independent of z.* The solution of this monodromy problem would imply a simple recipe to compute the $s^\alpha(t)$, and the explicit solution of the eCM model would then be given by $q^\alpha(t) = (1/c\bar{\Omega})\log s^\alpha(t)$.

It is tempting to conjecture that there exists some $z_0 \in \mathbb{C}$ and $\Omega \neq 0$ such that $V(t,z_0+\Omega) = W_\Omega(t)^{-1} V(t,z_0) W_\Omega(t)$ for some invertible $N\times N$ matrix $W_\Omega(t)$, e.g. $z_0 = -\Omega/2$. If that was true the above mentioned eigenvalues $s^\alpha(t)$ would be equal to the eigenvalues of the matrix $\exp(c\bar{\Omega}Q_0/2)\exp(c\bar{\Omega}B(z_0)t)\exp(c\bar{\Omega}Q_0/2)$. Even though this is true in the trigonometric limit and reproduces the well-known solution of the tCM model [23, 24] (see [11] for details), we checked that it is not true in the general elliptic case. However, we could derive Eq. (3.31) only due to several “miracles” which occur on the way, and we therefore feel that there should exist one more “miracle” which should allow to solve the monodromy problem stated above.

4 Generalization to integrable elliptic spin-particle systems

Recently it was shown that $\mathcal{N}=4$ supersymmetric Yang-Mills theory in 4 dimensions has an interesting relation to integrable spin systems [31], and this relation suggests that elliptic deformations of the Heisenberg spin chain [26] are of interest in this context; see [27] and references therein. It thus seems of interest to note that there exists a large class of integrable spin-particle systems of elliptic type which can be obtained by a straightforward generalization of our construction in Section 3. These models correspond to the elliptic deformation of the trigonometric systems derived and solved in [10, 11]. Since the derivation is a straightforward combination of the arguments in [10, 11] and Section 3, we only give the result: generalizing the ansatz in Eq. (3.7) to matter charges localized at an arbitrary number K of distinct points,

$$\rho^{\alpha\beta}(t,x,y) = (2\pi)^2 \sum_{k=1}^{K} \lambda_k^{\alpha\beta}(t)\delta(x-x_k)\delta(y-y_k) \tag{4.1}$$

where the $(x_k, y_k) \in [-\pi,\pi]^2$ are arbitrary, one obtains an integrable spin-particle systems which can be defined by the Hamiltonian[9]

$$H_K = \frac{1}{2}\sum_{\alpha=1}^{N}(p^\alpha)^2 + \frac{1}{2}\sum_{\substack{\beta=1\\ \alpha\neq\beta}}^{N}\sum_{j,k=1}^{K} v_{jk}(q^\alpha - q^\beta)\lambda_j^{\alpha\beta}\lambda_k^{\beta\alpha} + \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{j,k=1}^{N} c_{jk}\lambda_j^{\alpha\alpha}\lambda_k^{\alpha\alpha} \tag{4.2}$$

with

$$v_{jk}(q) = \sum_{m,n\in\mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}m(x_j-x_k)+\mathrm{i}n(y_j-y_k)}}{(am+\mathrm{i}bn+q)^2}, \quad c_{jk} = \sum_{\substack{m,n\in\mathbb{Z}\\ m^2+n^2\neq 0}} \frac{\mathrm{e}^{\mathrm{i}m(x_j-x_k)+\mathrm{i}n(y_j-y_k)}}{(am+\mathrm{i}bn)^2}, \tag{4.3}$$

and the following Poisson brackets,

$$\{p^\alpha, q^\alpha\} = \delta_{\alpha,\beta}, \quad \left\{\lambda_j^{\alpha\beta}, \lambda_k^{\alpha'\beta'}\right\} = \delta_{jk}\left(\delta_{\beta\alpha'}\lambda_j^{\alpha\beta'} - \delta_{\beta'\alpha}\lambda_j^{\alpha'\beta}\right); \tag{4.4}$$

all other Poisson brackets are trivial. There is also the constraint

$$\sum_{k=1}^{K}\lambda_k^{\alpha\alpha} = 0 \tag{4.5}$$

which is a remnant from Gauss' law.

We now give a heuristic argument that the model above can be reduced to spin models which provide a large generalization of Inozemtsev's [26]. We first discuss the case $K=1$ where the Hamiltonian above reduces to $H_1 = \frac{1}{2}\sum_\alpha (p^\alpha)^2 + \frac{1}{2}\sum_{\alpha\neq\beta}\lambda^{\alpha\beta}\lambda^{\beta\alpha}\wp_0(q^\alpha - q^\beta) + \frac{1}{2}\sum_\alpha c\lambda^{\alpha\alpha}\lambda^{\alpha\alpha}$ where $\wp_0(z) = \wp(z) + c$. We now impose the following reduction procedure: Assume $N = JL$ with integers $J, L > 0$ and write α as a pair of indices, $\alpha = (A,\ell)$ with $A = 1,2\ldots,J$ and $\ell = 1,2\ldots,L$, and similarly $\beta = (B,m)$. We now impose the conditions

$$\lambda^{(A,\ell)(B,m)} = (1-\delta_{\ell,m})\lambda_\ell^{AB}, \quad q^{(A,\ell)} = \ell a/L,$$

assuming that the particles degrees of freedom can be frozen to equidistant equilibrium positions (we assume here the existence of a classical counterpart of Polychronakos' freezing trick [32], waving our hands). This yields

$$H_1' = \sum_{A,B}\sum_{\ell\neq m}\wp_0([\ell-m]a/L)\lambda_\ell^{AB}\lambda_m^{BA} \tag{4.6}$$

[9] The following can be generalized by adding a term involving the diagonal zero model h^α of A_0, but we set this term to zero for simplicity.

where we dropped the kinetic energy terms for the particles. This is essentially the classical version of the Inozemtsev Hamiltonian. In a similar manner we formally obtain, for general K,

$$H'_K = \sum_{A,B} \sum_{j\neq k} v_{jk}([\ell - m]a/L)\lambda_{j,\ell}^{AB}\lambda_{k,m}^{BA} \tag{4.7}$$

which can be interpreted as an anisotropic 2D spin model. While the (x_j, y_j) are arbitrary in the spin-particle system above, we expect that one has to choose them equidistant, e.g. as $(0, 2\pi j/K)$, for the freezing trick to work. This choice yields a translation invariant model.

We finally sketch that our approach can also be used to obtain integrable spin-particle Hamiltonians involving higher powers of the spin interaction. The gauge theory Hamiltonian for our model is $\mathcal{H} = \frac{1}{2(2\pi)^2}\int \mathrm{d}^2x \, \mathrm{tr}\, E^2$, i.e.,

$$\mathcal{H} = \frac{1}{2(2\pi)^4} \sum_{m,n\in\mathbb{Z}} \sum_{\alpha,\beta=1}^{N} \hat{E}^{\alpha\beta}(m,n)\hat{E}^{\beta\alpha}(-m,-n), \tag{4.8}$$

and by inserting the solution of Gauss' law in the diagonal Coulomb gauge,

$$\hat{E}^{\alpha\beta}(m,n) = \delta_{mn}^{\alpha\beta}(2\pi)^2\, p^\alpha + \frac{(1-\delta_{mn}^{\alpha\beta})(2\pi)^2\hat{\lambda}^{\alpha\beta}(m,n)}{\mathrm{i}\,(am + \mathrm{i}bn + [q^\alpha - q^\beta])} \tag{4.9}$$

we obtain the Hamiltonian in Eq. (4.2) (see [11] for a detailed discussion of this method). Higher order integrable Hamiltonians can be obtained, in a similar manner, from the gauge theory Hamiltonian

$$\mathcal{H} = \int_{[-\pi,\pi]^2} \mathrm{d}^2x \sum_{k=2}^{\infty} a_k \, \mathrm{tr}\, E^k \tag{4.10}$$

with arbitrary real constants a_k. These correspond, of course, to the family of commuting "integrale of motion" for the elliptic spin-particles systems above. We expect that there should exist integrable reductions of these to spin systems.

The models discussed here are all classical, but it is plausible that they have integrable quantum counter parts. The discussion in [27] suggests that it would be interesting to study and solve the quantum versions of some of the models above. As far as we see, the method described in Section 3 allows to obtain information only about the *classical* models. However, as already mentioned, we have worked for a quite a while on another approach to the quantum versions of these models, and this has led to an explicit solution of the quantum eCM system recently [6]. We have wondered since long whether that approach can be related to the one discussed in the present paper. It seems that this question has become more interesting now.

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Epilogue. *As mentioned in the Prologue, this work on Calogero-Moser systems started a long time ago in a project on quantum chromodynamics in 1+1 dimensions which, at first sight, seems totally unrelated. I will now describe this story in more detail.*

As discovered by Schwinger [33], quantum electrodynamics with massless fermions in 1+1 dimensions can be solved exactly, and this so-called Schwinger model became famous since is allows to study several interesting aspects of quantum field theories in a precise mathematical framework (this is discussed beautifully in a paper by Manton [34]). In 1991 I became postdoc at UBC in Vancouver, and I started to collaborate with Gordon Semenoff on a project where we studied the non-Abelian analog of the Schwinger model: massless fermions coupled to a non-Abelian Yang-Mills field in (1+1)D. We were mainly interested in the short-distance aspects of this model and thus worked with a compact space equal to a circle, and we used a Hamiltonian framework allowing us to make precise mathematical sense of the model by well-defined operators on a certain Fock space [9]. At that time I believed that this model should be exactly solvable like the Schwinger model, and I invested quite some effort into trying to find a method of solution. After several assaults that led to nowhere I gave up and concluded that trying to solve a model was risky business since one can easily waste a year without getting anywhere. I thus decided to never again try to work on integrable systems. I was not aware that, at that very moment, I was already breaking this intention: one part of this project was on the problem of gauge fixing [7, 8], and the solution of this very problem implies, as a special case, the equivalence of the trigonometric CM system to a particular (1+1)D Yang-Mills theories [25].[10] *This result became well-known in the physics literature through a paper by Gorsky and Nekrasov [15] which appeared shortly after that. I should also mention closely related work by Hitchin [14] and Polychronakos and Minahan [12, 13] about which I learned later; see also [35, 36].*

After that my interests changed, and only a couple of years later I returned to this problem in a collaboration with Jonas Blom who was PhD student in our group. Our main motivation was to use this relation of CM systems to gauge theories to make integrability manifest and obtain explicit solutions. We were able to demonstrate the usefulness of this point of view by deriving and solving novel spin generalizations of the CM models [10, 11]. I also realized that this construction can be generalized naturally to the elliptic case, but shortly thereafter I found a paper by Gorsky and Nekrasov on the arXiv explaining how to generalize their approach to the elliptic case [21].[11] *Since my generalization of the trigonometric case seemed to me, at that time, rather obvious once the idea pointed out in [21] was realized, I wanted to produce more explicit results before publishing my notes. Somehow I never got back to working on this project again, and thus my notes remained unpublished.*[12] *In the beginning of this year Stefan Rauch invited me to nice meetings in Sardinia this June, and I was glad about this opportunity to present my results from 1999*

[10]I realized this relation in a discussion with Gorsky who visited UBC short after our paper [7] had appeared. I explained our results to him, and after a short discussion it was clear to both of us that a special case of Eq. (18) in [7] was (essentially) equivalent to the trigonometric CM system. I guess that this relation must have been known to him before since, to my knowledge, he never has acknowledged our discussion or our work. Unfortunately I did not write a paper on this at that time but only pointed out this relation in a short paragraph in a paper which was mainly on gauge fixing in Yang-Mills theory in higher dimensions; see Eq. (33) *ff* in [25].

[11]Curiously this paper never appeared in a journal.

[12]In July 2002 Harry Braden visited me and I showed him my notes. We were planning to collaborate and try to produce a fully explicit solution from my results, but this plan has remained a plan to this day.

to a few expert colleges including Francesco Calogero. Their positive feedback prompted me to write this paper.

References

[1] Calogero F, Solution of the one-dimensional N body problems with quadratic and/or inversely quadratic pair potentials, *J. Math. Phys.* **12** (1971) 419.

[2] Calogero F, One-dimensional many body problems with pair interactions whose exact ground state wave functions is of product type, *Lett. Nuovo Cim.* **13** (1975) 507.

[3] Calogero F, Exactly solvable one-dimensional many body problems, *Lett. Nuovo Cim.* **13** (1975) 411.

[4] F. Calogero, Motion of poles and zeros of special solutions of nonlinear and linear partial differential equations and related 'solvable' many body problems, *Nuovo Cim. B* **43**, 177 (1978).

[5] Carey A L and Langmann E, Loop groups, anyons and the Calogero-Sutherland model, *Commun. Math. Phys.* **201** (1999) 1 [arXiv:math-ph/9805010].

[6] Langmann E, An explicit solution of the (quantum) elliptic Calogero-Sutherland model, arXiv:math-ph/0407050.

[7] Langmann E and Semenoff G W, Gauge theories on a cylinder, *Phys. Lett. B* **296** (1992) 117 [arXiv:hep-th/9210011].

[8] Langmann E and Semenoff G W, Gribov ambiguity and nontrivial vacuum structure of gauge theories on a cylinder, *Phys. Lett. B* **303** (1993) 303 [arXiv:hep-th/9212038].

[9] Langmann E and Semenoff G W, Massless QCD in (1+1)-dimensions and gauge covariant Sugawara construction, *Phys. Lett. B* **341** (1994) 195 [arXiv:hep-th/9404159].

[10] Blom J and Langmann E, Novel integrable spin-particle models from gauge theories on a cylinder, *Phys. Lett. B* **429** (1998) 336.

[11] Blom J and Langmann E, Finding and solving Calogero-Moser type systems using Yang-Mills gauge theories, *Nucl. Phys. B* **563** (1999) 506 [arXiv:math-ph/9909019].

[12] Polychronakos A P, Integrable systems from gauged matrix models, *Phys. Lett. B* **266** (1991) 29.

[13] Minahan J A and Polychronakos A P, Equivalence of two-dimensional QCD and the c = 1 matrix model, *Phys. Lett. B* **312** (1993) 155 [arXiv:hep-th/9303153].

[14] Hitchin N, Stable bundles and integrable systems, *Duke Math. J.* **54** (1987) 91.

[15] Gorsky A and Nekrasov N, Hamiltonian systems of Calogero type and two-dimensional Yang-Mills theory, *Nucl. Phys. B* **414** (1994) 213 [arXiv:hep-th/9304047].

[16] Gavrilov L and Perelomov A, On the explicit solutions of the elliptic Calogero system, *J. Math. Phys.* **40**, 6339 (1999) [solv-int/9905011].

[17] Krichever I, Babelon O, Billey E and Talon M, Spin generalization of the Calogero-Moser system and the matrix KP equation, arXiv:hep-th/9411160.

[18] Nekrasov N, Holomorphic bundles and many body systems, *Commun. Math. Phys.* **180**, 587 (1996) [arXiv:hep-th/9503157].

[19] Enriquez B and Rubtsov V, Hitchin systems, higher Gaudin operators and r-matrices, alg-geom/9503010.

[20] Krichever M, Elliptic solutions of the Kadomtsev-Peviashvili equation and integrable systems of particles, *Funct. Anal. Appl.* **14**, 282 (1980).

[21] Gorsky A and Nekrasov N, Elliptic Calogero-Moser system from two-dimensional current algebra, arXiv:hep-th/9401021.

[22] Whitaker E T and Watson G N: *Course of modern analysis, 4th edition.* Cambridge Univ. Press (1958).

[23] Olshanetsky M A and Perelomov A M, Classical Integrable Finite Dimensional Systems Related To Lie Algebras, *Phys. Rep.* **71**, 313 (1981).

[24] Perelomov A M: *Integrable Systems of Classical Mechanics and Lie Algebras.* Vol. I, Birkhäuser (1990).

[25] Langmann E, Salmhofer M and Kovner A, Consistent axial - like gauge fixing on hypertori, *Mod. Phys. Lett. A* **9** (1994) 2913 [arXiv:hep-th/9308115].

[26] Inozemtsev V I, Integrable Heisenberg-van Vleck chains with variable range exchange, *Phys. Part. Nucl.* **34**, 166 (2003); *Fiz. Elem. Chast. Atom. Yadra* **34**, 332 (2003) [hep-th/0201001].

[27] Serban D and Staudacher M, Planar N = 4 gauge theory and the Inozemtsev long range spin chain, JHEP **0406**, 001 (2004) [hep-th/0401057].

[28] Gibbons J and Hermsen T, A generalisation of the Calogero-Moser system, *Physica* **11D**, 337 (1984).

[29] Wojciechowski S, An integrable marriage of the Euler equations with the Calogero-Moser system, *Phys. Lett.* **A 111**, 101 (1985).

[30] Polychronakos A P, Generalized Calogero models through reductions by discrete symmetries, *Nucl. Phys. B* **543**, 485 (1999) [arXiv:hep-th/9810211].

[31] Minahan J A and Zarembo K, The Bethe-ansatz for N = 4 super Yang-Mills, JHEP **0303**, 013(2003) [arXiv:hep-th/0212208].

[32] Polychronakos A P, Lattice integrable systems of Haldane-Shastry type, *Phys. Rev. Lett.* **70** (1993) 2329 [arXiv:hep-th/9210109].

[33] Schwinger J S, Gauge invariance and mass. II, *Phys. Rev.* **128** (1962) 2425.

[34] Manton N S, The Schwinger model and its axial anomaly, *Annals Phys.* **159** (1985) 220.

[35] Panzeri S: The c=1 matrix model formulation of two-dimensional Yang-Mills theories, *Mod. Phys. Lett. A* **8**, 3201 (1993) [arXiv:hep-th/9307173].

[36] Caselle M, D'Adda A, Magnea L. and Panzeri S., Two-dimensional QCD on the sphere and on the cylinder, arXiv:hep-th/9309107.