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Universality of Calogero-Moser Model

M A OLSHANETSKY

117333, ITEP, Moscow, Russia
RIMS, Kyoto Univ., Kyoto 606-8502, Japan
E-mail: olshanet@itep.ru

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Abstract

In this review we explain interrelations between the Elliptic Calogero-Moser model, the integrable Elliptic Euler-Arnold top, monodromy preserving equations and the Knizhnik-Zamolodchikov-Bernard equation on a torus.

1 Introduction

The Calogero Model was proposed first by Francesco Calogero as a model of exactly solvable one-dimensional nuclei [5, 6]. Later different generalizations of the model on the classical and quantum level were introduced in Refs. [24, 28, 36] (see also reviews [29, 30]). Nowadays these models, that we will call for brevity the Calogero models (CM), play a fundamental role in the contemporary theoretical physics. We shortly remind some of them. The first indication of this role came from the papers [1, 18] where interrelations between classical solutions of the rational and elliptic CM and special solutions of the KdV and KP hierarchies were established. Last fifteen years a wide range of applications was discovered. Among them are interrelations between the Calogero-Sutherland model [36] and the Fractional Quantum Hall Effect [2], integrable one-dimensional spin models with long-range interactions [16]. Important role plays the classical CM in the SUSY Yang-Mills theory [11] and in the string theory [37].

Most likely, the fundamental character of CM can be explained by their group-theoretical and geometrical nature. In the very beginning of seventies during Francesco Calogero visit to ITEP Ascold Perelomov and I have realized that the Calogero-Sutherland Hamiltonians up to a conjugation coincide with the radial parts of the second Casimir operators on $\mathrm{sl}(N,\mathbb{C})$ and $\mathrm{SL}(N,\mathbb{C})$. This observation was a starting point of our investigations of classical and quantum integrable systems, related to Lie algebras. According to this approach it was established that solutions of the classical rational and the trigonometric models come from a free motion on Lie algebras and Lie groups [17, 30, 31]. In this way their quantum counterparts are related to the representation theory of simple Lie algebras [32]. It implies, in particular, that the wave-functions are just some special matrix elements in irreducible representations.

In the elliptic case the situation is more elaborate. The classical elliptic Calogero-Moser model (ECMM) is a particular example of the Hitchin systems [13]. It is a wide class of classical integrable systems that can be derived from a topological 3d gauge theory. The inclusion of CM in the Hitchin theory was observed independently in Refs. [8, 12, 26, 27].

In this brief review we touch another facets of the classical and quantum ECCM. In Sect.1 we discuss equivalence of the classical ECMM and the so-called elliptic top (ET). The later describes the classical degrees of freedom that located on a vertex of the SL($N,\mathbb{C}$) generalization of the XYZ lattice model. For the two-particle case it leads to the equivalence between the two-dimensional version of ECMM [19, 23] and the Landau-Lifshitz model. This section is based on Ref. [23]. The correspondence between the classical ECMM for two particle case and the Painlevè VI equation [22] is discussed in Sect.2. Finally, in Sect.3 we present the interpretation of the Schrödinger equation corresponding to the quantum ECMM and the Knizhnik-Zamolodchikov-Bernard equation that arises in the Wess-Zumino-Witten model on a torus.

2 Calogero-Moser model and Integrable tops

2.1 ECMM with spin

Description of the system

The ECMM system is defined by the Hamiltonian

$$H^{CM} = \frac{1}{2}\sum_{j=1}^{N} v_j^2 + \nu^2 \sum_{j>k} \wp(u_j - u_k; \tau)\,. \tag{2.1}$$

Here $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ are coordinates and $\mathbf{v} = (v_1, v_2, \ldots, v_N)$ are their momenta with the canonical brackets

$$\{v_j, u_k\} = \delta_{j,k}\,. \tag{2.2}$$

In fact, instead of the potential in (2.1) we consider another double-periodic function

$$E_2(x;\tau) = \wp(x;\tau) + 2\eta_1(\tau)\,,$$

where E_2 is the second Eisenstein function and $\eta_1(\tau) = \zeta(\frac{1}{2})$. The additional constant becomes essential only on the quantum level.

Let $T_\tau^2 = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ be a torus endowed with a complex structure with parameter $\tau\,, \Im m\tau > 0$. The double-periodicity of the Weierstrass function implies that the particles lie on the torus $u_j \in T_\tau^2$, while $\mathbf{v} \in \mathbb{C}^N$. In what follows we assume that $\sum_j u_j = 0$, $\sum_j v_j = 0$. Thus, the phase space is $T^*(\oplus_{j=1}^N T_\tau^2/(\sum u_j = 0))$.

The system has the "spin" generalization [10]. Let $\mathbf{p}$ be an N-order matrix. We consider $\mathbf{p}$ as an element of the Lie algebra sl($N,\mathbb{C}$). The linear (Lie-Poisson) brackets on sl($N,\mathbb{C}$) for the matrix elements assume the form

$$\{p_{jk}, p_{mn}\} = p_{jn}\delta_{km} - p_{mk}\delta_{jn}\,. \tag{2.3}$$

Let $\mathcal{O}$ be a coadjoint orbit

$$\mathcal{O} = \{\mathbf{p} \in \mathrm{sl}(N,\mathbb{C}) \mid \mathbf{p} = h^{-1}\mathbf{p}^0 h,\ h \in \mathrm{SL}(N,\mathbb{C})\,, \mathbf{p}^0 \in D\}\,, \tag{2.4}$$

where D is the diagonal subgroup of $\mathrm{SL}(N,\mathbb{C})$. The phase space of the ECMM with spin is

$$\mathcal{R}^{CM_N} = \{T^*(\oplus_{j=1}^N T_\tau^2/(\sum u_j = 0)), \tilde{\mathcal{O}}\}\,, \tag{2.5}$$

where $\tilde{\mathcal{O}} = \mathcal{O}//D$ is the symplectic quotient with respect to the action of D. It implies i) the moment constraint $p_{jj} = 0$, ii) the gauge fixing, for example, as $p_{j,j+1} = p_{j+1,j}$. Note that

$$\dim(\mathcal{R}^{CM_N}) = 2N - 2 + \dim\mathcal{O} - 2\dim(D) = \dim\mathcal{O}\,. \tag{2.6}$$

The spin ECMM Hamiltonian has the form

$$H^{CM,spin} = \frac{1}{2}\sum_{j=1}^N v_j^2 + \sum_{j>k} p_{jk}p_{kj}E_2(u_j - u_k;\tau)\,. \tag{2.7}$$

The case (2.1) corresponds to the most degenerate nontrivial orbit $\mathcal{O} \sim T^*\mathbb{C}P^{N-1}$ when $N-1$ eigen-values of $\mathbf{p}$ coincide. In this case $\dim(\tilde{\mathcal{O}}) = 0$. The coupling constant ν^2 is proportional to $\mathrm{tr}(\mathbf{p}^2)$.

The equations of motion can be read off from (2.2), (2.3) and (2.7)

$$\partial_t u_j = v_j\,, \tag{2.8}$$

$$\partial_t v_n = -\sum_{j\neq n} p_{jk}p_{kj}\partial_{u_n}E_2(u_j - u_n;\tau)\,, \tag{2.9}$$

$$\partial_t \mathbf{p} = 2[\mathbf{J_u}(\mathbf{p}), \mathbf{p}]\,, \tag{2.10}$$

where the operator $\mathbf{J_u}$ acts on $\mathrm{sl}(N,\mathbb{C})$ as $\mathbf{J_u}\,:\ p_{jk} \to E_2(u_j - u_k)p_{jk}$.

Lax representation

The system has the Lax representation

$$\partial_t L^{CM} = [L^{CM}, M^{CM}]\,,$$

Introduce an auxiliary elliptic curve E_τ with the same modular parameter as above. It plays the role of the basic spectral curve with the spectral parameter z. The Lax matrix depends on z and has the form

$$L^{CM} = P + X,\quad P = \mathrm{diag}(v_1,\ldots,v_N),\quad X_{jk} = p_{jk}\phi(u_j - u_k, z)\,, \tag{2.11}$$

where

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}\,, \tag{2.12}$$

and $\vartheta(z) = \vartheta(z|\tau)$ is the odd theta-function

$$\vartheta(z|\tau) = q^{\frac{1}{8}}\sum_{n\in\mathbf{Z}}(-1)^n e^{\pi i(n(n+1)\tau + 2nz)}\,,\quad (q = \mathbf{e}(\tau) = \exp 2\pi i\tau)\,.$$

The matrix M^{CM} corresponding to the flow (2.8)–(2.10) takes the form

$$M^{CM} = -D + Y\,, \quad D = \mathrm{diag}(Z_1, \dots, Z_N)\,, \quad Y_{jk} = y(u_j - u_k, z)\,, \tag{2.13}$$

$$Z_j = \sum_{k \neq j} E_2(u_j - u_k)\,, \quad y(u, z) = \frac{\partial \phi(u, z)}{\partial u}\,.$$

The Lax matrix is a quasi-periodic meromorphic functions on the spectral curve E_τ taking values in the Lie algebra $\mathrm{sl}(N, \mathbb{C})$ with a simple pole at $z = 0$

$$\bar{\partial} L^{CM} = 0\,, \quad Res L^{CM}|_{z=0} = \mathbf{p}\,. \tag{2.14}$$

$$L^{CM}(z + 1) = L^{CM}(z)\,, \quad L^{CM}(z + \tau) = \mathrm{diag}(\mathbf{e}(\mathbf{u}))L^{CM}(z)\mathrm{diag}(-\mathbf{e}(\mathbf{u}))\,, \tag{2.15}$$

where $\mathrm{diag}(\mathbf{e}(\mathbf{u})) = \mathrm{diag}(\exp(2\pi i u_1, \dots, \exp(2\pi i u_N)).$ These conditions characterized uniquely the non-diagonal part X of L^{CM}.

The Lax equation is equivalent to the linear problem

$$(\lambda + L^{CM})Y = 0\,, \tag{2.16}$$

$$\partial_t + M^{CM})Y = 0\,. \tag{2.17}$$

The additional equation

$$\bar{\partial} Y = 0 \tag{2.18}$$

implies that M^{CM} is also meromorphic on E_τ.

2.2 Elliptic top on $\mathrm{SL}(N, \mathbb{C}\,)$

Description of the top

Consider the Euler-Arnold top (EAT) on the group $\mathrm{SL}(N, \mathbb{C}\,)$. Its phase space is embedded in the Lie coalgebra $\mathrm{sl}(N, \mathbb{C})^*$ as a coadjoint orbit. It is endowed with the Lie-Poisson brackets (2.3).

The EAT is determined by a symmetric operator $\mathbf{J} : \mathrm{sl}(N, \mathbb{C})^* \to \mathrm{sl}(N, \mathbb{C})$, that is called the inverse inertia operator. The Hamiltonian of the system is $H^{EAT} = \mathrm{tr}(\mathbf{SJ}(\mathbf{S}))$, where $\mathbf{S} \in \mathrm{sl}(N, \mathbb{C})^*$. A special choice of $\mathbf{J}$ leads to an integrable system. The elliptic top (ET) is an example of an integrable EAT.

To define the inverse inertia operator for ET we choose a special basis in $\mathrm{sl}(N, \mathbb{C})^* \sim \mathrm{sl}(N, \mathbb{C})$. Define two type of matrices

$$Q = \mathrm{diag}(\mathbf{e}_N(1), \dots, \mathbf{e}_N(m), \dots, 1)\,, \ (\,\mathbf{e}_N(z) = \exp(\frac{2\pi i}{N} z)\,)\,,$$

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Consider the two-dimensional lattice $\mathbb{Z}_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z})/(0,0)$ of order $N^2 - 1$. The matrices

$$T_\alpha = \frac{1}{2\pi i\theta}\mathbf{e}_N(\frac{\alpha_1\alpha_2}{2})Q^{\alpha_1}\Lambda^{\alpha_2}\,,\ (\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}^{(2)_N})$$

generate a basis in $\mathrm{sl}(N,\mathbb{C})$. The commutation relations in this basis assume the form

$$[T_\alpha, T_\beta] = \mathbf{C}_N(\alpha,\beta)T_{\alpha+\beta}\,,$$

where

$$\mathbf{C}_N(\alpha,\beta) = \frac{N}{\pi}\sin\frac{\pi(\alpha\times\beta)}{N}\,. \tag{2.19}$$

The Poisson structure on the dual space $\mathrm{sl}(N,\mathbb{C})^*$ is given by the linear Lie-Poisson brackets coming from (2.19)

$$\{S_\alpha, S_\beta\}_1 = \mathbf{C}(\alpha,\beta)S_{\alpha+\beta}\,. \tag{2.20}$$

Let $\mathbb{Z}_N^{(2)}(\tau) = \{\frac{\gamma_1+\gamma_2\tau}{N}\}$, $\gamma \in \mathbb{Z}_N^{(2)}$ be a regular lattice of order N^2-1 on T^2_τ. Introduce the following constant on $\mathbb{Z}_N^{(2)}(\tau)$: $E_2(\gamma) = E_2(\frac{\gamma_1+\gamma_2\tau}{N})$. Then the operator $\mathbf{J}$ for the ET is defined as

$$\mathbf{J}\,:\,S_\alpha \;\to\; E_2(\alpha)S_\alpha\,. \tag{2.21}$$

Let $\mathbf{S} = \sum_{\alpha\in\mathbb{Z}_N^{(2)}} S_{-\alpha}T_\alpha$. The Hamiltonian has the form

$$H^{ET} = -\frac{1}{2}\mathrm{tr}(\mathbf{S}\cdot\mathbf{J}(\mathbf{S})) \equiv -\frac{1}{2}\sum_{\gamma\in\mathbb{Z}_N^{(2)}} S_\gamma E_2(\gamma)S_{-\gamma}\,. \tag{2.22}$$

It defines the equations of motion

$$\partial_t\mathbf{S} = [\mathbf{J}(\mathbf{S}),\mathbf{S}]\,, \tag{2.23}$$

or

$$\partial_t S_\alpha = \sum_{\gamma\in\mathbb{Z}_N^{(2)}} S_{\alpha-\gamma}S_\gamma(E_2(\alpha-\gamma)-E_2(\gamma))\mathbf{C}_\theta(\gamma,\alpha)\,. \tag{2.24}$$

The phase space $\mathcal{R}^{ET}$ of ET is a coadjoint orbit of $\mathrm{SL}(N,\mathbb{C}\,)$

$$\mathcal{R}^{ET} = \mathcal{O}\,. \tag{2.25}$$

Note that it dimension coincides with $\dim(\mathcal{R}^{CM})$.

The Lax form of (2.23) is provided by the Lax matrix [35]

$$L^{ET} = \sum_\alpha S_\alpha\varphi(\alpha,z)T_\alpha\,,\quad \varphi(\gamma,z) = \mathbf{e}(\frac{\gamma_2 z}{N})\phi(\frac{\gamma_1+\gamma_2\tau}{N},z)\,, \tag{2.26}$$

and

$$M^{ET} = \sum_\alpha S_\alpha f(\alpha,z)T_\alpha\,,\quad f(\alpha,z) = \mathbf{e}(\frac{\alpha_2 z}{N})\partial_u\phi(u;z)|_{u=\frac{\alpha_1+\alpha_2\tau}{N}}\,. \tag{2.27}$$

The Lax matrix is characterized by the following conditions:

$$\bar\partial L^{ET} = 0\,,\quad Res L^{ET}|_{z=0} = \mathbf{S} = \sum S_{-\alpha}T_\alpha\,, \tag{2.28}$$

$$L^{ET}(z+1) = Q(\tau)L^{ET}(z)Q^{-1}(\tau)\,,\ L^{ET}(z+\tau) = \tilde\Lambda(z,\tau)L^{ET}(z)(\tilde\Lambda(z,\tau))^{-1}\,, \tag{2.29}$$

where $\tilde\Lambda(z,\tau) = -\mathbf{e}(\frac{-z-\frac{1}{2}\tau}{N})\Lambda$.

2.3 The map of the ECMM system to the ET system

The map is defined as the conjugation of L^{CM} by some matrix $\Xi(z)$:

$$L^{ET} = \Xi \times L^{CM} \times \Xi^{-1} \,. \tag{2.30}$$

The matrix $\Xi(z)$ is a meromorphic quasi-periodic map $E_\tau \to \mathrm{GL}(N,\mathbb{C})$. It is defined uniquely by its quasi-periodicity and the pole at $z=0$. The latter means that Ξ can be considered as a singular gauge transformation. Assume that an eigen-vector of the residue of $L^{CM_N} = \mathbf{p}$ at $z=0$ belongs to the one-dimensional kernel of $\Xi(z)$. Then it can be proved that (2.30) preserves the order of the pole.

The matrix Ξ has the following form. The quasi-periodicity of L^{CM} and L^{ET} leads to the following relations

$$\Xi(z+1,\tau) = Q \times \Xi(z,\tau)\,, \tag{2.31}$$

$$\Xi(z+\tau,\tau) = \tilde{\Lambda}(z,\tau) \times \Xi(z,\tau) \times \mathrm{diag}(\mathbf{e}(u_j))\,. \tag{2.32}$$

Let $\mathbf{p}^0$ be the diagonal matrix defining the coadjoint orbit (2.4) in the ECMM

$$Res\, L^{CM_N}|_{z=0} = \mathbf{p} = h^{-1}\mathbf{p}^0 h\,, \quad \mathbf{p}^0 = \mathrm{diag}(p_1^0,\ldots,p_N^0)\,. \tag{2.33}$$

Then $\Xi(z) = \Xi(z,\overrightarrow{\mathbf{r}}_j)$ depends on a choice of the eigen-vector $\overrightarrow{\mathbf{r}}_j = (r_{1,j},\ldots,r_{N,j})$ of the orbit matrix $\mathbf{p}$ corresponding to the eigenvalue p_j^0 (2.33). The vector $\overrightarrow{\mathbf{r}}_j)$ should lies in the kernel of Ξ. It has the form $\overrightarrow{\mathbf{r}}_j = h^{-1}(0,\ldots,0,1,0\ldots,0)^T$, where 1 stands on the j-th place.

We construct first $(N\times N)$- matrix $\tilde{\Xi}(z,u_1,\ldots,u_N;\tau)$ that satisfies (2.31) and (2.32) but has a special kernel:

$$\tilde{\Xi}_{ij}(z,u_1,\ldots,u_N;\tau) = \theta\begin{bmatrix} \frac{i}{N}-\frac{1}{2} \\ \frac{N}{2} \end{bmatrix}(z-Nu_j,N\tau), \tag{2.34}$$

where $\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)$ is the theta function with a characteristic

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{j\in\mathbb{Z}} \mathbf{e}\left((j+a)^2\frac{\tau}{2} + (j+a)(z+b)\right)\,.$$

It can be proved that the kernel of $\tilde{\Xi}$ at $z=0$ is generated by the following column-vector :

$$\left\{(-1)^l \prod_{j<k;j,k\neq l} \vartheta(u_k-u_j,\tau)\right\}, \quad l=1,2,\cdots,N.$$

Then the matrix $\Xi(z,\mathbf{u},\overrightarrow{\mathbf{r}}_i)$ assumes the form

$$\Xi(z,\mathbf{u},\overrightarrow{\mathbf{r}}_i) = \tilde{\Xi}(z)\times \mathrm{diag}\left(\frac{(-1)^l}{r_{l,i}} \prod_{j<k;j,k\neq l} \vartheta(u_k-u_j,\tau)\right)\,. \tag{2.35}$$

It leads to the map $\mathcal{R}^{CM_N} \to \mathcal{R}^{rot}$.

This transformation means that the particle coordinates and momenta of the ECMM $(\mathbf{v}, \mathbf{u})$ along with the spin variables $\mathbf{p}$ boil down to the orbit variables $(\mathbf{v}, \mathbf{u}, \mathbf{p}) \mapsto \mathbf{S}$. For the most degenerate orbit in the standard ECMM, defined by the coupling constant ν^2 this transformation leads to the degenerate orbit of the ET with the same value of the second Casimir.

Note that equation for ECMM with spin (2.9) remind the equation of motion for the EAT with the coordinate-dependent operator $\mathbf{J_u}$ (2.7). The only difference is the structure of the phase spaces $\mathcal{R}^{CM_N}$ (2.5) and $\mathcal{R}^{ET}$ (2.25). The gauge transform Ξ carries out the pass from $\mathcal{R}^{CM_N}$ to $\mathcal{R}^{ET}$. It depends only on the part of variablis on $\mathcal{R}^{CM_N}$, namely on $\mathbf{u}$ and $\mathbf{p}$ through the eigenvector $\overrightarrow{\mathbf{r}}_j$.

In geometrical terms the ECMM is the Hitchin type system related to vector holomorphic bundles of degree zero with the quasi-parabolic structure at the point $z = 0$. The coordinate variables $\mathbf{u}$ correspond to the holomorphic moduli, while the spin variables $\mathbf{p}$ are responsible for the quasi-parabolic structure. The ET system is defined in the similar way on the holomorphic bundle of degree one. There are no moduli in this case and all degrees of freedom $\mathbf{S}$ correspond to the quasi-parabolic structure. The transformation Ξ is the so-called *the upper modification*. It transforms the space of sections of the vector bundle of degree zero to the space of sections of the vector bundle of degree one. It can be lifted to the symplectic map (*the upper symplectic Hecke correspondence*) (2.30) of the phase space $\mathcal{R}^{CM_N}$ to the phase space $\mathcal{R}^{ET}$.

Consider in detail the case $N = 2$, when the system has the one degree of freedom. Let the eigen-vector of $\mathbf{p}$ has the form $(1,1)^T$ and put $\mathbf{S} = S_a\sigma_a$, where σ_a are the sigma matrices. Then the transformation has the form

$$\left\{ \begin{array}{l} S_1 = -v\frac{\theta_{10}(0)}{\vartheta'(0)}\frac{\theta_{10}(2u)}{\vartheta(2u)} - \nu\frac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)}\frac{\theta_{00}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}\,, \\ S_2 = -v\frac{\theta_{00}(0)}{i\vartheta'(0)}\frac{\theta_{00}(2u)}{\vartheta(2u)} - \nu\frac{\theta_{00}^2(0)}{i\theta_{10}(0)\theta_{01}(0)}\frac{\theta_{10}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}\,, \\ S_3 = -v\frac{\theta_{01}(0)}{\vartheta'(0)}\frac{\theta_{01}(2u)}{\vartheta(2u)} - \nu\frac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)}\frac{\theta_{00}(2u)\theta_{10}(2u)}{\vartheta^2(2u)}\,. \end{array} \right. \tag{2.36}$$

3 Calogero-Moser model and Isomonodromic deformations

The famous Painlevé VI equation depends on four free parameters ($\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$) and has the form

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$
$$+ \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right). \tag{3.1}$$

It can be transformed to the elliptic form [3, 24, 34] that we will use. Let $\omega_0 = 0$, $\omega_{1,2}$ are the half periods of the elliptic curve E_τ, $\omega_3 = \omega_1 + \omega_2$ and

$$\nu_0 = \alpha\,,\ \nu_1 = -\beta\,,\ \nu_2 = \gamma\,,\ \nu_3 = \frac{1}{2} - \delta\,.$$

Then (3.1) takes the form

$$\partial_\tau u = -\sum_{j=0}^{3} \partial_u \nu_j^2 E_2(u + \omega_j)\,, \tag{3.2}$$

where the variables are replaced as

$$(u,\tau) \to \left(X = \frac{E_2(u|\tau) - e_1}{e_2 - e_1}\,,\ t = \frac{e_3 - e_1}{e_2 - e_1} \right)\,,\ e_j = E_2(\omega_j)\,.$$

We will not consider here the general case and restrict ourselves to the case $\nu_j = \frac{\nu}{4}$ [1]. Then PVI assumes the form

$$\partial_\tau^2 u = -\partial_u \nu^2 E_2(2u)\,.$$

It is a non-autonomous Hamiltonian system with the same Hamiltonian as for the two-body ECMM

$$H = \frac{1}{2}v^2 + \nu^2 E_2(2u)\,,$$

but now the module τ plays the role of the time.

Let us introduce the new parameter κ and consider the equation

$$\kappa^2 \frac{d^2 u}{d\tau^2} = -\partial_u \nu^2 E_2(2u|\tau)\,. \tag{3.3}$$

The case $\kappa = 1$ can be achieved by the rescaling the dynamical variables (v,u) and the half-periods ω_1, ω_2

$$v \to \kappa^{-\frac{1}{2}} v\,,\ u \to \kappa^{\frac{1}{2}} u\,,\ \omega_j \to \kappa^{\frac{1}{2}} \omega_j\,. \tag{3.4}$$

The equation (3.3) has the Lax representation

$$\kappa \partial_\tau L^P - \kappa \partial M^P + [M^P, L^P] = 0\,.$$

Let $\mu = \frac{\tau-\tau_0}{\tau-\bar{\tau}_0}$, $x(u,w,\bar{w}) = \frac{\nu}{2\pi i(1-\mu)}\phi(u,w)$, where τ_0 corresponds to some fixed module and $\phi(u,w)$ is defined by (2.12), $y(u,w,\bar{w}) = \frac{\tau_0-\bar{\tau}_0}{2\pi i\kappa(\tau-\bar{\tau}_0)}\partial_u x(u,w,\bar{w})$. The Lax matrices assume the form

$$L^P = \begin{pmatrix} \frac{v}{1-\tilde{\mu}_\tau} & x(u,w,\bar{w}) \\ x(-u,w,\bar{w}) & -\frac{v}{1-\tilde{\mu}_\tau} \end{pmatrix}\,,\quad M^P = \begin{pmatrix} 0 & y(2u,w,\bar{w}) \\ y(-2u,w,\bar{w}) & 0 \end{pmatrix}\,, \tag{3.5}$$

The Lax equation can be considered as the consistency condition for the linear system

$$(\kappa\partial + L^P)\Psi = 0\,, \tag{3.6}$$

$$(\kappa\partial_\tau + M^P)\Psi = 0\,, \tag{3.7}$$

$$(\bar{\partial} + \mu\partial)\Psi = 0\,,\quad \mu = \frac{\tau - \tau_0}{\tau - \bar{\tau}_0}\,, \tag{3.8}$$

where (3.8) implies the holomorphity of the Baker-Akhiezer function Ψ in the coordinates deformed by μ: $w = z - \frac{\tau-\bar{\tau}_0}{\rho}(\bar{z} - z)$, $\bar{w} = \bar{z}$, $(\rho = \tau_0 - \bar{\tau}_0)$.

We will prove that the linear problem for the two-body ECMM (2.16)–(2.18) coincides with (3.6)–(3.7) in the limit $\kappa \to 0$. The constant κ plays the role of the Planck constant and (2.16)–(2.18) is the result of the quasi-classical limit. Define the time t corresponding

[1] The general case was investigated in [38].

to the two-body ECMM Hamiltonian as $\tau = \tau_0 + \kappa t$, and represent the Baker-Akhiezer function in the WKB approximation form

$$\Psi = \Phi \exp(\frac{\mathcal{S}^{(0)}}{\kappa} + \mathcal{S}^{(1)}), \tag{3.9}$$

where Φ is a group valued function and $\mathcal{S}^{(0)}$, $\mathcal{S}^{(1)}$ are diagonal matrices. Substitute (3.9) in the linear system (3.6),(3.8),(3.7). If $\frac{\partial}{\partial \bar{w}_0}\mathcal{S}^{(0)} = 0$ and $\frac{\partial}{\partial t}\mathcal{S}^{(0)} = 0$, there are no terms of order κ^{-1}. In the quasi-classical limit we put $\partial\mathcal{S}^{(0)} = \lambda$. In the zero order approximation we come to the linear system of the two-body ECMM (2.16)–(2.18). The Baker-Akhiezer function Y takes the form

$$Y = \Phi e^{t\frac{\partial}{\partial \tau_0}\mathcal{S}^{(0)}}.$$

This passage from the autonomous two-body ECCM to the Painlevé VI equation is an example of the Whitham quantization. The quasi-classical limit of the full PVI yields the generalization of ECMM [15].

We can consider the SL$(N,\mathbb{C})$ generalization of the isomonodromy problem. The related Lax matrix has the form

$$L = P + X\,, \quad P = 2\pi i\,\mathrm{diag}\,\frac{\mathbf{v}}{1-\mu}\,,$$

$$X_{jk} = \{x_\alpha\} = (\tau - \bar{\tau}_0)\nu\phi(u_j - u_k, w).$$

The multi-component analog of the Painlevé VI equation is the monodromy preserving equation

$$\frac{\kappa^2 d^2 u_j}{d\tau^2} = -\frac{\nu^2}{(2\pi i)^2}\sum_{k\neq j}^{N} \partial_{u_j} E_2(u_j - u_k|\tau)\,. \tag{3.10}$$

In the quasi-classical limit $\kappa \to 0$ we come to the linear problem for the N-body ECMM (2.16)–(2.18).

4 Calogero-Moser model and Knizhnik-Zamolodchikov-Bernard equation

The Knizhnik-Zamolodchikov-Bernard equation (KZB) is the generalization on a torus of the Knizhnik-Zamolodchikov equation [20] obtained by D.Bernard [4]. Its solutions are correlation functions of the Wess-Zumino-Witten model on a the torus with n marked points. The KZB equation has the form of the non-stationer Schrödinger equation where the role of times is played by the module τ and the position of $n-1$ points. We consider here the elliptic curve with $n = 1$. The general case ($n > 1$) was considered in Ref. [14, 21]. The correlation function F depends on a finite-dimensional representation V attributing to the marked point. The KZB equation has the form

$$\left(\kappa^{quant}\partial_\tau + \frac{1}{2}\sum_{j=1}^{N}\partial_{u_j}^2 + \sum_{j>k}\hat{e}_{jk}\hat{e}_{kj}E_2(u_j - u_k;\tau)\right)F = 0\,, \tag{4.1}$$

where $\hat{e}_{kj}$ are generators of the matrix elements e_{kj} in V and $\kappa^{quant} = \kappa + N$. To pass to the classical limit in the KZB equations we replace the conformal block by its quasi-classical expression

$$F = \exp \frac{\mathcal{F}}{\hbar}, \tag{4.2}$$

where $\hbar = (\kappa^{quant})^{-1}$. Consider the classical limit $\kappa^{quant} \to \infty$ and assume that values of the Casimirs C_a^i, $(i = 1, \dots, \mathrm{rank}G,\ a = 1, \dots, n)$ corresponding to the irreducible representations also go to infinity. Let all values $\lim \frac{C_a^i}{\kappa^{quant}}$ are finite. It allows to fix the coadjoint orbits in the marked point. In the classical limit (4.1) is transformed to the Hamilton-Jacobi equation for the action $\mathcal{F}$

$$\partial_\tau \mathcal{F} - H^{CM,spin}(\partial_{\mathbf{u}}\mathcal{F}, \mathbf{u}) = 0\,.$$

In this way we come to isomonodromy preseving case.

On the critical level $\kappa^{quant} = 0$ we come to the groundstate problem for the quantum ECMM for the zero eigenvalue. It allows to describe the wave-functions of the quantum ECMM [7, 9].

We summarize the result of last two sections in the following diagram.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{KZB eq.}\\ (\kappa^{quant}\partial_\tau + \hat{H})F = 0\end{array}} \arrow[r, "\kappa^{quant}\to 0"] \arrow[d, "\hbar\to 0"] & \boxed{\begin{array}{c}\text{KZB eq. on the critical level,}\\ \hat{H}F = 0,\end{array}} \arrow[d, "\hbar\to 0"] \\
\boxed{\text{Multicomp. Painlevé VI}} \arrow[r, "\kappa\to 0"] & \boxed{\text{Classical ECMM}}
\end{tikzcd}$$

Here before going from PVI to the classical ECMM we renormalize the variables and the half-periods according with (3.4).

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