Journal of Nonlinear Mathematical Physics Volume 12, Supplement 1 (2005), 212–227 **Birthday Issue**

# Explicit integration of the Hénon-Heiles Hamiltonians [1]

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*This article is part of the special issue published in honour of Francesco Calogero on the occasion of his 70th birthday*

**Abstract**

We consider the cubic and quartic Hénon-Heiles Hamiltonians with additional inverse square terms, which pass the Painlevé test for only seven sets of coefficients. For all the not yet integrated cases we prove the singlevaluedness of the general solution. The seven Hamiltonians enjoy two properties: meromorphy of the general solution, which is hyperelliptic with genus two and completeness in the Painlevé sense (impossibility to add any term to the Hamiltonian without destroying the Painlevé property).

## 1 Introduction

The "Hénon-Heiles Hamiltonian" (HH) [28] originally denoted a two-degree of freedom classical Hamiltonian, the sum of a kinetic energy and a potential energy, in which the potential is a cubic polynomial in the position variables $q_1, q_2$,

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1 q_2^2 - \frac{1}{3} q_1^3, \tag{1.1}$$

i.e. the simplest one after two coupled harmonic oscillators. This system, which describes the motion of a star in the axisymmetric potential of the galaxy ($q_1$ is the radius, $q_2$ is the altitude), happens to be nonintegrable and to display a strange attractor. However, if one changes the numerical coefficients, the system may become integrable in some sense, and this question (to find all the integrable cases and to integrate them) has attracted a lot of activity in the last thirty years.

The present article is a self-contained paper which reviews the current state of this problem, restricted here to the autonomous case. It covers some old results for completeness and it presents an explicit integration for all the cases which have not yet been



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integrated. To summarize the possibly integrable cases (in the Liouville sense or in the Painlevé sense) have been isolated long ago and the explicit integration of all these cases is now achieved in the Painlevé sense (finding a closed-form single-valued expression for the general solution) but not yet in the Hamilton-Jacobi sense (finding the separating variables of the Hamilton-Jacobi equation).

In section 2 we briefly recall the three main accepted meanings of the word *integrability* for Hamiltonian systems. In section 3, taking one degree of freedom as an example, we relax the requirement on $q_j$ by accepting the singlevaluedness of some integer power of $q_j$. In section 4 one recalls the results of the Painlevé test for two degrees of freedom, i.e. the selection of three "cubic" cases plus four "quartic" cases. In section 5 we recall the link of these seven cases with soliton systems and we present in the quartic case a fourth-order first-degree ordinary differential equation (ODE) equivalent to the Hamilton's equations. In section 6 we enumerate three possible strategies to prove the singlevaluedness of the general solution. In section 7 we briefly recall the separation of variables in the two Stäckel cases (one cubic and one quartic). In section 8 the two remaining cubic cases are integrated by separating the Hamilton-Jacobi equation. In the three remaining quartic cases, sections 9 and 10, the singlevaluedness is proven by building a birational transformation to ODEs in the classification of Cosgrove [17].

## 2 Integrability of Hamiltonian systems

Given a Hamiltonian system with a finite number $N$ of degrees of freedom, three main definitions of *integrability* are known for it,

1. the one in the sense of Liouville, that is the existence of $N$ independent invariants $K_j$ the pairwise Poisson brackets of which vanish, $\{K_j, K_l\} = 0$,

2. the one in the sense of Hamilton-Jacobi, which is to find explicitly some canonical variables $s_j, r_j, j = 1, N$ which "separate" the *Hamilton-Jacobi equation* for the action $S$ [3, chap. 10], which for two degrees of freedom is

$$H(q_1, q_2, p_1, p_2) - E = 0,\ p_1 = \frac{\partial S}{\partial q_1},\ p_2 = \frac{\partial S}{\partial q_2}, \tag{2.1}$$

3. the one in the sense of Painlevé [12], i.e. the proof that the general solution $q_j(t)$ is a single-valued expression of the time $t$, represented either by an explicit, closed-form expression, or by the solution of a Jacobi inversion problem, see e.g. (7.8) below. In the particular case $N = 2$, the inversion of the system of two integrals

$$C_1 = \int_\infty^{s_1} \frac{\mathrm{d}s}{\sqrt{P(s)}} + \int_\infty^{s_2} \frac{\mathrm{d}s}{\sqrt{P(s)}}, \qquad t + C_2 = \int_\infty^{s_1} \frac{s\mathrm{d}s}{\sqrt{P(s)}} + \int_\infty^{s_2} \frac{s\mathrm{d}s}{\sqrt{P(s)}}, \tag{2.2}$$

   in which $P$ is a polynomial of degree 5 or 6, and $C_1, C_2$ are two constants of integration, leads to symmetric functions of $s_1, s_2$ being meromorphic in $t$.

*Remark.* One can prove [4] that every Liouville integrable system has a Lax pair, and the Lax pair is indeed the starting point of a powerful method [43] to compute the separating variables.

The goal here is to integrate in the sense of Painlevé, and ideally to find the separating variables of the Hamilton-Jacobi equation.

# 3 The case of one degree of freedom

In this case, $H = p^2/2 + V(q)$, the Hamilton's equations of motion admit a singlevalued general solution if and only if $V$ is a polynomial of degree at most four, in which case $q(t)$ is an elliptic function or one of its degeneracies. If one slightly relaxes the requirement on $q$ and accepts that only some integer power $q^n$ be singlevalued, then one additional case arises [26],

$n = \pm 2$ and $V$ the sum of even terms [22, 36],

$$H = \frac{p^2}{2} + aq^2 + bq^4 + cq^{-2}, \tag{3.1}$$

in which case $q^2$ obeys either a linear equation ($b = 0$) or the Weierstrass elliptic equation ($b \neq 0$).

# 4 Two degrees of freedom: the seven Hénon-Heiles Hamiltonians

If one considers the most general two-degree of freedom classical autonomous Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2) \tag{4.1}$$

and if one requires the existence of some singlevalued integer powers $q_1^{n_1}, q_2^{n_2}$, it is then necessary that the Hamilton's equations of motion, when written in these variables, pass the Painlevé test. The application of this test isolates two classes of potentials $V$, called "cubic" and "quartic" for simplification.

1. In the cubic case HH3 the admissible Hamiltonians are, [10, 23, 13],

$$H = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1 q_1^2 + \omega_2 q_2^2) + \alpha q_1 q_2^2 - \frac{1}{3}\beta q_1^3 + \frac{1}{2}\gamma q_2^{-2},\ \alpha \neq 0 \tag{4.2}$$

$$q_1'' + \omega_1 q_1 - \beta q_1^2 + \alpha q_2^2 = 0, \tag{4.3}$$

$$q_2'' + \omega_2 q_2 + 2\alpha q_1 q_2 - \gamma q_2^{-3} = 0, \tag{4.4}$$

in which the constants $\alpha, \beta, \omega_1, \omega_2$ and $\gamma$ can only take the three sets of values,

$$(\mathrm{SK}): \quad \beta/\alpha = -1, \omega_1 = \omega_2, \tag{4.5}$$

$$(\mathrm{KdV5}): \quad \beta/\alpha = -6, \tag{4.6}$$

$$(\mathrm{KK}): \quad \beta/\alpha = -16, \omega_1 = 16\omega_2. \tag{4.7}$$

The meaning of the labels SK, KdV5, KK are explained in section 5.

2. In the quartic case HH4 the admissible Hamiltonians are, [37, 27],

$$\begin{aligned} H &= \frac{1}{2}(P_1^2+P_2^2+\Omega_1 Q_1^2+\Omega_2 Q_2^2)+CQ_1^4+BQ_1^2Q_2^2+AQ_2^4 \\ &\quad +\frac{1}{2}\left(\frac{\alpha}{Q_1^2}+\frac{\beta}{Q_2^2}\right)+\gamma Q_1,\ B\neq 0, \end{aligned} \tag{4.8}$$

$$Q_1''+\Omega_1 Q_1+4CQ_1^3+2BQ_1Q_2^2-\alpha Q_1^{-3}+\gamma=0, \tag{4.9}$$

$$Q_2''+\Omega_2 Q_2+4AQ_2^3+2BQ_2Q_1^2-\beta Q_2^{-3}=0, \tag{4.10}$$

in which the constants $A,B,C,\alpha,\beta,\gamma,\Omega_1$ and $\Omega_2$ can only take the four values (the notation $A:B:C=p:q:r$ stands for $A/p=B/q=C/r=$ arbitrary),

$$\left\{\begin{array}{l} A:B:C=1:2:1,\ \gamma=0, \\ A:B:C=1:6:1,\ \gamma=0,\ \Omega_1=\Omega_2, \\ A:B:C=1:6:8,\ \alpha=0,\ \Omega_1=4\Omega_2, \\ A:B:C=1:12:16,\ \gamma=0,\ \Omega_1=4\Omega_2. \end{array}\right. \tag{4.11}$$

For each of the seven cases so isolated there exists a second constant of the motion $K$ [19, 6, 29] [30, 5, 6] which commutes with the Hamiltonian,

$$\begin{aligned} (\mathrm{SK}):K &= K_0^2+3\gamma(3p_1^2q_2^{-2}+4\alpha q_1+2\omega_2), && (4.12)\\ & K_0=3p_1p_2+\alpha q_2(3q_1^2+q_2^2)+3\omega_2 q_1 q_2, \\ (\mathrm{KdV5}):K &= 4\alpha p_2(q_2p_1-q_1p_2)+(4\omega_2-\omega_1)(p_2^2+\omega_2 q_2^2+\gamma q_2^{-2}) \\ &\quad +\alpha^2q_2^2(4q_1^2+q_2^2)+4\alpha q_1(\omega_2q_2^2-\gamma q_2^{-2}), && (4.13)\\ (\mathrm{KK}):K &= (3p_2^2+3\omega_2q_2^2+3\gamma q_2^{-2})^2+12\alpha p_2q_2^2(3q_1p_2-q_2p_1) \\ &\quad -2\alpha^2q_2^4(6q_1^2+q_2^2)+12\alpha q_1(-\omega_2q_2^4+\gamma)-12\omega_2\gamma, && (4.14)\\ \mathrm{quartic}:K &= \text{see } (7.11),(9.1),(9.2),(10.1). && (4.15) \end{aligned}$$

Therefore one should be able to integrate both in the Hamilton-Jacobi sense (separation of variables) and in the Painlevé sense (closed-form single-valued general solution). This invariant $K(q_1,q_2,p_1,p_2)$ is polynomial in the momenta $p_1,p_2$, with the degrees 2 (KdV5 and 1:2:1 cases) and 4 (the five other cases) and the difficulty to perform the separation of variables is intimately related to the degree of $K$ in the momenta.

# 5 Link to soliton equations

In the cubic case it is possible to build [23] by elimination of $q_2$ between the three equations (4.2)–(4.4), a fourth-order ODE for $q_1(t)$ with two nice properties:

1. $q_1''''$ is a polynomial in $q_1''',q_1'',q_1',q_1$, without the $q_1'''^2$ term,
2. this fourth-order ODE is, in each of the three cases, the traveling wave reduction of a fifth-order soliton equation.

This ODE, namely

$$
\begin{aligned}
& q_1'''' + (8\alpha - 2\beta) q_1 q_1'' - 2(\alpha+\beta) q_1'^2 - \frac{20}{3}\alpha\beta q_1^3 \\
& +(\omega_1 + 4\omega_2) q_1'' + (6\alpha\omega_1 - 4\beta\omega_2) q_1^2 + 4\omega_1\omega_2 q_1 + 4\alpha E = 0,
\end{aligned}
\tag{5.1}
$$

is independent of the coefficient $\gamma$ of the nonpolynomial term $q_2^{-2}$ and it depends on the constant value $E$ of the Hamiltonian $H$.

This elimination establishes the identification [23] of the HH3 Hamiltonian system with the traveling wave reduction $u(x,t) = U(x-ct)$ of the fifth-order conservative partial differential equation (PDE)

$$
u_t + \left( u_{xxxx} + (8\alpha - 2\beta) u u_{xx} - 2(\alpha+\beta) u_x^2 - \frac{20}{3}\alpha\beta u^3 \right)_x = 0,
\tag{5.2}
$$

and the three values of $\beta/\alpha, \omega_1$ and $\omega_2$ for which HH3 passes the Painlevé test are precisely the only values for which the PDE (5.2) is a soliton equation, respectively called the Sawada-Kotera (SK) [42] , fifth-order KdV (KdV5) [34] and Kaup-Kupershmidt (KK) [32, 24] equations.

In each of the four quartic cases one can similarly establish a link [25, 5] with a soliton system made of two coupled PDEs, most of them appearing in lists established from group theory [20]. However, the elimination of $Q_2$ in a way similar to the cubic case leads to

$$
\begin{aligned}
& -Q_1'''' + 2\frac{Q_1' Q_1'''}{Q_1} + \left(1 + 6\frac{A}{B}\right)\frac{Q_1''^2}{Q_1} - 2\frac{{Q_1'}^2 Q_1''}{Q_1^2} \\
& \quad + 8\left(6\frac{AC}{B} - B - C\right) Q_1^2 Q_1'' + 4(B-2C) Q_1 {Q_1'}^2 + 24C\left(4\frac{AC}{B} - B\right) Q_1^5 \\
& \quad + \left[12\frac{A}{B}\omega_1 - 4\omega_2 + \left(1 + 12\frac{A}{B}\right)\frac{\gamma}{Q_1} - 4\left(1 + 3\frac{A}{B}\right)\frac{\alpha}{Q_1^4}\right] Q_1'' \\
& \quad + 6\frac{A}{B}\frac{\alpha^2}{Q_1^7} + 20\frac{\alpha}{Q_1^5}{Q_1'}^2 - 12\frac{A}{B}\frac{\gamma\alpha}{Q_1^4} + 4\left(3\frac{A}{B}\omega_1 - \omega_2\right)\left(\gamma - \frac{\alpha}{Q_1^3}\right) - 2\gamma\frac{{Q_1'}^2}{Q_1^2} \\
& \quad + 6\left(\frac{A}{B}\gamma^2 + 2B\alpha - 8\frac{AC}{B}\alpha\right)\frac{1}{Q_1} + \left(6\frac{A}{B}\omega_1^2 - 4\omega_1\omega_2 - 8BE\right) Q_1 \\
& \quad + 48\frac{AC}{B}\gamma Q_1^2 + 4\left(12\frac{AC}{B} - B - 4C\right)\omega_1 Q_1^3 = 0.
\end{aligned}
\tag{5.3}
$$

This ODE, which depends on $E$ but not on $\beta$, is equivalent to the Hamilton's equations. Therefore this would be the most suitable ODE to which to apply the Painlevé test.

In the 1:12:16 case with the constraint $\alpha = 0$ this ODE is identical to the autonomous restriction of [33, Eq. (5.9)], an equation linked to the hierarchy of the second Painlevé equation, reproduced as [18, Eq. (7.141)]. The Hamiltonian system equivalent to this ODE is easily integrated by the method of separation of variables [1], see section 10. The results to be displayed in next sections show that, in the four HH4 cases, the general solution $Q_1^2$ of (5.3) is single-valued, with in addition $Q_1$ single-valued in the 1:6:8 case.

## 6 Strategies to perform the explicit integration

In order to find the general solution in closed form for each of the seven cases one can think of three strategies. By decreasing order of elegance these are the following.

1. Take advantage of the knowledge of the second invariant $K$ (integrability in the Liouville sense) to find a canonical transformation to separating variables, i.e. to integrate in the Hamilton-Jacobi sense and then prove that the Hamilton's equations written for the separating variables have a single-valued general solution. This is the natural strategy, but is also the most difficult one.

2. To eliminate one of the two variables, say $q_2(t)$, between the two Hamilton's equations and the two constants of the motion and to identify one of the three resulting ODEs for, say $q_1(t)$, as a member in a list of ODEs already classified and integrated by classical authors like Chazy [11], Bureau [9] or Cosgrove [17, 18]. The main difficulty is that, since systems of two coupled second-order ODEs have not yet been classified, one must eliminate one of the two variables, which in the quartic case generates a nonpolynomial ODE such as (5.3), which has not yet been classified.

3. To establish a birational transformation between a classified ODE and one of the seven cases and then carry out the solution.

# 7 Integration of the HH3-KdV5 and HH4-1:2:1 cases

When the degree of $K$ is two, there exists a general method [44] to find the separating variables and we just recall its results for completeness.

## 7.1 The cubic case $\beta/\alpha = -6$ (KdV5)

Under the canonical transformation to parabolic coordinates [19, 2, 48],

$$(q_1, q_2, p_1, p_2) \to (s_1, s_2, r_1, r_2), \tag{7.1}$$
$$q_1 = -(s_1 + s_2 + \omega_1 - 4\omega_2)/(4\alpha),\ q_2^2 = -s_1 s_2/(4\alpha^2), \tag{7.2}$$
$$p_1 = -4\alpha \frac{s_1 r_1 - s_2 r_2}{s_1 - s_2},\ p_2^2 = -16\alpha^2 \frac{s_1 s_2 (r_1 - r_2)^2}{(s_1 - s_2)^2}, \tag{7.3}$$

the Hamiltonian takes the form

$$H = \frac{f(s_1, r_1) - f(s_2, r_2)}{s_1 - s_2}, \tag{7.4}$$
$$f(s, r) = -\frac{s^2(s + \omega_1 - 4\omega_2)^2(s - 4\omega_2) - 64\alpha^4\gamma}{32\alpha^2 s} + 8\alpha^2 r^2 s. \tag{7.5}$$

Therefore the Hamilton-Jacobi equation (2.1) allows the introduction of a separating constant $K$ identical to the second constant of the motion (4.13) so that

$$f(s_j, r_j) - E s_j + \frac{K}{2} = 0,\ j = 1, 2. \tag{7.6}$$

The transformed Hamilton's equations

$$s_1' = \frac{\partial H}{\partial r_1} = 16\alpha^2 \frac{s_1}{s_1 - s_2} r_1,\ s_2' = \frac{\partial H}{\partial r_2} = 16\alpha^2 \frac{s_2}{s_2 - s_1} r_2, \tag{7.7}$$

are equivalently written as

$$(s_1 - s_2)s_1' = \sqrt{P(s_1)},\ (s_2 - s_1)s_2' = \sqrt{P(s_2)}, \tag{7.8}$$
$$P(s) = s^2(s + \omega_1 - 4\omega_2)^2(s - 4\omega_2) + 32\alpha^2 E s^2 - 16\alpha^2 K s - 64\alpha^4\gamma, \tag{7.9}$$

called a *hyperelliptic system* of genus two. The variables $q_1$ and $q_2^2$ are meromorphic and the Hamiltonian system has the Painlevé property.

### 7.2 The quartic case 1:2:1

$$\begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\omega_1 q_1^2 + \omega_2 q_2^2) + \frac{1}{2}(q_1^4 + 2q_1^2 q_2^2 + q_2^4) \\ &\quad + \frac{1}{2}\left(\frac{\alpha}{q_1^2} + \frac{\beta}{q_2^2}\right) = E, \end{aligned} \tag{7.10}$$
$$\begin{aligned} K &= (q_2 p_1 - q_1 p_2)^2 + q_2^2 \frac{\alpha}{q_1^2} + q_1^2 \frac{\beta}{q_2^2} \\ &\quad - \frac{\omega_1 - \omega_2}{2}\left(p_1^2 - p_2^2 + q_1^4 - q_2^4 + \omega_1 q_1^2 - \omega_2 q_2^2 + \frac{\alpha}{q_1^2} - \frac{\beta}{q_2^2}\right). \end{aligned} \tag{7.11}$$

Quite similarly the canonical transformation to elliptic coordinates [49],

$$\begin{cases} q_j^2 = (-1)^j \dfrac{(s_1 + \omega_j)(s_2 + \omega_j)}{\omega_1 - \omega_2},\ j = 1, 2, \\ p_j = 2q_j \dfrac{\omega_{3-j}(r_2 - r_1) - s_1 r_1 + s_2 r_2}{s_1 - s_2},\ j = 1, 2, \end{cases} \tag{7.12}$$

maps the Hamilton's equations to the hyperelliptic system (7.8) with

$$\begin{aligned} P(s) &= s(s + \omega_1)^2(s + \omega_2)^2 - \alpha(s + \omega_2)^2 - \beta(s + \omega_1)^2 \\ &\quad -(s + \omega_1)(s + \omega_2)\left[E(2s + \omega_1 + \omega_2) - K\right]. \end{aligned} \tag{7.13}$$

We remark that the variable $x = q_1^2 + q_2^2$ obeys the fourth-order ODE

$$\begin{aligned} &x'''' + (20x + 4\omega_1 + 4\omega_2)x'' + 10x'^2 + 40x^3 \\ &+8(\omega_1 + \omega_2)(3x^2 - E) + (16\omega_1\omega_2 - E)x - 8(\alpha + \beta + K) = 0, \end{aligned} \tag{7.14}$$

which, up to some translation, is identical to the ODE (5.1) in the KdV5 case.

## 8 Integration of the cubic cases SK and KK

$$H_{\mathrm{SK}} = \frac{1}{2}(P_1^2 + P_2^2) + \frac{\Omega_1}{2}(Q_1^2 + Q_2^2) + \frac{1}{2}Q_1 Q_2^2 + \frac{1}{6}Q_1^3 + \frac{\lambda^2}{8}Q_2^{-2}, \tag{8.1}$$
$$H_{\mathrm{KK}} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega_2}{2}(16q_1^2 + q_2^2) + \frac{1}{4}q_1 q_2^2 + \frac{4}{3}q_1^3 + \frac{\lambda^2}{2}q_2^{-2}. \tag{8.2}$$

These two cases are equivalent under a birational canonical transformation [7, 40], which exchanges the two sets $(H, K, \Omega_1, \lambda^2)_{\mathrm{SK}}$ and $(H, K, \omega_2, \lambda^2)_{\mathrm{KK}}$. The two Hamilton-Jacobi equations are simultaneously separated as follows [38, 46].

1. One introduces the canonical transformation to Cartesian coordinates

$$\begin{cases} \tilde{Q}_1 = Q_1 + \Omega_1 + Q_2, \ \tilde{P}_1 = (P_1 + P_2)/2, \\ \tilde{Q}_2 = Q_1 + \Omega_1 - Q_2, \ \tilde{P}_2 = (P_1 - P_2)/2, \end{cases} \tag{8.3}$$

which trivially separates $H_{\mathrm{SK}}$ for $\lambda = 0$,

$$\lambda = 0: \ H_{\mathrm{SK}} = \tilde{P}_1^2 + \tilde{P}_2^2 + \frac{1}{12}(\tilde{Q}_1^3 + \tilde{Q}_2^3) - 4\Omega_1^2(\tilde{Q}_1 + \tilde{Q}_2). \tag{8.4}$$

2. One then applies to $H_{\mathrm{KK}}$ two canonical transformations, firstly the transformation $(q_j, p_j) \to (Q_j, P_j)$ taken for $\lambda = 0$ and secondly the rotation (8.3), which results in

$$q_1 = -6\left(\frac{\tilde{P}_1 - \tilde{P}_2}{\tilde{Q}_1 - \tilde{Q}_2}\right)^2 - \frac{\tilde{Q}_1 + \tilde{Q}_2}{2}, \ q_2^2 = 24\frac{f(\tilde{Q}_1, \tilde{P}_1) - f(\tilde{Q}_2, \tilde{P}_2)}{\tilde{Q}_1 - \tilde{Q}_2}, \tag{8.5}$$

$$p_1 = -4\tilde{Q}_1\frac{\tilde{P}_1 - \tilde{P}_2}{\tilde{Q}_1 - \tilde{Q}_2} - 2\frac{\tilde{Q}_1\tilde{P}_2 - \tilde{Q}_2\tilde{P}_1}{\tilde{Q}_1 - \tilde{Q}_2}, \ p_2 = \tilde{Q}_2\frac{\tilde{P}_1 - \tilde{P}_2}{\tilde{Q}_1 - \tilde{Q}_2}, \tag{8.6}$$

$$H_{\mathrm{KK}} = f(\tilde{Q}_1, \tilde{P}_1) + f(\tilde{Q}_2, \tilde{P}_2) + \frac{\lambda^2}{24}\frac{\tilde{Q}_1 - \tilde{Q}_2}{f(\tilde{Q}_1, \tilde{P}_1) - f(\tilde{Q}_2, \tilde{P}_2)}, \tag{8.7}$$

$$f(q, p) = p^2 + \frac{1}{12}q^3 - 4\omega_2^2 q. \tag{8.8}$$

Therefore both Hamilton-Jacobi equations are separated [46], viz.

$$(f(\tilde{Q}_j, \tilde{P}_j) - E/2)^2 + \lambda^2\tilde{Q}_j/24 + K = 0, \ j = 1, 2, \tag{8.9}$$

with $K$ the second integral of the motion (4.12) or (4.14). In the particular case $\lambda = 0$, the Hamiltonians themselves are separated [38].

3. Finally the Hamilton's equations in the variables $(\tilde{Q}_1, \tilde{Q}_2, \tilde{P}_1, \tilde{P}_2)$ are identified [46] to a hyperelliptic system of the canonical form (7.8),

$$\begin{cases} \tilde{Q}_1 = s_1^2 - \dfrac{3K}{\lambda^2}, \ \tilde{Q}_2 = s_2^2 - \dfrac{3K}{\lambda^2}, \ \tilde{P}_1 = \dfrac{r_1}{2s_1}, \ \tilde{P}_2 = \dfrac{r_2}{2s_2}, \\ P(s) = -\dfrac{1}{3}\left(s^2 - 3\dfrac{K}{\lambda^2}\right)^3 + \Omega_1^2\left(s^2 - 3\dfrac{K}{\lambda^2}\right) + \dfrac{\lambda}{\sqrt{3}}s + 2E, \end{cases} \tag{8.10}$$

thus providing the meromorphic general solution

$$\begin{cases} q_1 = -\dfrac{s_1^2 + s_2^2}{2} - \dfrac{3}{2}\left(\dfrac{s_1' + s_2'}{s_1 + s_2}\right)^2 + \dfrac{3K}{\lambda^2}, \\ q_2^{-2} = \dfrac{s_1 + s_2}{2\sqrt{3}\lambda}, \\ Q_1 = \sqrt{3}(s_1' + s_2') + s_1^2 + s_2^2 + s_1 s_2 - \dfrac{3K}{\lambda^2}, \\ Q_2^2 = -2\sqrt{3}(s_1 + s_2)(s_1 s_1' + s_2 s_2') + 2(s_1 + s_2)^2\left(s_1^2 + s_2^2 - \dfrac{9K}{2\lambda^2}\right). \end{cases} \tag{8.11}$$

*Remark.* Cosgrove [17] was the first to obtain the above hyperelliptic expressions for $q_1$ and $Q_1$, by a direct integration of the fourth-order ODE (5.1) in the KK and SK cases. They are respectively denoted F-III and F-IV in his classification and $\lambda^2$ is a first integral of the ODE. Therefore setting $\lambda = 0$ would prevent finding its general solution.

## 9 Integration of the quartic 1:6:1 and 1:6:8 cases

$$1:6:1 \left\{ \begin{array}{l} H = \frac{1}{2}(P_1^2 + P_2^2) + \frac{\omega_1}{2}(Q_1^2 + Q_2^2) - \frac{1}{32}(Q_1^4 + 6Q_1^2Q_2^2 + Q_2^4) \\ \quad - \frac{1}{2}\left(\frac{\kappa_1^2}{Q_1^2} + \frac{\kappa_2^2}{Q_2^2}\right) = E, \\ K = \left(P_1P_2 + Q_1Q_2\left(-\frac{Q_1^2 + Q_2^2}{8} + \omega_1\right)\right)^2 \\ \quad - P_2^2\frac{\kappa_1^2}{Q_1^2} - P_1^2\frac{\kappa_2^2}{Q_2^2} + \frac{1}{4}\left(\kappa_1^2 Q_2^2 + \kappa_2^2 Q_1^2\right) + \frac{\kappa_1^2\kappa_2^2}{Q_1^2Q_2^2} \end{array} \right. \tag{9.1}$$

and

$$1:6:8 \left\{ \begin{array}{l} H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega_2}{2}(4q_1^2 + q_2^2) - \frac{1}{16}(8q_1^4 + 6q_1^2q_2^2 + q_2^4) \\ \quad + \gamma q_1 + \frac{\beta}{2q_2^2} = E, \\ K = \left(p_2^2 - \frac{q_2^2}{16}(2q_2^2 + 4q_1^2 + \omega_2) + \frac{\beta}{q_2^2}\right)^2 - \frac{1}{4}q_2^2(q_2p_1 - 2q_1p_2)^2 \\ \quad + \gamma\left(-2\gamma q_2^2 - 4q_2p_1p_2 + \frac{1}{2}q_1q_2^4 + q_1^3q_2^2 + 4q_1p_2^2 - 4\omega_2q_1q_2^2 + 4q_1\frac{\beta}{q_2^2}\right). \end{array} \right. \tag{9.2}$$

The situation is similar to that for the cubic SK and KK cases. There is a canonical transformation [5] between the 1:6:1 and 1:6:8 cases which maps the constants as follows

$$H_{1:6:8} = H_{1:6:1},\ K_{1:6:8} = K_{1:6:1},\ \omega_2 = \omega_1,\ \gamma = \frac{\kappa_1 + \kappa_2}{2},\ \beta = -(\kappa_1 - \kappa_2)^2. \tag{9.3}$$

However, the separating variables have only been found for $\beta\gamma = 0$. In the case $\beta = \gamma = 0$ [8, 2] the canonical transformation,

$$\left\{ \begin{array}{l} \tilde{Q}_1 = \frac{1}{2}(Q_1 + Q_2)^2,\ \tilde{Q}_2 = \frac{1}{2}(Q_1 - Q_2)^2, \\ \tilde{P}_1 = \frac{P_1 + P_2}{2(Q_1 + Q_2)},\ \tilde{P}_2 = \frac{P_1 - P_2}{2(Q_1 - Q_2)}, \end{array} \right. \tag{9.4}$$

separates the Hamiltonian $H_{1:6:1}$

$$\kappa_1 = \kappa_2 = 0:\ H_{1:6:1} = f(\tilde{Q}_1, \tilde{P}_1) + f(\tilde{Q}_2, \tilde{P}_2),\ f(q,p) = 2qp^2 - \frac{1}{16}q^2 + \frac{\omega_1}{2}q \tag{9.5}$$

and leads to elliptic functions for $Q_1, Q_2, q_1, q_2^2$. In the generic case the best achievement to date for the separating variables [39] is to proceed as in the cubic SK-KK case. After applying two canonical transformations, firstly the transformation $(q_j, p_j) \to (Q_j, P_j)$ taken for $\beta = \gamma = \kappa_1 = \kappa_2 = 0$ and secondly the transformation (9.4), the Hamilton-Jacobi equation $H_{1:6:8} - E = 0$ becomes

$$\left\{ \begin{array}{l} g(\tilde{Q}_1, \tilde{P}_1) - g(\tilde{Q}_2, \tilde{P}_2) - \gamma\sqrt{\tilde{Q}_1\tilde{Q}_2}\frac{\tilde{P}_1 - \tilde{P}_2}{\tilde{Q}_1 - \tilde{Q}_2}\left(f(\tilde{Q}_1, \tilde{P}_1) - f(\tilde{Q}_2, \tilde{P}_2)\right) = 0, \\ g(q,p) = \frac{1}{4}f(q,p)^2 - Ef(q,p) + \frac{\beta}{8}q,\ f(q,p) = 2qp^2 - \frac{1}{16}q^2 + \frac{\omega_1}{2}q, \end{array} \right. \tag{9.6}$$

i.e., it separates only for $\gamma = 0$. The Hamilton's equations in the variables $(\tilde{Q}_1, \tilde{Q}_2, \tilde{P}_1, \tilde{P}_2)$ can then be identified [45] to a hyperelliptic system of the canonical form (7.8)

$$\gamma = 0 : \begin{cases} \tilde{Q}_1 = s_1^2 - \dfrac{K}{2\kappa_1^2},\ \tilde{Q}_2 = s_2^2 - \dfrac{K}{2\kappa_1^2}. \\ P(s) = \dfrac{1}{2}\left(s^2 - \dfrac{K}{2\kappa_1^2}\right)^3 - 4\omega_1^2\left(s^2 - \dfrac{K}{2\kappa_1^2}\right)^2 + \left(4E + 2\sqrt{2}\kappa_1 s\right)\left(s^2 - \dfrac{K}{2\kappa_1^2}\right) \end{cases} \tag{9.7}$$

and thus provides the meromorphic general solution

$$\gamma = 0 :\ q_1^2 = -\frac{s_1^2 + s_2^2}{2} + \left(\frac{s_1' + s_2'}{s_1 + s_2}\right)^2 - \frac{2\sqrt{2}\kappa_1}{s_1 + s_2} + \frac{K}{2\kappa_1^2} + 4\omega_1,\ q_2^2 = \frac{4\sqrt{2}\kappa_1}{s_1 + s_2}. \tag{9.8}$$

In the generic case $\beta\gamma \neq 0$ the second strategy (see section 6) cannot be used since the ODE (5.3) belongs to a class not yet investigated for the Painlevé property. Fortunately the third strategy succeeds in performing the integration and one can establish a birational transformation between the ODE (5.3) and the autonomous F-VI equation (a-FVI) in the classification of Cosgrove [17], viz.

$$\text{a-F-VI} :\ y'''' = 18yy'' + 9y'^2 - 24y^3 + \alpha_{\mathrm{VI}} y^2 + \frac{\alpha_{\mathrm{VI}}^2}{9} y + \kappa t + \beta_{\mathrm{VI}},\ \kappa = 0, \tag{9.9}$$

an ODE the general solution of which is meromorphic and expressed with genus two hyperelliptic functions [17, Eq. (7.26)]. The principle, explained in [35, 47], is to remark that the 1:6:8 Hamilton's equations and the a-F-VI ODE are the traveling wave reduction of two soliton systems linked by a Bäcklund transformation (BT). These are, respectively, the coupled KdV system denoted c-KdV$_1$ [6, 5], viz.

$$\begin{cases} f_\tau + \left(f_{xx} + \dfrac{3}{2} f f_x - \dfrac{1}{2} f^3 + 3fg\right)_x = 0, \\ -2g_\tau + g_{xxx} + 6gg_x + 3fg_{xx} + 6gf_{xx} + 9f_x g_x - 3f^2 g_x \\ \qquad + \dfrac{3}{2} f_{xxxx} + \dfrac{3}{2} f f_{xxx} + 9 f_x f_{xx} - 3f^2 f_{xx} - 3f f_x^2 = 0, \end{cases} \tag{9.10}$$

and another system of the c-KdV type, denoted bi-SH system [20, 41, 31, 21],

$$\begin{cases} -2u_\tau + \left(u_{xx} + u^2 + 6v\right)_x = 0, \\ v_\tau + v_{xxx} + uv_x = 0. \end{cases} \tag{9.11}$$

This BT is defined by the Miura transformation

$$\begin{cases} u = \dfrac{3}{2}\left(2g - f_x - f^2\right), \\ \\ v = \dfrac{3}{4}\left(2f_{xxx} + 4ff_{xx} + 8gf_x + 4fg_x + 3f_x^2 - 2f^2 f_x - f^4 + 4gf^2\right). \end{cases} \tag{9.12}$$

Under the reduction $x - c\tau = t$ the Bäcklund transformation between the two PDE systems becomes a birational transformation between 1:6:8 and the a-F-VI equation, see details in

[16]. The result is a meromorphic general solution $Q_1^2, Q_2^2, q_1, q_2^2$, rationally expressed as

$$\left\{\begin{aligned} & q_1 = \frac{W'}{2W} + \frac{\gamma}{W}\left[9j - 3\left(y + \frac{4}{9}\omega_2\right)(h+E) - \frac{9}{4}\gamma^2\right], \\ & q_2^2 = -16\left(y - \frac{5}{9}\omega_2\right) \\ & \quad + \frac{1}{W}\Bigg[12\left(y' + \frac{\gamma}{2}\right)^2 - 48y^3 - 16\omega_2 y^2 + \left(24E + \frac{128}{9}\omega_2^2\right)y + \frac{1280}{243}\omega_2^3 \\ & \qquad - \frac{40}{3}\omega_2 E + \frac{3}{4}\beta - 24\gamma\left(y - \frac{5}{9}\omega_2\right)h' - 144\gamma^2\left(y - \frac{5}{9}\omega_2\right)^2\Bigg], \\ & W = (h+E)^2 - 9\gamma^2\left(y - \frac{5}{9}\omega_2\right), \\ & \alpha_{\rm VI} = 4\omega_2,\ \beta_{\rm VI} = \frac{3}{4}\gamma^2 + 2\omega_2 E - \frac{3}{16}\beta - \frac{512}{243}\omega_2^3, \end{aligned}\right. \tag{9.13}$$

in which $h$ and $j$ are convenient auxiliary variables [17, Eqs. (7.4)–(7.5)],

$$\left\{\begin{aligned} & h = y'' - 6y^2 - \frac{4}{3}\omega_2 y + \frac{16}{27}\omega_2^2, \\ & j = \left(y - \frac{2}{9}\omega_2\right)y'' - \frac{1}{2}y'^2 - 4y^3 \\ & \quad + \frac{1}{6}\left(4\omega_2 y^2 + \frac{16}{9}\omega_2^2 y - \frac{512}{243}\omega_2^3 + 2\omega_2 E + \frac{3}{4}\gamma^2 - \frac{3}{16}\beta\right). \end{aligned}\right. \tag{9.14}$$

This shows that $q_1^2$ in (9.8) is the square of the single-valued expression

$$\gamma = 0: \ q_1 = \frac{h'}{h+E},\ q_2^2 = \frac{E}{s_1 + s_2}. \tag{9.15}$$

Contrary to previous cases the coefficients of the hyperelliptic curve [17, Eq. (7.23)] depend algebraically [15] on the parameters $\beta, \gamma, \kappa_1, \kappa_2$ of the Hamiltonians, and this could explain the difficulty to separate the variables in the Hamilton-Jacobi equation. Note that, in the particular case $\beta\gamma = 0$, i.e. $\kappa_1^2 = \kappa_2^2$, these coefficients become rational as in (9.7).

## 10 Integration of the quartic 1:12:16 case

$$\left\{\begin{aligned} & H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega_1}{8}(4q_1^2 + q_2^2) - \frac{1}{32}(16q_1^4 + 12q_1^2 q_2^2 + q_2^4) \\ & \qquad + \frac{1}{2}\left(\frac{\alpha}{q_1^2} + \frac{\beta}{q_2^2}\right) = E, \\ & K = \left(8(q_2 p_1 - q_1 p_2)p_2 - q_1 q_2^4 - 2q_1^3 q_2^2 + 2\omega_1 q_1 q_2^2 - 8q_1\frac{\beta}{q_2^2}\right)^2 \\ & \qquad + \frac{32\alpha}{5}\left(q_2^4 + 10\frac{q_2^2 p_2^2}{q_1^2}\right). \end{aligned}\right. \tag{10.1}$$

Up to now separating variables are only known in the case $\alpha\beta = 0$. The case $\alpha = 0$ belongs to the Stäckel class (two invariants quadratic in $p_1, p_2$). Under the canonical transformation to parabolic coordinates,

$$q_1 = s_1 + s_2,\ q_2^2 = -4s_1 s_2,\ p_1 = \frac{s_1 r_1 - s_2 r_2}{s_1 - s_2},\ p_2 = q_2\frac{r_1 - r_2}{2(s_1 - s_2)}, \tag{10.2}$$

the Hamilton-Jacobi equation is separated and $(s_1, s_2)$ obey the system (7.8) with

$$\alpha = 0: \; P(s) = s^6 - \omega_1 s^3 + 2Es^2 + \frac{K}{20}s - \frac{\beta}{4}. \tag{10.3}$$

In the case $\beta = 0$ the Hamilton-Jacobi equation is separated [45] by two successive canonical transformations which yield a similar hyperelliptic curve

$$\beta = 0: \; P(s) = s^6 - \omega_1 s^3 + 2Es^2 + \frac{K}{20}s - \alpha. \tag{10.4}$$

In the generic case $\alpha\beta \neq 0$, following the third strategy, one has found [5, 45, 35] a path the segments of which are either traveling wave reductions or Bäcklund transformations, linking the 1:12:16 Hamiltonian to a hyperelliptic system of the canonical form (7.8) with the hyperelliptic curve (8.10), which separates both the cubic SK and KK cases. However, since the curve (8.10) contains neither (10.3) nor (10.4), this path is certainly not the optimal one to reach the separating variables. Nevertheless this proves the singlevaluedness of the general solution $q_1^2, q_2^2$, the explicit expression of which results from the product of the six pieces [5, 45, 47]

$$\left\{\begin{array}{l} q_1^2 = \frac{1}{5}(2R - 6S + \omega_1), \; q_2^2 = \frac{4}{5}(-3R - 4S + \omega_1), \\ R = W_1' - W_1^2, \; S = -W_2' - \frac{1}{2}W_2^2, \\ W_1 = \frac{Q_1}{2} + \frac{Q_2'}{Q_2} - \frac{K_3}{Q_2^2}, \; W_2 = \frac{Q_1}{2} - \frac{Q_2'}{Q_2} + \frac{K_3}{Q_2^2}, \; K_3 = \sqrt{-\alpha} + \frac{1}{2}\sqrt{-\beta}, \\ Q_1 = F, \; Q_2^2 = \frac{2}{5}\left(F' - 2F^2 - G + \omega_1\right), \\ F, G = \text{see below, Eqs. (10.7)}, \\ U = -3\left(y - \frac{\omega_1}{30}\right), \; V = -6y'' + 18y^2 - \frac{9}{5}\omega_1 y + \frac{1}{10}\omega_1^2 - \frac{3}{5}E, \end{array}\right. \tag{10.5}$$

where $y$ obeys the F-IV ODE [17], integrated with genus two hyperelliptic functions.

In the fifth line of (10.5) the expressions result from the inversion of the reduction $(u, v, f, g)(x, \tau) = (U, V, F, G)(x + \omega_1\tau)$ of the Miura transformation

$$u = \frac{3}{10}\left(3f_x - f^2 + 2g\right), \; v = \frac{9}{10}\left(f_{xxx} + g_{xx} + f_x g - f g_x - f f_{xx} + g^2\right), \tag{10.6}$$

i.e.,

$$\left\{\begin{array}{l} F = -\frac{W'}{2W} + K_{1,\mathrm{a}}X_2, \\ G = -F^2 - X_1X_2 + K_{1,\mathrm{a}}\frac{54U'}{X_1} - 54K_{1,\mathrm{a}}\left(U + \frac{3\omega_1}{20}\right)\frac{W'}{WX_1} + \frac{2}{3}\left(U + \frac{9\omega_1}{10}\right), \\ W = X_1^2 + 108K_{1,\mathrm{a}}^2\left(U + \frac{3\omega_1}{20}\right), \\ X_1 = V + 2U^2 - 3\omega_1 U + \frac{9}{50}\omega_1^2 - \frac{27}{5}E, \\ X_2 = 9\left(-4U'^2 + \frac{8}{3}UV - \frac{8}{25}\omega_1 U^2 + \frac{2}{5}\omega_1 V + \frac{48}{5}EU\right. \\ \qquad\qquad \left. - \frac{42}{25}\omega_1^2 U + \frac{9}{8}(4\alpha + \beta) - \frac{9}{2}K_{1,\mathrm{a}}^2 + \frac{36}{25}\omega_1 E - \frac{27}{125}\omega_1^3\right). \\ K_{1,\mathrm{a}} = \sqrt{-\alpha} - \frac{1}{2}\sqrt{-\beta}. \end{array}\right. \tag{10.7}$$

# 11 Conclusion and open problems

The present results are twofold.

1. In the seven cases the general solution is reducible to a canonical hyperelliptic system with genus two and therefore meromorphic.

2. Since each of the seven cases can be mapped to a fourth-order ODE which is complete in the Painlevé sense, it is impossible to add any term to the Hamiltonian without destroying the Painlevé property. The seven Hénon-Heiles Hamiltonians are *complete*.

The main open problems are to find the separating variables in three of the generic quartic cases.

## Acknowledgments

The authors acknowledge the financial support of the Tournesol grant no. T2003.09 between Belgium and France. Our thanks also go to the referee who helped us to improve the text.

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