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Some Remarks on Materials with Memory: Heat Conduction and Viscoelasticity

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Abstract

Materials with memory are here considered. The introduction of the dependence on time not only via the present, but also, via the past time represents a way, alternative to the introduction of possible non linearities, when the physical problem under investigation cannot be suitably described by any linear model. Specifically, the two different models of a rigid heat conductor, on one side, and of a viscoelastic body, on the other one, are analyzed. In them both, to evaluate the quantities of physical interest a key role is played by the past history of the material and, accordingly, the behaviour of such materials is characterized by suitable constitutive equations where Volterra type kernels appear. Specifically, in the heat conduction problem, the heat flux is related to the history of the temperature-gradient while, in isothermal viscoelasticity, the stress tensor is related to the strain history. Then, the notion of equivalence is considered to single out and associate together all those different thermal histories, or, in turn, strain histories, which produce the same work. The corresponding explicit expressions of the minimum free energy are compared.

1 Introduction

The introduction of a Volterra type integral to model a rigid heat conductor in such a way to avoid the infinte speed heat propagation goes back to Cattaneo [3], who suggested a new generalized Fourier's law which linearly relates the heat flux, its time derivative and the temperature-gradient. Subsequently, Coleman's results [4] concerning materials with memory, induced Gurtin and Pipkin [20], to proposed a non-linear model, as well as a linearized version of it, which generalizes the previous ones.

Among the many results, since then, in the study of heat transfer phenomena, those more directly of interest in connection to the present approach have been obtained by Gurtin [19] and by Coleman and Dill [5]. They studied properties of free energy functionals in the case of materials with memory and, subsequently, Giorgi and Gentili [16] investigated the heat conduction problem in a fading memory material. The thermodynamical model here adopted is the same studied by Fabrizio, Gentili and Reynolds [11], who considered the thermodynamics of a rigid homogeneous linear heat conductor with memory. On the other hand, as far as the viscoelasticity problem is concerned, in materials with memory, here Noll's definition of state [23] is adopted. Indeed Noll takes the material response as the basis for the definition of state: if an arbitrary continuation of different given histories leads to the same response of the material, then the given histories are equivalent and the state is represented as the class of all the equivalent histories. Accordingly, Fabrizio and Golden [12] introduced the notion of the minimal state. This idea has been applied in [18] to the case of linear viscoelasticity with scalar relaxation functions given by a sum of exponentials.

The material in the present work is organized as follows. The opening Section 2 is devoted to briefly recall the model, according to [11], of a rigid heat conductor with memory. Specifically, the *thermodynamic state function* is defined and, subsequently, the meaning of what termed *process*, *process prolongation* and some related quantities, are exploited. A key role is played by the notion of equivalence between different thermal histories, stated on identification of all those ones associated to the same heat flux.

The thermal work associated to a process is considered in the subsequent Section 3 where results obtained in [2] are recalled. In particular, given two different, but equivalent, thermal processes, the thermal work related to them both is the same (and viceversa). Hence, given a thermal history, the concept of equivalence can be stated in terms of the thermal work instead than of the heat flux. A representation of the functional space of all finite thermal work processes is given.

The subsequent Section 4, is concerned about an introduction on the model of isothermal linear viscoelastic solid with memory according to the approach widely exposed by Fabrizio and Morro in [14]. Some remarks on the comparison between the heat conduction and the viscoelastic cases are also comprised.

The subject of the subsequent Section 5 is the definition of process, process prolongation and, hence, of equivalence in terms of the strain history associated to each process. Accordingly, [15], the same strain history corresponds to equivalent viscoelastic processes. Then, the viscoelastic work is written, under the form obtained by Gentili [15]; it is compared with that one [2], obtained in the case of the thermal work functional. An interesting connection between thermodynamics of rigid bodies and isothermal viscoelasticity appears. Indeed, not only the work functionals here considered, but also the minimum free energies assume, in the two different cases, expressions which show many analogies. The comparison between them follows when recent investigations obtained in the framework of linear isothermal viscoelasticity are recalled. In particular, the results of Deseri, Gentili and Golden [8], Fabrizio and Golden [12] [13], Golden [17], Gentili [15], and Fabrizio [10] are here referred to. Furthermore, it should be remarked that Gurtin and Pipkin in [20] already pointed out analytical analogies between the two models of heat conduction with memory, on one side, and of longitudinal motion of a viscoelastic bar, on the other one; indeed, the integro-differential equation which follows from the energy equation is of the same form in both cases.

An Appendix comprises some of the background definitions required throughout the previous Sections.

2 Heat conduction in a rigid solid

This section is devoted to describe the model of a rigid heat conductor with memory. In particular, the approach presented by Fabrizio, Gentili and Reynolds in [11], and, subsequently, in [2] is adopted. Accordingly, the key notions comprised in [2] are briefly reviewed.

The simplifying assumption that the internal energy and the relative temperature are linearly related is adopted. That is, the internal energy e is assumed of the form

$$e(\mathbf{x},t) = \alpha_0 u(\mathbf{x},t) \quad , \tag{2.1}$$

where α_0 represents the *energy relaxation function*, here assumed to be constant, $\mathbf{x} \in \mathbb{R}^3$ denotes the position within the conductor[1], $t \in \mathbb{R}^+$ denotes the time variable[2], and $u := \theta - \theta_0$ the temperature difference with respect to a fixed reference temperature θ_0. The heat flux $\mathbf{q} \in \mathbb{R}^3$, when the *heat flux relaxation function* denoted by $k(\mathbf{x},t)$, is assumed to satisfy the constitutive equation

$$\mathbf{q}(\mathbf{x},t) - \int_0^\infty k(\mathbf{x},\tau)\nabla\theta(\mathbf{x},t-\tau)\, d\tau \quad , \tag{2.2}$$

which shows that the heat flux $\mathbf{q}$ depends on the time variable not only via the *present* time t, but also via its past *history*. In addition, according to [11] and [2], the rigid heat conductor is understood to be an isotropic material, and, hence, focussing the attention on a generic element of the conductor, no dependence on the position in the conductor is considered. Then, the energy e, the heat flux $\mathbf{q}$, the temperature θ and, consequently, all the other quantities are represented by functions of the time variable alone. Thus, formulae (2.1, 2.2) can be rewritten omitting the $\mathbf{x}-$dependence; and, in particular, the heat flux relaxation function, which reads $k(t)$, can be represented by

$$k(t) = k_0 + \int_0^t \dot{k}(s)\, ds \quad , \tag{2.3}$$

where $k_0 \equiv k(0)$ denotes the initial (positive) value of the heat flux relaxation function, thus termed *initial heat flux relaxation coefficient*. It is further required that

$$\dot{k} \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \qquad \textit{and} \qquad k \in L^1(\mathbb{R}^+) \tag{2.4}$$

which imply $k(\infty) := \lim_{t\to\infty} k(t) = 0$. These assumptions can be physically interpreted recalling that there is no heat flux when, at infinity, the thermal equilibrium is reached.

The constitutive equations (2.1) and (2.2) can be re-written, in turn, as follows

$$e(t) = \alpha_0 \left[\theta(t) - \theta_0\right] \tag{2.5}$$

and, when the heat flux relaxation function k satisfies both conditions (2.3) and (2.4), the heat flux $\mathbf{q}$ can be written in the following two equivalent forms:

$$\mathbf{q}(t) - \int_0^\infty k(\tau)\mathbf{g}(t-\tau)\, d\tau \quad \textit{or} \quad \mathbf{q}(t) \int_0^\infty \dot{k}(\tau)\bar{\mathbf{g}}^t(\tau)\, d\tau; \tag{2.6}$$

[1]more precisely, it should be required that $\mathbf{x} \in \mathcal{B} \subset \mathbb{R}^3$ where $\mathcal{B}$ denotes the bounded closed set in $\mathbb{R}^3$ which represents the configuration domain of the conductor, here not specified since of no interest in the present study.

[2]**notation remark:** throughout the whole paper $\mathbb{R}^+ := [0,\infty)$ while $\mathbb{R}^{++} := (0,\infty)$.

where $\mathbf{g} := \nabla\theta$ denotes the temperature-gradient, and

$$\bar{\mathbf{g}}^t(\tau)\int_{t-\tau}^{t} \mathbf{g}(s)\ ds \tag{2.7}$$

represents the integrated history of the temperature-gradient. The thermodynamical state of the conductor is characterized when, according to [11], [22] and [2], the *thermodynamic state function* $\sigma : \mathbb{R} \to \mathbb{R}\times\mathbb{R}^3$ which associates $t \longmapsto \sigma(t) \equiv \big(\theta(t), \bar{\mathbf{g}}^t\big)$ is given; that is, when the temperature and the integrated history of the temperature-gradient, $\bar{\mathbf{g}}^t$ which belong to a suitable Hilbert space, are assigned. In addition, physically admissible thermodynamical phenomena, namely associated to finite heat flux, are those ones which belong to the following vectorial space

$$\Gamma := \left\{ \bar{\mathbf{g}}^t : (0,\infty) \to \mathbb{R}^3 : \ \left| \int_0^\infty \dot{k}(s+\tau)\, \bar{\mathbf{g}}^t(s)\ ds \right| < \infty \ \ , \ \ \forall\, \tau \geq 0 \right\} . \tag{2.8}$$

According to [15], the property $\dot{k} \in L^1(\mathbb{R})$ is termed *fading memory*: it implies that corresponding to any arbitrary $\varepsilon > 0$ there exists a positive constant $\tilde{a} = a(\varepsilon, \bar{\mathbf{g}}^t)$ such that

$$\left| \int_0^\infty \dot{k}(s+a)\bar{\mathbf{g}}^t(s)\ ds \right| < \varepsilon \ \Leftrightarrow \left| \int_0^\infty k(s+a)\mathbf{g}(t-s)\ ds \right| < \varepsilon \ , \ \ \forall\, a > \tilde{a}\, , \tag{2.9}$$

whose thermodynamical meaning is that, when the conductor is approaching its asymptotic thermal equilibrium, the heat flux becomes smaller and smaller in agreement with the prescription of no heat flux at equilibrium.

Hence, the set of all possible heat fluxes, corresponding to different integrated histories of the temperature-gradient, can be represented via the linear functional

$$\widetilde{Q}\{\bar{\mathbf{g}}^t\} := \int_0^\infty \dot{k}(s)\, \bar{\mathbf{g}}^t(s)\ ds \implies \forall T > 0, \ \ \widetilde{Q}\{\bar{\mathbf{g}}^{t+T}\}\int_0^\infty \dot{k}(s)\, \bar{\mathbf{g}}^{t+T}(s)\ ds, \tag{2.10}$$

and, by definition (2.7), when an integrated history of the temperature-gradient $\bar{\mathbf{g}}^t$ is choosen, the corresponding heat flux at time $t+T$ follows

$$\mathbf{q}(t+T) : \int_0^\infty \dot{k}(s)\, \bar{\mathbf{g}}^{t+T}(s)\ ds\ . \tag{2.11}$$

The definitions of process and of the set of admissible states, given in [11], see also [2], are here only briefly recalled. An integrable function $P : [0,T) \to \mathbb{R}\times\mathbb{R}^3$ such that $P(\tau) = \Big(\dot{\theta}_P(\tau), \mathbf{g}_P(\tau)\Big) \quad \forall\ \tau \in [0,T)$ is termed *process of duration* $T > 0$. Accordingly, a process P of duration $T < \infty$ is known when the two applications $\dot{\theta}_P : [0,T) \to \mathbb{R}$ and $\mathbf{g}_P \ : [0,T) \to \mathbb{R}^3$ are assigned, and thus the state function $\sigma(t)$ follows via integration from the initial time $t = 0$ to the generic time t. Indeed, as pointed out in [11], given its initial value $\sigma(0) = \big(\theta_\star(0), \bar{\mathbf{g}}^0_\star\big)$, where, in turn, $\theta_\star(0)$ denotes the temperature, and $\bar{\mathbf{g}}^0_\star$ the integrated history of the temperature-gradient at time $t = 0$, then a process P delivers the state function $\sigma(t) = \big(\theta(t), \bar{\mathbf{g}}^t\big)$,$\forall\ t \in [0,T)$, defined by

$$\theta(t)\theta_\star(0) + \int_0^t \dot{\theta}_P(\xi)\ d\xi \ \ , \quad \bar{\mathbf{g}}^t(s) = \begin{cases} \displaystyle\int_{t-s}^{t} \mathbf{g}_P(\xi)\ d\xi & 0 \leq s < t \\ \displaystyle\int_0^t \mathbf{g}_P(\xi)\ d\xi + \bar{\mathbf{g}}^0_\star(s-t) & s \geq t\ , \end{cases} \tag{2.12}$$

which is continuous at $s = t$. A key tool is represented by the definition of "prolongation" (denoted via $\star$) [2] of a generic process of known history, via a given one, characterized by an assigned $\mathbf{g}_P$. According to [2], a prolongation of the integrated history, which is continuous at $s = T$, and hence $\left(\mathbf{g}_P \star \bar{\mathbf{g}}_i\right)^T(T) = \bar{\mathbf{g}}_P^T(T)$, is introduced as follows.

Definition 1. Given a process P, hence assigned $\mathbf{g}_P$, its duration T and a set of integrated histories of the temperature-gradient $\bar{\mathbf{g}}_i^t(s)$, corresponding to $\mathbf{g}_i^t(s),\ i = 1, \ldots, n,\ n \in \mathbb{N}$, then the prolongation of the history $\bar{\mathbf{g}}_i^t(s)$, induced by the process P, is defined by

$$\left(\mathbf{g}_P \star \bar{\mathbf{g}}_i\right)^{t+T}(s) := \begin{cases} \bar{\mathbf{g}}_P^T(s) = \displaystyle\int_{T-s}^{T} \mathbf{g}_P(\xi)\ d\xi & 0 \leq s < T \\ \bar{\mathbf{g}}_P^T(T) + \bar{\mathbf{g}}_i^t(s-T) = \displaystyle\int_0^T \mathbf{g}_P(\xi)\ d\xi + \int_{t+T-s}^{t} \mathbf{g}_i(\xi)\ d\xi & s \geq T\ . \end{cases} \tag{2.13}$$

Then, as pointed out in [2], the notion of equivalence between two thermodynamical states can be established. Precisely,

Definition 2. two thermodynamical state functions assigned via $\sigma_1(t) = (\theta_1(t), \bar{\mathbf{g}}_1^t)$ and $\sigma_2(t)(\theta_2(t), \bar{\mathbf{g}}_2^t)$ are *equivalent* if $\forall\ \mathbf{g}_P\ : [0, T) \to \mathbb{R}^3\ ,\ \ \forall\ \ T > 0$, the corresponding heat flux is the same, namely $\widetilde{Q}\left\{\left(\mathbf{g}_P \star \bar{\mathbf{g}}_1\right)^{t+T}\right\} = \widetilde{Q}\left\{\left(\mathbf{g}_P \star \bar{\mathbf{g}}_2\right)^{t+T}\right\}$.
This definition, see [2] for details, can also be rewritten replacing the thermodynamical state functions with the integrated histories of the temperature-gradient, sometimes for short termed thermal histories. Thus, two different integrated histories of the temperature-gradient, or two thermodynamical states, are equivalent when, for any prolongation of any time duration, the same heat flux corresponds to both of them. Under a physical viewpoint, this equivalence is introduced exactly to find and join together all those states associated to the same heat flux, no matter the specific temperature evolution $\theta(t)$.

A detailed analysis is comprised in [2], where are considered both the case of a generic history as well as the special one, termed *zero history*, characterized by a constant temperature $\theta(t) = \theta_0$ and, hence, a zero temperature-gradient so that the corresponding integrated history of the temperature-gradient, denote as $\bar{\mathbf{g}}_0^t$, is also zero. Accordingly, the corresponding state function is $\sigma_0 : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^3$ which associates $t \longmapsto \sigma_0(t) \equiv (\theta_0, \mathbf{0})$. In particular, [2], a prolongation of the zero history via any assigned process, characterized by the duration τ, $\tau \leq T < \infty$ and $\mathbf{g}_P : [0, \tau) \to \mathbb{R}^3$, is given by

$$\left(\mathbf{g}_P \star \bar{\mathbf{g}}_0\right)^{t+\tau}(s) = \begin{cases} \bar{\mathbf{g}}_P^\tau(s) = \displaystyle\int_{\tau-s}^{\tau} \mathbf{g}_P(\xi)\ d\xi & 0 \leq s < \tau \\ \bar{\mathbf{g}}_P^\tau(\tau) & s \geq \tau\ . \end{cases} \tag{2.14}$$

The set of all admissible states, [2], is the set comprising all those states which correspond to a finite heat flux, namely

$$\Sigma := \left\{\sigma(t) \in \mathbb{R} \times \mathbb{R}^3\ :\ \left|\ \int_0^\infty \dot{k}(s)\, \bar{\mathbf{g}}^t(s) ds\right| < \infty \Leftrightarrow \left|\ \int_0^\infty k(s)\, \mathbf{g}(t-s) ds\right| < \infty\right\}, \tag{2.15}$$

Notably, if $\sigma \in \Sigma$, it follows that both the heat flux $\mathbf{q}$, as well as the internal energy e, given, in turn, by (2.6) and (2.1), are finite. Indeed, according to remarks in [11] and [2], those processes with a physical meaning, namely which admit associated finite thermodynamic potentials, are exactly those ones characterized by a state $\sigma(0) \in \Sigma$ whose time evolution $\sigma(t)$ belongs to Σ itself.

3 Thermal Work

This Section is devoted to show, according to the results obtained in [2], a convenient form in which the thermal work functional can be written. Remarkably, the thermal work associated to equivalent processes turns out to be the same.

The relevance of the concept of work in the study of any physical problem, and in particular a thermodynamical one [11], is well known. Thus, this Section is concerned about the notion of work, termed thermal work, in the case of the thermodynamical model under investigation. Thus, for later convenience, the following definition [2] is recalled.

Definition 3. Given any initial state, defined on assigning the state function $\sigma(t) = (\theta(t), \bar{\mathbf{g}}^t)$, and any prolongation process P of arbitrary finite duration τ, where $0 \le \tau < T$, $P(\tau) = \left(\dot{\theta}_P(\tau), \mathbf{g}_P(\tau)\right)$, then the thermal work associated to the time interval $[0, T)$ is represented by the functional

$$\widetilde{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\} := -\int_0^T \widetilde{Q}\left\{(\mathbf{g}_P \star \bar{\mathbf{g}})^{t+\tau}\right\} \cdot \mathbf{g}_P(\tau)\ d\tau \quad . \tag{3.1}$$

The value of the thermal work, $\widetilde{W}(\bar{\mathbf{g}}^t; \mathbf{g}_P)$, which corresponds to a specific choice of the initial state and of the prolongation process can be evaluated via

$$\widetilde{W}\left(\bar{\mathbf{g}}^t; \mathbf{g}_P\right) = -\int_0^T \mathbf{q}\,(t+\tau) \cdot \mathbf{g}_P(\tau)\ d\tau \tag{3.2}$$

where $\mathbf{q}\,(t+\tau)$, given by (2.11), represents the heat flux at time $t+\tau$.
In the case when the *zero history*, characterized by the state function $\sigma(t) \equiv (\theta_0, \mathbf{0})$, is prolonged via any assigned process P according to (2.14), then the thermal work functional is given by

$$\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\} = -\int_0^T \int_0^\infty \dot{k}(s)\,(\mathbf{g}_P \star \bar{\mathbf{g}}_0)^{t+\tau}(s)\ \cdot \mathbf{g}_P(\tau)\, ds d\tau \quad . \tag{3.3}$$

Different expressions of the thermal work functional can be obtained, according to the detailed computations in [2]. In particular, when an arbitrary process is assigned, i.e. $\mathbf{g}_P$ is known, the thermal work functional follows from the state to be prolonged: two different cases arise. They correspond to prolong via the assigned $\mathbf{g}_P\}$, in turn, the *zero history* characterized by the state function $\sigma_0 \equiv (\theta_0, \mathbf{0})$ (trivial prolongation), and any fixed state function $\sigma(t) \equiv \left(\theta(t), \bar{\mathbf{g}}^t\right)$ (generic prolongation). Convenient forms of the thermal work functional, for any given process P, and hence any $\mathbf{g}_P(\tau)$, and for all $0 \le \tau < T < \infty$, are, see [2], $W\{\mathbf{0}; \mathbf{g}_P\}$ in (3.3), when the zero history is prolonged and

$$\widetilde{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\} = \int_0^\infty \int_0^\infty \left[\frac{1}{2} k(|\tau - s|)\mathbf{g}_P(s) - \dot{k}(\tau + s)\bar{\mathbf{g}}^t(s)\right] \cdot \mathbf{g}_P(\tau)\, ds d\tau\ , \tag{3.4}$$

when the generic case is considered. Then, the introduction of

$$\mathbf{I}(\tau, \bar{\mathbf{g}}^t) := \int_0^\infty \dot{k}(\tau + s)\, \bar{\mathbf{g}}^t(s) ds \quad , \tag{3.5}$$

allows to write $\widetilde{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}$ as follows

$$\widetilde{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}=\int_0^\infty\left[\frac{1}{2}\int_0^\infty k(|\tau-s|)\mathbf{g}_P(s)ds-\mathbf{I}(\tau,\bar{\mathbf{g}}^t)\right]\cdot\mathbf{g}_P(\tau)d\tau\quad . \tag{3.6}$$

Furthermore, see [2], via the introduction of Fourier transforms, the thermal work functional $\widetilde{W}$ can be written

$$\widehat{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}=\frac{1}{2\pi}\left[\int_{-\infty}^{+\infty}k_c(\omega)\mathbf{g}_+(\omega)\cdot\overline{\mathbf{g}_+(\omega)}\,d\omega-\int_{-\infty}^{+\infty}\mathbf{I}_+(\omega)\cdot\overline{\mathbf{g}_+(\omega)}\,d\omega\right] \tag{3.7}$$

where $\mathbf{g}_+(\omega)$ and $\mathbf{I}_+(\omega)$, in turn, according to the usual definitions also recalled in the Appendix ((6.1), (6.3)), represent the Fourier Transforms

$$\mathbf{I}_+(\omega):=\int_0^\infty\mathbf{I}(\tau,\bar{\mathbf{g}}^t)\,e^{-i\omega\tau}d\tau\quad,\quad\mathbf{g}_+(\omega):=\int_0^\infty\mathbf{g}_P(\tau)\,e^{-i\omega\tau}d\tau\ , \tag{3.8}$$

of the non negative real valued functions $\mathbf{I}(\tau,\bar{\mathbf{g}}^t)$ and of $\mathbf{g}_P$, both defined over $\mathbb{R}^+$. In addition, $\overline{\mathbf{g}_+(\omega)}$ denotes the complex conjugate of $\mathbf{g}_+(\omega)$, and $k_c(\omega)$ the Fourier cosine transform, given by (6.1), of $k(|\tau-s|)$, which, by definition, is an even function of its real non negative argument since $k_c(0)>0$, as prescribed by the adopted assumptions [11] and the thermodynamical restrictions on the constitutive equation. Hence, the analytical form of $\widetilde{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}$ and $\widehat{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}$ suggests to introduce the following inner product

$$\langle f,\phi\rangle:\int_0^{+\infty}f(t)\cdot\phi(t)\ dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f_+(\omega)\cdot\overline{\phi_+(\omega)}\ d\omega\ , \tag{3.9}$$

whenever the integral is finite. It is clear, since Plancharel's theorem implies the identity of norms in spaces dual with respect to Fourier transforms, that $\widetilde{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}=\widehat{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}$; hence, $\widetilde{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}$ is finite whenever $\widehat{W}\{\bar{\mathbf{g}}^t;\mathbf{g}_P\}$ is finite; accordingly, inner products can be evaluated in both the spaces related to each other via duality under Fourier transform.

Definition 4. A process is termed *finite thermal work process* whenever the functional $\widetilde{W}\{\mathbf{0};\mathbf{g}_P\}$ is finite; namely, on account of (3.3), when

$$\widetilde{W}\{\mathbf{0};\mathbf{g}_P\}=\frac{1}{2}\int_0^\infty\int_0^\infty k(|\tau-s|)\,\mathbf{g}_P(s)\cdot\mathbf{g}_P(\tau)dsd\tau<\infty\quad . \tag{3.10}$$

That is, when the zero history is prolonged via an arbitrary finite thermal work process, then, the functional $\widetilde{W}\{\mathbf{0};\mathbf{g}_P\}$ is finite. It, on application of the duality via Fourier transform, can be also expressed as $\widehat{W}\{\mathbf{0};\mathbf{g}_P\}$, given by

$$\widehat{W}\{\mathbf{0};\mathbf{g}_P\}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}k_c(\omega)\,\mathbf{g}_+(\omega)\cdot\overline{\mathbf{g}_+(\omega)}\,d\omega. \tag{3.11}$$

Hence, the *set of all finite thermal work states* can be represented on introduction of the functional space

$$\mathcal{H}(\mathbb{R}^+,\mathbb{R}^3):\left\{\phi:\mathbb{R}^+\to\mathbb{R}^3:\left|\int_{-\infty}^{+\infty}k_c(\omega)\phi_+(\omega)\cdot\overline{\phi_+(\omega)}d\omega\right|<\infty\right\}. \tag{3.12}$$

Since $k_c(\omega)$ is a real even function, according to Section 2, which, since $k(0) = k_0$ is positive, and in agreement with [11] is assumed to be positive valued, then, the functional space $\mathcal{H}(\mathbb{R}^+, \mathbb{R}^3)$, can be equipped by the following inner product

$$\langle f, \phi \rangle_k : \int_{-\infty}^{+\infty} k_c(\omega)\, f_+(\omega) \cdot \overline{\phi_+(\omega)}\; d\omega \tag{3.13}$$

and by the induced *norm*

$$\|\phi\|_k := \langle \phi, \phi \rangle_k \int_{-\infty}^{+\infty} k_c(\omega)\, \phi_+(\omega) \cdot \overline{\phi_+(\omega)}\; d\omega \quad . \tag{3.14}$$

Hence, the functional space which comprises all admissible thermal states, denoted as $\mathcal{H}_k(\mathbb{R}^+, \mathbb{R}^3)$, can be characterized as the *completion with respect to the norm* $\|\cdot\|_k$ of the functional space $\mathcal{H}(\mathbb{R}^+, \mathbb{R}^3)$. Specifically, the following definition is adopted.

Definition 5. The set of all finite thermal work states is the set $\Sigma_0 \subset \Sigma$ comprising all those thermal states which correspond to a finite thermal work. Thus, according to this Definition, $\Sigma_0 \subset \Sigma$ represents a subset of the set Σ of admissible states, given by (2.15); in particular, Σ_0 comprises all those admissible states, i.e. all couples $(\theta(t), \bar{\mathbf{g}}^t(\tau)) \in \Sigma \subset \mathbb{R} \times \mathbb{R}^3$, such that, in addition, the thermal work $\widetilde{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\}$ is finite for any choice of the prolongation process P characterized by $\mathbf{g}_P \in \mathcal{H}_k(\mathbb{R}^+, \mathbb{R}^3)$.

Now, in the case of a generic thermal state, the corresponding thermal work (3.6) needs to be considered. Specifically, the boundedness of $\widetilde{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\}$ is required to characterize the admissibility of a generic thermal state. Comparison between $\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\}$ and $\widetilde{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\}$, given, respectively, by (3.10) and (3.6), shows that when $\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\}$ is bounded, then, the first term of $\widetilde{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\}$ is bounded too, and, accordingly, only the second one remains to be considered. Again, via Fourier transform, $\widehat{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\}$ in (3.7) is obtained; the latter compared with $\widehat{W}\{\mathbf{0}; \mathbf{g}_P\}$ shows that the boundedness condition is satisfied as soon as

$$\langle \mathbf{I}, \mathbf{g} \rangle : \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{I}_+(\omega) \cdot \overline{\mathbf{g}_+(\omega)}\, d\omega \ , \tag{3.15}$$

where $\mathbf{I}_+(\omega)$ is given in (3.8), is finite. Hence, when a generic thermal state is considered, it can be termed *admissible* whenever its prolongation via any process P produces a finite thermal work: precisely, the following definition is equivalent to the previous one.

Definition 6. The set of all admissible thermal states is the set of all states $\sigma(t) \in \Sigma_0$, such that, for any choice of the prolongation process P, such that $\mathbf{g}_P \in \mathcal{H}_k$, it follows that $\mathbf{I}_+(\omega)$ belongs to the space

$$\mathcal{H}'_k(\mathbb{R}^+, \mathbb{R}^3) : \left\{ f : \mathbb{R}^+ \to \mathbb{R}^3 \text{ s.t. } |\langle f, \phi \rangle_k| < \infty,\ \forall\, \phi \in \mathcal{H}_k(\mathbb{R}^+, \mathbb{R}^3) \right\} , \tag{3.16}$$

dual of $\mathcal{H}_k(\mathbb{R}^+, \mathbb{R}^3)$ with respect to the inner product $\langle \cdot, \cdot \rangle_k$ defined in (3.13). This definition implies that, $\forall\ \sigma(t) \in \Sigma_0$, it follows that $\mathbf{I}_+(\omega) \in \mathcal{H}'_k(\mathbb{R}^+, \mathbb{R}^3)$. Consequently, the notion of equivalence between two different states can be stated on the basis of the related thermal work.

Definition 7. Two integrated histories of the temperature-gradient $\bar{\mathbf{g}}_1^t$ and $\bar{\mathbf{g}}_2^t$, which correspond to the states $(\theta_1(t), \bar{\mathbf{g}}_1^t)$ and $(\theta_2(t), \bar{\mathbf{g}}_2^t)$, are termed *w-equivalent* if

$$\forall\ \mathbf{g}_P\ : [0,\tau) \to \mathbb{R}^3\ \ ,\ \ \forall\ \ \tau > 0\ \Longrightarrow\ \widetilde{W}\{\bar{\mathbf{g}}_1^t; \mathbf{g}_P\} = \widetilde{W}\{\bar{\mathbf{g}}_2^t; \mathbf{g}_P\}\ \ ; \tag{3.17}$$

that is, on application of Plancharel's theorem,

$$\forall\ \mathbf{g}_P\ : [0,\tau) \to \mathbb{R}^3\ \ ,\ \ \forall\ \ \tau > 0\ \Longrightarrow\ \widehat{W}\{\bar{\mathbf{g}}_1^t; \mathbf{g}_P\} = \widehat{W}\{\bar{\mathbf{g}}_2^t; \mathbf{g}_P\}\ \ . \tag{3.18}$$

In [2] is proved, that two arbitrary histories of the temperature-gradient, corresponding to the states $(\theta_1(t), \bar{\mathbf{g}}_1^t)$ and $(\theta_2(t), \bar{\mathbf{g}}_2^t)$, are *w-equivalent* if and only if they are *equivalent* according to Definition 2. That is, $\forall\ \mathbf{g}_P\ : [0,\tau) \to \mathbb{R}^3$, $\forall\ \tau > 0$

$$\widetilde{Q}\left\{(\mathbf{g}_P \star \bar{\mathbf{g}}_1)^{t+\tau}\right\} = \widetilde{Q}\left\{(\mathbf{g}_P \star \bar{\mathbf{g}}_2)^{t+\tau}\right\} \Longleftrightarrow \widetilde{W}\{\bar{\mathbf{g}}_1^t; \mathbf{g}_P\} = \widetilde{W}\{\bar{\mathbf{g}}_2^t; \mathbf{g}_P\}. \tag{3.19}$$

Consequently, according to the result in [2], equivalence can be stated in terms of heat flux (equivalence) or of thermal work (w-equivalence); namely, if two different equivalent processes, i.e. associated to the same heat flux, are considered, then the corresponding thermal work is also the same. Thus, the thermal work functional can be regarded as a state function, that, on application of Plancharel's formulae, can be written

$$\mathcal{W}(\sigma(t), \mathbf{g}_P) := \widetilde{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\} \quad \Longleftrightarrow \quad \mathcal{W}(\sigma(t), \mathbf{g}_P)\widehat{W}\{\bar{\mathbf{g}}^t; \mathbf{g}_P\}. \tag{3.20}$$

The introduction of $\mathcal{W}(\sigma(t), \mathbf{g}_P)$ emphasizes that, when a process P is assigned via $\mathbf{g}_P$, the thermal work which corresponds to all those states identified by the same state function $\sigma(t)$ is the same. Finally, via the approach adopted in [2], the expression of the minimum free energy can be readily obtained to be [2] represented by

$$\psi_m(\sigma) = \frac{1}{2}\int_0^\infty\!\!\int_0^\infty k(|\tau - s|)\mathbf{g}^{(m)}(s) \cdot \mathbf{g}^{(m)}(\tau)\ dsd\tau\ \ , \tag{3.21}$$

where $\mathbf{g}^{(m)}(\cdot)$ represents the temperature gradient, solution of a suitable [2] Wiener-Hopf equation.

4 Isothermal viscoelastic body with memory

The aim of this Section is to briefly summarize the analytical description of the model of an isothermal viscoelastic material with memory. Indeed, the historical as well as phenomenological ideas which have been developed throughout the literature are far beyond the present study. Conversely, here the main goal is to show the analytical analogy between the two models with memory: rigid heat conductor, on one hand, and isothermal viscoelastic body, on the other one. An overview concerning isothermal viscoelasticity is comprised in [15] and in [7] as well as in references therein. The work by Gentili [15] is devoted to minimum free energy and its connection to maximum recoverable work, while Deseri, Fabrizio and Golden, in [7], studied free energies, again in isothermal viscoelasticity, with the aim to treat applications to partial differential equations.

In this Section, following the lines of Section 2, only those definitions which are needed to establish equivalence between two different viscoelastic processes are considered. Indeed, the idea is to associate into equivalence classes all those processes which produce the same material response according to the approach proposed by Noll [23].

Hence, following to the same way of thinking adopted in connection to the heat conduction problem, no space variable dependence is considered under the assumption the body is isotropic. The constitutive equation which characterizes a linear viscoelastic material is the classical Boltzmann-Volterra that relates the stress tensor $\mathbf{T}(t) \in Sym$ to the strain history tensor $E : (-\infty, t] \to Sym$:

$$\mathbf{T}\,(t) = \int_0^\infty \mathbb{G}(\tau)\,\dot{\mathbf{E}}\,(t-\tau)\;d\tau \quad \mathit{or} \quad \mathbf{T}(t) = \mathbb{G}_0 E(t) + \int_0^\infty \dot{\mathbb{G}}(\tau)\mathbf{E}(t-\tau)\;d\tau \tag{4.1}$$

where the fourth order tensor $\mathbb{G}(t)$ denotes the elastic modulus, $\mathbf{E}(t)$ the value of the strain at the time t and $\mathbf{E}^t$ the past history defined by

$$\begin{aligned} \mathbf{E}^t :\ & (0,\infty) \to Sym \\ & t \longmapsto \mathbf{E}^t(s) := \mathbf{E}(t-s)\ . \end{aligned} \tag{4.2}$$

The elastic modulus is assumed such that its time derivative $\dot{\mathbb{G}(t)} \in L^1(\mathbb{R}^+, Lin(Sym))$ so that, for all positive t,

$$\mathbb{G}(t) = \mathbb{G}_0 + \int_0^t \dot{\mathbb{G}}(s)ds \quad , \quad \mathbb{G}_0 := \mathbb{G}(0) \tag{4.3}$$

where the initial value of the elastic modulus $\mathbb{G}_0$ is termed *instantaneous elastic modulus* [15]; furthermore, since $\dot{\mathbb{G}} \in L^1(\mathbb{R}^+)$, then

$$\mathbb{G}(\infty) := \lim_{t\to\infty} \mathbb{G}(t) \in Lin(Sym) \tag{4.4}$$

which represents the *equilibrium elastic modulus*. Hence, the tensor defined via

$$\breve{\mathbb{G}}(t) = \mathbb{G}(t) - \mathbb{G}(\infty) \quad \Longrightarrow \lim_{t\to\infty} \breve{\mathbb{G}}(t) = 0, \tag{4.5}$$

is symmetric, $\breve{\mathbb{G}}^T(t) = \breve{\mathbb{G}}(t)$ and, by definition, belongs to $L^1(\mathbb{R}^+)$. The state and the strain history of a viscoelastic body is characterized, according to [15], [14] and [7], by a *viscoelastic state function* $\sigma : \mathbb{R} \to Sym \times Sym$, which associates $t \mapsto \sigma(t) :\equiv (\mathbf{E}(t), \mathbf{E}^t)$. Hence, the viscoelastic state function is known when the strain tensor $\mathbf{E}(t)$ and the strain past history, $\mathbf{E}^t$ which belong to a suitable Hilbert space, are assigned. Physically meaningful viscoelastic phenomena, are characterized by a finite stress tensor $\mathbf{T}(t)$ for all times t and thus they belong to the vectorial space

$$\Gamma := \left\{ \mathbf{E}^t : (0,\infty) \to Sym : \left| \int_0^\infty \dot{\mathbb{G}}(s+\tau)\,\mathbf{E}^t(s)\;ds \right| < \infty \quad , \quad \forall\, \tau \geq 0 \right\}. \tag{4.6}$$

According to [15], the property $\dot{\mathbb{G}} \in L^1(\mathbb{R}^+)$ is termed *fading memory* and it implies that, corresponding to any arbitrary $\varepsilon > 0$, there exists a positive constant $\tilde{a} = a(\varepsilon, \mathbf{E}^t)$ such that

$$\left| \int_0^\infty \dot{\mathbb{G}}(s+a)\mathbf{E}^t(s)\;ds \right| = \left| \int_0^\infty \dot{\mathbb{G}}(s+a)\mathbf{E}(t-s)\;ds \right| < \varepsilon\ , \quad \forall\, a > \tilde{a}\,. \tag{4.7}$$

Now, as far as the setting of the models is concerned, some remarks close this Section. Specifically, the expression (4.1), which gives the stress $\mathbf{T}$ when the strain history is known, is of the same form of (2.6) where the heat flux is written when the history of the temperature gradient is assigned. A comparison between such two formulae shows, further than the obvious dimensionality difference, they cannot be obtained one from the other one via simple substitutions. Indeed, comparison between the first two formulae (2.6) and (4.1) show that, in turn, the time derivative of the strain history and the elastic modulus play the same role of the history of the temperature gradient and the heat flux relaxation function. The sign difference follows from the different time behavior of the elastic modulus and the heat flux relaxation function; specifically, the latter is a non increasing monotone function which admits an initial positive value k_0 while the behaviour of the fourth order tensor $\mathbb{G}_0$, described for instance by Gentili in [15], is not the same. Indeed, if the symmetric tensor $\tilde{\mathbb{G}}(t) := -\breve{\mathbb{G}}(t)$, is introduced, then it shares with $k(t)$ the property to be a non increasing function of t admitting an initial positive value and converging to zero when $t \to \infty$. Since its definition combined with (4.5) implies $\tilde{\mathbb{G}}(t) := \mathbb{G}(\infty) - \mathbb{G}(t)$: the "minus" sign which appears in (2.6) and doesn't in (4.1) is explained. Furthermore, comparison between the second two expressions in (2.6) and (4.1) shows that it can be expected the role, in turn, played by the stress tensor, the relaxation modulus and the time derivative of the strain tensor in isothermal viscoelasticity to be the same, respectively, of the heat flux, the heat flux relaxation function and the temperature gradient in heat conduction in a rigid body. Indeed, the *elastic modulus* is a non decreasing function which admits the finite limit value $\mathbb{G}(\infty) \in Lin(Sym)$. In particular, the behaviour of $\tilde{\mathbb{G}}$ follows immediately from its definition.

Finally, note that the definition of the function space Γ exhibits interesting similarities with the corresponding definition (2.6), which characterizes admissible states of a rigid heat conductor with memory. Indeed, definition (2.9), follows from definition (2.6), as soon as the heat flux relaxation function and the integrated history of the temperature gradient are, in turn, replaced by the elastic modulus and the strain history.

5 Viscoelastic Work

The present Section is concerned about the notions of process, process prolongation and, subsequently, of work, here termed viscoelastic work, to emphasize that it is referred to the linear viscoelasticity model. The Section closes with a brief comparison with the case of the thermal work.

In investigations concerning a linear viscoelastic solid, various authors [23, 18, 9, 6], have been concerned about the notions of state and process. Here, the definitions adopted in [15] of process and of process prolongation are recalled

Definition 8. Given a process P, hence assigned $\mathbf{E}_P$, its duration T and $\sigma(t) := (\mathbf{E}(t), \mathbf{E}^t)$, then the prolongation of the history $\mathbf{E}^t(s)$, induced by the process P, is defined by

$$\left(\mathbf{E}_P \star \mathbf{E}\right)^{t+T}(s) := \begin{cases} \mathbf{E}_P^T(s) & 0 \le s < T \\ \mathbf{E}^{t+T}(s) & s \ge T\ . \end{cases} \tag{5.1}$$

Remark. *Note that this definition of prolongation is quite different from that one (2.13) referring to rigid heat conduction; indeed, the two models differ the most as far as the definition of prolongation is concerned. This follows since the role played by the strain history* $E^t(\tau) := E(t-\tau)$ *in the viscoelastic model is the same the integrated history of the temperature gradient* $\bar{\mathbf{g}}^t(\tau) \int_{t-\tau}^t \mathbf{g}(s)\, ds$ *plays in rigid heat conduction.*

The subsequent step is the introduction of the following definition of equivalence, see Gentili [15].

Definition 9. two given two strain histories assigned via $\sigma_i(t) := (\mathbf{E}_i(t), \mathbf{E}_i^t), i = 1, 2$, they are termed to be equivalent if $\forall\, \mathbf{E}_P : [0,T) \to Sym \;\;,\;\; \forall \;\; T > 0$, are *equivalent* if $\forall\, \mathbf{E}_P \; : [0,T) \to Sym \;\;,\;\; \forall \;\; T > 0$, the corresponding stress tensor is the same, namely $\widetilde{\mathbf{T}} \left\{ \mathbf{E}_P(t), (\mathbf{E}_P \star \mathbf{E}_1)^{t+T} \right\} = \widetilde{\mathbf{T}} \left\{ \mathbf{E}_P(t), (\mathbf{E}_P \star \mathbf{E}_2)^{t+T} \right\}$.

This relation plays the same role in the isothermal viscoelastic model that the equivalence between two different thermal history, obtained in [2], plays in the rigid linear heat conduction problem. Hence, strain histories are divided into equivalence classes depending on the associated stress.

Then, the viscoelastic work can be introduced. In particular, two different situations are considered: on one hand, the viscoelastic work is evaluated when the process P is prolonged via the null strain history or via a generic strain history. The expressions obtained in the two different cases show interesting similarities with the corresponding ones when the thermal work is studied. In particular, the "zero" prolongation is characterized by the state function $\sigma(t) := (\mathbf{0}(t), \mathbf{0}^t)$ so that (5.1) implies

$$\mathbf{E}_0(t) := \int_0^t \dot{\mathbf{E}}(s) ds \;\;,\;\; (\mathbf{E}_P \star \mathbf{E})^{t+T}(s) := \begin{cases} \mathbf{E}_P^T(s) & 0 \le s < T \\ \mathbf{E}^{t+T}(s) & s \ge T \;. \end{cases} \tag{5.2}$$

It can be proved, see [15], that the corresponding viscoelastic work reads

$$\widetilde{W}\{\mathbf{0}, \mathbf{0}; \dot{\mathbf{E}}_P\} = \frac{1}{2} \int_0^\infty \int_0^\infty \mathbb{G}(|\tau - s|) \dot{\mathbf{E}}_P(s) \cdot \dot{\mathbf{E}}_P(\tau)\, ds d\tau \;\;, \tag{5.3}$$

in the case of the trivial prolongation, and

$$\widetilde{W}\{\mathbf{E}(t), \mathbf{E}^t; \dot{\mathbf{E}}_P\} = \int_0^\infty \left[\frac{1}{2} \int_0^\infty \mathbb{G}(|\tau - s|) \dot{\mathbf{E}}_P(s) ds - \mathbf{I}(\tau, \mathbf{E}(t), \mathbf{E}^t) \right] \cdot \dot{\mathbf{E}}_P(\tau) d\tau \;\;, \tag{5.4}$$

where

$$\mathbf{I}(\tau, \mathbf{E}(t), \mathbf{E}^t) := -\mathbb{G}(\tau)\, \mathbf{E}(t) - \int_0^\infty \dot{\mathbb{G}}(\tau + s)\, \mathbf{E}^t(s) ds \;\;, \tag{5.5}$$

when the generic prolongation is considered. Then, on introduction of $\breve{\mathbb{G}}$, $\widetilde{W}$ in (5.3) reads

$$\widetilde{W}\{\mathbf{0}, \mathbf{0}; \dot{\mathbf{E}}_P\} = \frac{1}{2} \mathbb{G}_\infty \mathbf{E}_0(T) \cdot \mathbf{E}_0(T) + \frac{1}{2} \int_0^\infty \int_0^\infty \breve{\mathbb{G}}(|\tau - s|) \dot{\mathbf{E}}_P(s) \cdot \dot{\mathbf{E}}_P(\tau)\, ds d\tau \;\;, \tag{5.6}$$

which, see [15], via the introduction of

$$\breve{\mathbf{I}}(\tau, \breve{\mathbf{E}}^t) := -\int_0^\infty \dot{\mathbb{G}}(\tau + s)\, \breve{\mathbf{E}}^t(s) ds = \mathbb{G}_\infty\, \mathbf{E}(t) + \mathbf{I}(\tau, \mathbf{E}(t), \mathbf{E}^t) \;\;,\;\; \tau \ge 0 \;, \tag{5.7}$$

can be rewritten in the form

$$\begin{aligned}\widetilde{W}\{\mathbf{0},\mathbf{0};\dot{\mathbf{E}}_P\} &= \frac{1}{2}\mathbb{G}_\infty \mathbf{E}_0(T)\cdot\mathbf{E}_0(T)+\frac{1}{2}\int_0^\infty\!\!\int_0^\infty \breve{\mathbb{G}}(|\tau-s|)\dot{\mathbf{E}}_P(s)\cdot\dot{\mathbf{E}}_P(\tau)\,dsd\tau \\ &+\frac{1}{2}\mathbb{G}_\infty \mathbf{E}_0(t)\cdot\mathbf{E}_0(T)-\frac{1}{2}\int_0^\infty\!\!\int_0^\infty \breve{\mathbf{I}}(\tau,\breve{\mathbf{E}}^t)\cdot\dot{\mathbf{E}}_P(\tau)\,dsd\tau\ ,\end{aligned}\tag{5.8}$$

where

$$\breve{\mathbf{I}}(\tau,\breve{\mathbf{E}}^t) := -\int_0^\infty \dot{\mathbb{G}}(\tau+s)\,\breve{\mathbf{E}}^t(s)ds = \mathbb{G}_\infty\,\mathbf{E}(t)+\mathbf{I}(\tau,\mathbf{E}(t),\mathbf{E}^t)\quad,\quad \tau\geq 0\ .\tag{5.9}$$

On introduction of the following transforms

$$\breve{\mathbf{I}}_+(\omega,\breve{\mathbf{E}}^t) := \int_0^\infty \breve{\mathbf{I}}(\tau,\breve{\mathbf{E}}^t)\,e^{-i\omega\tau}d\tau\quad,\quad \dot{\mathbf{E}}_+(\omega) := \int_0^\infty \dot{\mathbf{E}}_P(\tau)\,e^{-i\omega\tau}d\tau\ ,\tag{5.10}$$

the viscoelastic work can be written as follows

$$\widetilde{W}\{\mathbf{0},\mathbf{0};\dot{\mathbf{E}}_P\} = \frac{1}{2}\mathbb{G}_\infty\mathbf{E}_0(T)\cdot\mathbf{E}_0(T)+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\breve{\mathbb{G}}(\omega)\dot{\mathbf{E}}_+(\omega)\cdot\overline{\dot{\mathbf{E}}_+(\omega)}\,d\omega\tag{5.11}$$

in the case of the trivial zero prolongation, where $\dot{\mathbf{E}}_+(\omega)$ represent the Fourier Transforms, $\overline{\dot{\mathbf{E}}_+(\omega)}$ its complex conjugate and $\breve{\mathbb{G}}_c(\omega)$, according to [15], the Fourier cosine transform of $\breve{\mathbb{G}}(|\tau-s|)$. In the general case, the viscoelastic work, again in the form obtained by Gentili [15], reads

$$\begin{aligned}\widehat{W}\{\mathbf{E}(t),\breve{\mathbf{E}}^t;\dot{\mathbf{E}}_P\} &= \frac{1}{2}\mathbb{G}_\infty\mathbf{E}_0(T)\cdot\mathbf{E}_0(T)+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\breve{\mathbb{G}}(\omega)\dot{\mathbf{E}}_+(\omega)\cdot\overline{\dot{\mathbf{E}}_+(\omega)}\,d\omega \\ &+\frac{1}{2}\mathbb{G}_\infty\mathbf{E}_0(t)\cdot\mathbf{E}_0(T)-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\breve{\mathbf{I}}_+(\omega,\breve{\mathbf{E}}^t)\cdot\overline{\dot{\mathbf{E}}_+(\omega)}\,d\omega\ .\end{aligned}\tag{5.12}$$

Again, exactly in the same way as in Section 3, in the case when the thermal work was considered, the inner product (3.9) can be introduced whenever the integral is finite. Then, on application of Plancharel's theorem, the identity of norms in spaces dual with respect to Fourier transforms, implies that $\widetilde{W}\{\mathbf{E}(t),\mathbf{E}^t;\dot{\mathbf{E}}_P\}=\widehat{W}\{\mathbf{E}(t),\breve{\mathbf{E}}^t;\dot{\mathbf{E}}_P\}$; hence, the latter is finite if, and only if, the first one is finite; accordingly, inner products can be evaluated in both the spaces related to each other via duality under Fourier transform.

The same way of reasoning followed in Section 3, see [15], allows to establish the admissible states are only those ones related to a finite viscoelastic work, and the equivalence between any couple of different states as those ones associated to the same value of the viscoelastic work. Finally, also in viscoelasticity, the work functional can be proved to represent, [15], a state function. Finally, also in the case of viscoelasticity, the minimum free energy follows to be [15] represented by

$$\psi_m(\sigma) = \frac{1}{2}\mathbb{G}_\infty\mathbf{E}_0(t)\cdot\mathbf{E}_0(t)+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\breve{\mathbb{G}}(\omega)\dot{\mathbf{E}}_+^{(m)}(\omega)\cdot\overline{\dot{\mathbf{E}}_+^{(m)}(\omega)}\,d\omega\tag{5.13}$$

where $\mathbf{E}^{(m)}(\cdot)$ denotes the strain, solution of a suitable [15] Wiener-Hopf equation. Note that, the two forms assumed by the free energy in thermodynamics (3.21) and in viscoelasticity (5.13) are of the same form except for the extra term which appears in the latter, due to the different behaviour at infinity of the heat flux relaxation function k and the relaxation modulus $\mathbb{G}$: that is $\lim_{t\to\infty}k(t)=0$, while $\lim_{t\to\infty}\mathbb{G}(t)=\mathbb{G}(\infty)\neq 0$.

6 Appendix

This short Section is devoted to briefly recall some notions, in the classical notation, required throughout the whole paper. Accordingly, (see for instance [1]), the following Fourier transforms are defined. Given a function $f : [0, \infty) \to \mathbb{R}^n$, the formal half range Fourier sine and cosine transforms are defined, in turn, by

$$f_s(\omega)\int_0^\infty f(\tau)\sin\ \omega\tau\ d\tau \quad , \quad f_c(\omega)\int_0^\infty f(\tau)\cos\ \omega\tau\ d\tau \quad . \tag{6.1}$$

In the case when its trivial prolongation over the whole $\mathbb{R}$, denoted again as f,

$$f(\tau) := \begin{cases} f(\tau) & \tau \geq 0 \\ 0 & \tau < 0\ , \end{cases} \tag{6.2}$$

is considered, then, the corresponding Fourier transform is given by

$$\tilde{f}(\omega)\int_{-\infty}^\infty f(\tau)e^{-i\omega\tau}\ d\tau \qquad hence \qquad \tilde{f}(\omega) = f_c(\omega) - i f_s(\omega) \quad , \tag{6.3}$$

where the relation of Fourier transform with Fourier sine and cosine transforms is also shown. In addition, when the even prolongation of f over the whole $\mathbb{R}$, denoted as $f(|\tau|)$, is considered

$$f(|\tau|) := \begin{cases} f(\tau) & \tau \geq 0 \\ f(-\tau) & \tau < 0\ , \end{cases} \tag{6.4}$$

then, since $f(|\tau|)$, even by definition (6.4), is real valued, then the corresponding Fourier transform, given by (6.3), is an even real valued function, namely $\Im(\omega) = 0$ implies $\Im(\tilde{f}(\omega)) = 0$ (see for instance [1]), related to Fourier cosine transforms via $\tilde{f}(\omega) = 2f_c(\omega)$. As far as the tensor quantities which appear in the viscoelastic model, Sym denotes the space of symmetric second order tensors acting on $\mathbb{R}^3$, i.e. $Sym := \{\mathbf{M} \in Lin(\mathbb{R}^3) : \mathbf{M} = \mathbf{M}^\top\}$, where the superscript "$^\top$" stands for transposition. The space of the fourth order tensors defined over Sym is indicated as $Lin(Sym)$, thus Sym is isomorphic to $\mathbb{R}^6$. In particular, for every $\mathbf{L}, \mathbf{M} \in Sym$, if $\mathbf{C}_i$, $i = 1, ..., 6$ is an orthonormal basis of Sym with respect to the usual inner product in $Lin(\mathbb{R}^3)$, namely $tr(\mathbf{L}\mathbf{M}^\top)$, then

$$\mathbf{L} = \sum_{i=1}^{6} L_i \mathbf{C}_i\ , \qquad\qquad \mathbf{M} = \sum_{i=1}^{6} M_i \mathbf{C}_i \tag{6.5}$$

and $tr(\mathbf{L}\mathbf{M}^\top) = \sum_{i=1}^{6} L_i M_i$. As a consequence, each tensor of Sym can be treated as a vector in $\mathbb{R}^6$, in addition, $\mathbf{L} \cdot \mathbf{M}$ denotes the inner product between elements of Sym, namely

$$\mathbf{L} \cdot \mathbf{M} = tr(\mathbf{L}\mathbf{M}^\top) = tr(\mathbf{L}\mathbf{M}) = \sum_{i=1}^{6} L_i M_i \tag{6.6}$$

and $|\mathbf{M}|^2 = \mathbf{M} \cdot \mathbf{M}$. Hence [21], given any fourth order tensor $\mathbb{K} \in Lin(Sym)$ it can be represented as

$$\mathbb{K} = \sum_{i,i=1}^{6} K_{ij} \mathbf{C}_i \otimes \mathbf{C}_j \ , \tag{6.7}$$

where $\mathbb{K}^\top$ denotes the transpose of $\mathbb{K}$, as an element of $Lin(\mathbb{R}^6)$. The norm $|\mathbb{K}|$ of $\mathbb{K} \in Lin(Sym)$ follows

$$|\mathbb{K}|^2 = tr\left(\mathbb{K}\mathbb{K}^\top\right) = \left(\sum_{i,j=1}^{6} K_{ij} K_{ji}\right) .$$

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