Some Holey designs and Incomplete designs for the join graph of K_1 and C_4 with a pendent edge

Xiaoshan Liu [a]
Department of Mathematics & Physics
Shijiazhuang University of Economics
China, Shijiazhuang
liuxiaoshan80617@163.com

Qi Wang [b]
Graduate School
Hebei University of Economics & Business
China, Shijiazhuang
wangqi80617@163.com

***Abstract*-A G -design of λK_v is a pair (X, B), where X is the vertex set of K_v and B is a collection of subgraphs of K_v, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exact (at most, at least) λ blocks of B. In this paper, we will discuss some holey designs and incomplete designs for the join graph of K_1 and C_4 with a pendent edge for $\lambda = 1$.**

Keywords-* G*-packing design,* G *-covering design, Holey* G *-design

I. INTRODUCTION

A complete multigraph of order v and index λ, denoted by λK_v, is an undirected graph with v vertices, where any two distinct vertices x and y are joined by λ edges (x, y). Let G be a finite simple graph. A G -design G - $GD_\lambda(v)$ (G -packing design G - $PD_\lambda(v)$, G -covering design G - $CD_\lambda(v)$) of λK_v is a pair (X, B), where X is the vertex set of K_v and B is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exact (at most, at least) λ blocks of B. A packing (covering) design is said to be maximum (minimum) if no other such packing (covering) design of the same order has more (fewer) blocks. The number of blocks in a maximum packing design (minimum covering design), denoted by $p(v, G, \lambda)$ ($c(v, G, \lambda)$), is called the packing number (covering number). Obviously,

$$p(v,G,\lambda) \le U(v,G,\lambda) = \left\lfloor \frac{\lambda v(v-1)}{2|E(G)|} \right\rfloor \le \left\lceil \frac{\lambda v(v-1)}{2|E(G)|} \right\rceil = V(v,G,\lambda) \le c(v,G,\lambda),$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest (lest) integer y such that $y \le x$ ($y \ge x$). A G - $PD_\lambda(v)$ (G - $CD_\lambda(v)$) is called optimal and is denoted by G - $OPD_\lambda(v)$ (G - $OCD_\lambda(v)$) if the left (right) equality in above inequality holds. Obviously, there exists a G - $GD_\lambda(v)$ if and only if $p(v, G, \lambda) = c(v, G, \lambda)$. So a G - $GD_\lambda(v)$ can be regarded as a G - $OPD_\lambda(v)$ or a G - $OCD_\lambda(v)$. The leave $L_\lambda(P)$ of a packing design G - $PD_\lambda(v)$ = (v, P) is a subgraph of λK_v and its edges are the supplement of P in λK_v. When P is maximum, $|L_\lambda(P)|$ is called leave-edges number and is denoted by $l_\lambda(v)$. Similarly, the repeat-edge graph $R_\lambda(C)$ of a covering design G - $CD_\lambda(v)$ $= (v, C)$ is a subgraph of λK_v and its edges are the supplement of λK_v in C. When C is minimum, $|R_\lambda(C)|$ is called repeat-edges number and is denoted by $r_\lambda(v)$. Generally, the symbols $L_\lambda(P)$ and $l_\lambda(v)$ can be denoted by L_λ and l_λ briefly, while $R_\lambda(P)$ and $r_\lambda(v)$ can be denoted by R_λ and r_λ correspondingly.

Let $X = \bigcup_{i=1}^{t} X_i$ be the vertex set of $K_{n_1, n_1, \cdots, n_t}$, a complete multipartite graph consisting of t parts with size $n_1, n_2, \cdots, n_t$ respectively, where the sets X_i

$(1 \le i \le t)$ are disjoint and $|X_i| = n_i$. Denote $v = \sum_{i=1}^{n} n_i$ and $G = \{X_1, X_2, \cdots, X_t\}$. For any given graph G, if the edges of $\lambda K_{n_1, n_1, \cdots, n_t}$, a t-partite graph with replication λ, can be decomposed into edge-disjoint subgraphs A, each of which is isomorphic to G and is called as block, then the system (X, G, A) is called a holey G-design with index λ, denoted by $G\text{-}HD_\lambda(T)$, where $T = n_1^1, n_2^1, \cdots, n_t^1$ is the type of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $n_1^{k_1}, n_2^{k_2}, \cdots, n_m^{k_m}$ denotes n_1 occurrences of k_1, n_2 occurrences of k_2, $\cdots$, n_m occurrences of k_m. A $G\text{-}HD_\lambda(1^{v-w}w^1)$ is called an incomplete G-design, denoted by $G\text{-}ID_\lambda(v,w) = (V, W, A)$, where $|V| = v$, $|W| = w$ and $W \subset V$. For $\lambda = 1$, the index λ of GD_λ HD_λ ID_λ OPD_λ OCD_λ is often omitted.

Lemma 1.1[7] There exists $G\text{-}GD_\lambda(v) \Leftrightarrow \begin{cases} v \ge 6; \\ \lambda v(v-1) \equiv 0 \pmod{18}; \\ (\lambda, v) \ne (1,9). \end{cases}$

Nonexistences and some constructions of the maximum packing designs and the minimum covering designs for the join graph of K_1 and C_4 with a pendent edge for $\lambda = 1$, will be given out as follows. For convenience, as a block in graph design G is denoted as following vertex-labels.

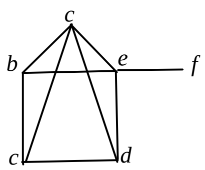

Figure 1. Graph G

II. MAIN METHOD

Lemma 2.1[2] Given positive integers $h, w, \lambda, m,$ if there exist $G\text{-}HD(h^m)$ and $G\text{-}ID_\lambda(h+w, w)$ then

(1). Suppose there exists $G\text{-}OPD_\lambda(w)$ or $G\text{-}OPD_\lambda(h+w)$, so does $G\text{-}OPD_\lambda(mh+w)$.

(2). Suppose there exists $G\text{-}OCD_\lambda(w)$ or $G\text{-}OCD_\lambda(h+w)$, so does $G\text{-}OCD_\lambda(mh+w)$.

Lemma 2.2[2] Given positive integers v, λ, u. Let X be a v set, then

(1). Suppose there exists $G\text{-}OPD_\lambda(v) = (X, P)$ with the leave $L_\lambda(P) \subset G$, then there exists $G\text{-}OCD_\lambda(v)$ with the repeat-edge graph $G \setminus L_\lambda(P)$.

(2). Suppose there exist both $G\text{-}OPD_\lambda(v) = (X, P)$ and $G\text{-}OPD_u(v) = (X, P')$ with leaves $L_\lambda(P)$ and $L_u(P')$ respectively. If $|L_\lambda(P)| + |L_u(P')| = l_{\lambda+u}$, then exists $G\text{-}OPD_{\lambda+u}(v) = (X, P \cup P')$ with the leave $L_\lambda(P) \cup L_u(P')$.

III. CONSTRUCTIONS FOR HOLEY DESIGNS

Lemma 3.1[5] There exist $G\text{-}HD(9^{2t+1})$ and $G\text{-}HD(18^{t+2})$ for $t \ge 1$.

Theorem 3.2 There exists $G\text{-}HD(9^4)$.

Proof. Give the direct construction of $G\text{-}HD(9^4)$ on vertex set $Z_9 \times Z_4$ and blocks are:

$(0_0, 0_1, 2_2, 1_1, 0_2, 4_0)$, $(0_0, 2_1, 6_2, 3_1, 8_2, 4_0)$, $(0_0, 3_2, 3_3, 5_1, 4_3, 2_0)$, $(0_1, 7_2, 1_3, 2_0, 0_3, 7_1)$, $(0_0, 5_3, 1_2, 6_3, 7_2, 4_3)$, $(0_0, 0_3, 4_1, 1_3, 6_1, 7_0)$ mod (9, −).

IV. CONSTRUCTIONS FOR ID

Theorem 4.1 There exists $G\text{-}ID(9+\varpi, \varpi)$ for $\varpi = 2, 3, \cdots, 7, 8, 12$.

Proof. There are $\varpi + 4$ blocks in each $G\text{-}ID(9+\varpi, \varpi)$.

$\varpi = 2: Z_3 \times Z_3 \cup \{x_1, x_2\}$. $(0_1, 1_0, 2_0, x_1, 1_2, 1_1)$, $(0_0, 2_2, 1_2, x_2, 0_1, 1_1)$ mod (3, −).

$\varpi = 3:$ $Z_9 \cup \{x_1, x_2, x_3\}$.

$(0, x_1, 1, x_2, 2, 6)$, $(0, x_3, 3, 6, 4, 1)$, $(6, x_1, 7, x_2, 8, 3)$, $(2, 5, 4, 8, 7, 0)$, $(3, x_1, 4, x_2, 5, 0)$, $(1, x_3, 2, 3, 7, 4)$, $(5, x_3, 6, 1, 8, 0)$.

$\varpi = 4: Z_9 \cup \{x_1, x_2, x_3, x_4\}$.

$(2, x_1, 1, x_2, 0, 4)$, $(3, x_1, 4, x_2, 5, 6)$, $(6, x_1, 7, x_2, 8, 4)$, $(0, 6, 1, 8, 3, 7)$, $(3, x_3, 1, x_4, 2, 6)$, $(7, x_3, 8, x_4, 0, 5)$, $(4, x_3, 6, x_4, 5, 8)$, $(7, 5, 1, 4, 2, 8)$.

$\varpi = 5: Z_3 \times Z_3 \cup \{x_1, x_2, \cdots, x_5\}$.

$(0_1, x_1, 0_0, x_2, 2_2, 1_2)$, $(0_0, x_3, 1_2, x_4, 1_1, 2_2)$, $(2_2, 0_0, 2_0, x_5, 2_1, 1_1)$ mod(3,−).

$\varpi = 6: Z_9 \cup \{x_1, x_2, \cdots, x_6\}$.

$(6,x_1,3,x_2,4,8),(5,x_1,7,x_2,0,8),(1,x_1,8,x_2,2,5),(5,x_3,6,x_4,8,2),(0,x_5,1,x_6,2,4),(2,x_3,7,x_4,3,4),(4,x_3,0,x_4,1,5),(8,x_5,7,x_6,6,2),(5,x_5,3,x_6,4,7),(7,0,6,1,3,8)$.

$\varpi = 7: Z_9 \cup \{x_1, x_2, \cdots, x_7\}$.

$(0,x_1,6,x_2,1,5),(4,x_1,5,x_2,7,6),(8,x_1,3,x_2,2,6),(2,x_3,3,x_4,1,6),(5,x_5,6,x_6,0,8),(4,x_5,2,x_6,3,5),(7,x_3,5,x_4,0,2),(x_7,1,4,0,3,6),(2,x_3,8,x_4,6,x_7),(1,x_5,7,x_6,8,6),(x_7,2,5,8,7,3)$.

$\varpi = 8: Z_3 \times Z_3 \cup \{x_1, x_2, \cdots, x_8\}$.

$(0_0,x_1,0_2,x_2,0_1,10),(1_1,x_3,0_0,x_4,1_2,2_2),(1_2,x_5,0_1,x_6,0_0,1_0),(2_2,x_7,0_0,x_8,0_1,1_1)$ mod(3,−).

$\varpi = 12: Z_9 \cup \{x_1, x_2, \cdots, x_{12}\}$.

$(6,x_9,2,x_{10},4,3),(8,x_5,1,x_6,6,x_{12}),(7,x_5,0,x_6,4,x_{12}),(0,x_{11},1,x_{12},2,8),(0,x_1,4,x_2,3,x_{11}),(1,x_1,2,x_2,5,x_{11}),(7,x_1,8,x_2,6,x_{11}),(5,x_5,2,x_6,3,x_{12}),(1,x_3,6,x_4,4,x_{11}),(2,x_3,3,x_4,7,x_{11}),(5,x_3,0,x_4,8,x_{11}),(3,x_7,1,x_8,7,x_{12}),(7,x_9,1,x_{10},5,4),(6,x_7,0,x_8,5,x_{12}),(4,x_7,2,x_8,8,x_{12}),(8,x_9,0,x_{10},3,6)$.

Theorem 4.2 There exists G - $ID(18+\varpi,\varpi)$ for $\varpi = 2,4,5,6,7,8$.

Proof. There are $2\varpi+17$ blocks in each G - $ID(18+\varpi,\varpi)$.

$\varpi = 2: Z_3 \times Z_6 \cup \{x_1, x_2\}$.

$(0_0,x_1,1_1,x_2,1_2,1_0),(0_3,x_1,1_4,x_2,0_5,0_0),(0_0,1_0,0_1,2_2,0_3,2_0),(0_3,2_1,1_3,1_2,0_4,1_0),(0_0,1_3,0_4,1_4,2_5,0_3),(0_1,1_2,0_2,0_4,0_5,2_0),(0_5,1_5,0_2,1_4,2_1,3_1)$ mod (3, −).

$\varpi = 4: Z_{18} \cup \{x_1, x_2, x_3, x_4\}$.

$(4,x_3,5,x_4,6,14),(16,x_3,17,x_4,0,15),(2,4,9,5,11,3),(3,x_1,5,x_2,4,14),(3,9,15,12,17,0),(1,x_3,2,x_4,3,10),(9,x_1,10,x_2,11,16),(0,7,3,6,5,16),(6,x_1,8,x_2,7,14),(12,x_1,13,x_2,14,16),(7,x_3,9,x_4,8,16),(15,x_1,17,x_2,16,9),(2,6,12,7,13,3),(1,8,11,12,5,14),(11,6,15,7,17,1),(0,x_1,2,x_2,1,14),(13,x_3,15,x_4,14,9),(1,16,6,9,13,8),(1,10,15,4,7,16),(17,5,13,4,10,6),(2,3,8,10,16,4),(14,2,17,8,15,5),(0,4,8,9,12,8),(0,10,1_3,11,14,3),(10,x_3,11,x_4,12,16)$.

$\varpi = 5: Z_3 \times Z_6 \cup \{x_1, x_2, \cdots, x_5\}$.

$(0_2,x_5,1_3,1_1,2_4,1_4),(0_0,0_3,0_4,0_2,1_0,2_4),(0_0,x_1,2_1,x_2,1_2,2_5),(0_3,x_1,1_4,x_2,2_5,0_2),(0_1,x_3,1_2,x_4,2_3,1_3),(0_5,x_5,2_0,0_5,2_1,0_3),(0_4,x_3,2_5,x_4,1_0,2_3),(0_1,0_0,0_5,0_4,1_1,1_2),(0_3,0_2,1_2,2_4,0_5,2_3)$ mod (3, −).

$\varpi = 6: Z_{18} \cup \{x_1, x_2, \cdots, x_6\}$.

$(0,4,7,10,13,5),(17,x_5,0,x_6,1,10),(15,x_1,16,x_2,17,6),(7,x_3,9,x_4,8,14),(1,x_3,2,x_4,3,13),(8,x_5,10,x_6,9,2),(5,x_5,7,x_6,6,16),(12,x_1,14,x_2,13,17),(1,6,11,7,14,2),(2,x_5,13,x_6,4,14),(0,14,5,8,11,16),(1,15,4,8,12,17),(10,x_3,12,x_4,11,17),(1,5,9,13,16,2),(11,x_5,12,x_6,13,6),(4,9,16,10,17,5),(3,9,14,10,15,5),(14,x_5,15,x_6,16,12),(0,9,12,3,6,15),(16,x_3,0,x_4,17,14),(0,x_1,2,x_2,1,11),(4,x_3,6,x_4,5,11),(13,x_3,14,x_4,15,11),(2,7,13,8,15,0),(3,x_1,5,x_2,4,11),(6,x_1,8,x_2,7,12),(9,x_1,10,x_2,11,3),(3,7,16,8,17,2),(2,5,10,6,12,4)$.

$\varpi = 7: Z_{18} \cup \{x_1, x_2, \cdots, x_7\}$.

$(13,x_3,14,x_4,15,12),(14,x_5,15,x_6,16,11),(15,x_1,16,x_2,17,10),(17,x_5,0,x_6,1,8),(1,x_3,2,x_4,3,7),(6,13,x_7,12,1,16),(4,x_3,6,x_4,5,17),(7,x_7,14,0,15,8),(10,3,14,5,15,1),(4,9,12,16,13,5),(12,x_1,13,x_2,14,8),(1,5,9,14,11,0),(2,x_5,3,x_6,4,15),(2,15,9,16,6,10),(3,x_1,4,x_2,5,12),(10,x_3,11,x_4,12,3),(2,x_7,5,8,11,3),(6,x_1,7,x_2,8,12),(2,7,12,17,13,3),(1,10,x_7,4,7,17),(11,x_5,13,x_6,12,0),(0,4,8,13,10,2),(5,x_5,6,x_6,7,16),(17,6,14,4,11,15),(9,x_1,10,x_2,11,7),(0,x_7,3,6,9,17),(0,x_1,1,x_2,2,14),(8,3,17,x_7,16,5),(16,x_3,17,x_4,0,5),(8,x_5,9,x_6,10,16),(7,x_3,8,x_4,9,3)$.

$\varpi = 8: Z_3 \times Z_6 \cup \{x_1, x_2, \cdots, x_8\}$.

$(0_0,x_1,1_1,x_2,1_2,10),(0_3,x_1,0_5,x_2,1_4,0_1),(0_1,x_3,1_2,x_4,2_3,0_0),(0_2,0_5,1_0,0_1,1_5,2_3),(0_4,x_3,1_5,x_4,1_0,0_0),(0_2,x_5,1_3,x_6,1_4,0_1),(0_5,x_5,2_0,x_6,0_1,1_1),(0_1,x_7,2_2,x_8,2_5,4_3),(0_0,x_7,0_3,x_8,1_4,2_2),(0_1,0_4,0_0,1_3,0_3,1_2),(0_5,1_3,0_4,1_4,1_2,0_2)$ mod (3, −).

V. ACKNOWLEDGMENT

[a]This author was supported by the funds from Education Department Foundation of Hebei Province under fund number Z2012057 and Foundation of Shijiazhuang University of Economics under fund number ZR201103.Tel.: (+86)13463955288 *E-mail address:* liuxiaoshan80617@163.com.

[b]This author was supported by the funds from Nature Science Foundation of Hebei Province under fund number A2011207003 and Outstanding Youth Fund Project of Scientific Research in Colleges and Universities of Hebei Department of Education under fund number Y2011115
Tel:(+86)13463955388; *E-mail address:*stwangqi@heuet.edu.cn

REFERENCES

[1] B. Alspach and H. Gavlas, "Cycle decompositions of Kn and Kn−I", Journal of Combinatorial Theory (B), Vol. 21, 2000, pp. 146-155.

[2] J. C. Bermond, C. Huang, A. Rosa and D. Sotteau, "Decomposition of complete graphs into isomorphic subgraphs with five vertices", Ars Combinatoria, Vol. 10, 1980, pp. 211-254.

[3] J. C. Bermond and J. Sch¨onheim, "G-decomposition of Kn, where G has four vertices or less", Discrete Math. Vol. 19,1977, pp. 113-120.

[4] A. Blinco, "On diagonal cycle systems", Australasian Journal of Combinatorics, Vol. 24, 2001, pp. 221-230.

[5] Q. Kang, Y. Du and Z. Tian, "Decomposition of λK_v into some graph with six vertices and seven edges", unpublished.

[6] Q. Kang, Y. Du and Z. Tian, "Decomposition of complete graph into isomorphic subgraphs with six vertices and seven edges", unpublished.

[7] C. P. Ma, "The graph designs for six graphs with six vertices and nine edges". Master thesis, in press.