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A Hamiltonian Formulation for Free Surface Water Waves with Non-Vanishing Vorticity

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Abstract

We describe the derivation of a formalism in the context of the governing equations for two-dimensional water waves propagating over a flat bed in a flow with non-vanishing vorticity. This consists in providing a Hamiltonian structure in terms of two variables which are scalar functions.

1 Introduction

In the case of irrotational flow (i.e. zero vorticity), the formulation of the water wave dynamics as a Hamiltonian system is due to Zakharov [32], Milder [27], Miles [28] *et. al.* (see the review [13]). For this Hamiltonian structure, the velocity potential for the irrotational flow and the surface elevation are the canonically conjugate variables. For free surface water flows with vorticity, Maddocks and Pego [26] expressed the governing equations as a canonical Hamiltonian system, the two variables being vector functions (see also related considerations provided in [1, 20]). Due to the lack of existence of a velocity potential in the case when vorticity is present in the flow, a representation of the governing equations for water waves with a free surface in Hamiltonian form in terms of scalar variables seems confined to the irrotational case. In this paper, combining the elegant viewpoint due to Maddocks and Pego [26] with some ideas advocated by Seliger and Whitham [30] in the context of water flows without a free boundary, we will express the governing equations for water waves propagating over a flat bed in a flow with non-vanishing vorticity as a Hamiltonian system, the two variables being scalar functions. In Section 2 we describe the derivation of the Hamiltonian system. Section 3 is devoted to an illustration of the proposed Hamiltonian formulation in the context of shear flow of constant vorticity over a flat bed and with a flat free surface. We would like to point out that our aim is to put forward a point of view offering a simplified set of coordinates that have a certain elegance - we are not reporting on analytical results, leaving for example the choice of the appropriate function spaces unspecified. The novelty with respect the other approaches in [1, 4, 26, 20] consists either in the fact that the variables are scalar and not vector functions or in that earlier considerations were made for irrotational and/or steady flows [3, 5, 6, 13, 15, 16, 17, 27, 28, 32].

2 Description

Let us first recall the governing equations for the propagation of two-dimensional water waves over a flat bed (see [19]). The two-dimensional character means that the motion is identical in any direction parallel to the crest line so that a full picture is provided by analyzing a cross section of the flow that is perpendicular to the wave crests. Choosing spatial coordinates (x, y) so that the horizontal x-axis is in the direction of wave propagation and the y-axis points vertically upwards, with t being time, the fluid domain is bounded below by a flat rigid bottom $y = -d$ and above by a free surface $y = \eta(t, x)$. The only external force whose influence on the water we take into consideration is gravity. If $(u(t, x, y),\ v(t, x, y))$ is the velocity field, the equation of motion is Euler's equation

$$u_t + uu_x + vu_y = -P_x, \tag{2.1a}$$

$$v_t + uv_x + vv_y = -P_y - g, \tag{2.1b}$$

where $P(t, x, y)$ denotes the pressure and g is the gravitational constant of acceleration. Euler's equation is indicative of the inviscid setting, which is appropriate for gravity water waves [21]. Another realistic assumption for gravity water waves is constant density [21], which implies the equation of mass conservation

$$u_x + v_y = 0. \tag{2.2}$$

The boundary conditions for the water wave problem are the dynamic boundary condition

$$P = 0 \quad \text{on} \quad y = \eta(t, x) \tag{2.3}$$

which decouples the motion of the overlying air from that of the water, as well as the kinematic boundary conditions

$$v = \eta_t + u\eta_x \quad \text{on} \quad y = \eta(t, x) \tag{2.4}$$

and

$$v = 0 \quad \text{on} \quad y = -d. \tag{2.5}$$

Relation (2.5) expresses the fact that the rigid bottom is impermeable so that the flow is tangent to the horizontal bed $y = -d$, while (2.4) ensures that the same particles always form the free surface [19]. The vorticity of the water flow is given by

$$\omega = v_x - u_y. \tag{2.6}$$

The equations (2.1)-(2.6) are the governing equations for a two-dimensional free surface water flow with vorticity ω. We would like to point out that while most studies of water waves are devoted to irrotational flows (that is, $\omega \equiv 0$), waves with vorticity (rotational waves) are commonly seen in nature, for example, in sea regions with shear currents (e.g. the assumption of constant non-zero vorticity is appropriate for tidal flows [14]). Moreover, field evidence indicates that the assumption of irrotational flow is inappropriate even for waves advancing into still water - a situation where zero vorticity is usually deemed suitable

- since vorticity is generated at the free surface and it propagates slowly downwards in the fluid [23, 24].

While for the study of the existence of solutions to the water wave problem the formulation (2.1)-(2.6) in Eulerian coordinates (that is, from the viewpoint of a fixed oberver noticing the flow characteristics at time t and spatial location (x, y)) is sometimes advantageous[1], in certain circumstances[2] it is useful to adopt the Lagrangian viewpoint (that is, the flow pattern is obtained by describing the path of each individual water particle). The main advantage of the Lagrangian viewpoint is that it transforms the free boundary value problem (2.1)-(2.6) into a problem in a fixed domain in the space of Lagrangian indexes. Indeed, let

$$\Omega = \{(a, b) \in \mathbb{R}^2 : \; a \in \mathbb{R},\, b \in [-d, 0]\} \tag{2.7}$$

be the label domain for the water particles $(x(t, a, b),\, y(t, a, b))$. We would like to point out that the label (a, b) does not represent the initial state of a particle, it just marks that specific particle. If we denote by $\dot{f}$ the material time derivative of the function f, given by $\dot{f} = f_t + u f_x + v f_y$, then

$$\dot{x} = u, \qquad \dot{y} = v, \tag{2.8}$$

and we can recast the Euler equation (2.1) as

$$\dot{u} = -P_x, \qquad \dot{v} = -P_y - g. \tag{2.9}$$

In the Lagrangian setting we do not encounter the kinematic boundary conditions (2.4)-(2.5) since $b = 0$ represents the free surface and $b = -d$ the flat bed. The dynamic boundary condition (2.3) becomes

$$P(t, a, 0) = 0 \tag{2.10}$$

and the vorticity is given by (2.6), with u and v defined in (2.8). Finally, the equation of mass conservation, expressing incompressibility, is equivalent to the requirement that the determinant of the Jacobian matrix J for the coordinate change $(a, b) \mapsto (x, y)$ is time invariant. The Lagrangian labels can be chosen in such a way that

$$\det J = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} \end{vmatrix} = 1. \tag{2.11}$$

Hamilton's principle is to find stationary points of an action integral which is usually of the form $\displaystyle\int_{t_1}^{t_2} \mathcal{L}\, dt$. Let

$$\mathcal{L}(x, y, u, v, \lambda) = \int\int_{\Omega} \Big(\frac{u^2 + v^2}{2} + gy - \lambda(u_x + v_y) \Big)\, da\, db \tag{2.12}$$

[1]See [2, 9, 10, 11, 12, 31] for the existence of steady solutions. For aspects of the Cauchy problem associated to (2.1)-(2.6) we refer to [22] and references therein.

[2]For example, to describe explicit, non-flat two or three dimensionial water waves propagating in water of infinite depth or parallel to a beach of constant slope cf. [7, 8].

be the Lagrangian. The scalar function $\lambda = \lambda(t, a, b)$, called *striction* in [26], plays the role of a Lagrange multiplier by means of which the incompressibility of the flow, expressed by (2.2), is enforced. According to Hamilton's principle the variation of the action must be zero. We obtain (see [26] for the details of the computation) the Euler-Lagrange equations

$$-\dot{\Big(\frac{\delta\mathcal{L}}{\delta u}\Big)} + \frac{\delta\mathcal{L}}{\delta x} = 0, \qquad -\dot{\Big(\frac{\delta\mathcal{L}}{\delta v}\Big)} + \frac{\delta\mathcal{L}}{\delta y} = 0, \tag{2.13}$$

with the constraint

$$\frac{\delta\mathcal{L}}{\delta\lambda} = 0. \tag{2.14}$$

The Euler-Lagrange equations (2.13) can be recognized as being precisely the Euler equation (2.9) with pressure

$$P = \dot{\lambda} \tag{2.15}$$

since

$$\frac{\delta\mathcal{L}}{\delta u} = u + \lambda_x, \qquad \frac{\delta\mathcal{L}}{\delta v} = v + \lambda_y. \tag{2.16}$$

The constraint (2.14) is nothing but the incompressibility condition (2.2). Requiring

$$\lambda = 0 \quad \text{for} \quad b = 0. \tag{2.17}$$

we recover the dynamic boundary condition (2.10). Indeed, since $\lambda(t, x, \eta(t, x)) = 0$, we have

$$\lambda_t + \lambda_y \eta_t = 0 \quad \text{on} \quad y = \eta(t, x), \tag{2.18}$$

and

$$\lambda_x + \lambda_y \eta_x = 0 \quad \text{on} \quad y = \eta(t, x). \tag{2.19}$$

Taking into account (2.4) in combination with (2.18)-(2.19), we infer that

$$\dot{\lambda} = \lambda_t + u\lambda_x + v\lambda_y = \lambda_t - u\lambda_y\eta_x + v\lambda_y = \lambda_t + \eta_t\lambda_y = 0$$

on the free surface $y = \eta(t, x)$. That is, (2.10) holds in view of (2.15).

In this context, Maddocks and Pego [26] introduced the *impetus*, defined through

$$\xi_1 = \frac{\delta\mathcal{L}}{\delta u}, \qquad \xi_2 = \frac{\delta\mathcal{L}}{\delta v}. \tag{2.20}$$

Taking into account (2.16), the equations (2.20) can be inverted to yield

$$u = \xi_1 - \lambda_x, \qquad v = \xi_2 - \lambda_y, \tag{2.21}$$

and the Legendre transform

$$\tilde{\mathcal{H}}(x, y, \xi_1, \xi_2, \lambda) = \int\int_\Omega (\xi_1, \xi_2) \cdot (u, v)\, da\, db - \mathcal{L}(x, y, u, v, \lambda) \tag{2.22}$$

yields

$$\tilde{\mathcal{H}}(x,y,\xi_1,\xi_2,\lambda) = \int\int_\Omega \Big(\frac{u^2+v^2}{2} - gy\Big)\, da\, db. \tag{2.23}$$

The identity (2.23) is obtained from (2.22) using integration by parts and taking into account the boundary conditions (2.5) and (2.17). Define the Hamiltonian by

$$\mathcal{H}(x,y,\xi_1,\xi_2) = \min_{\{\lambda:\ \lambda=0 \ \text{ for } \ b=0\}} \int\int_\Omega \Big(\frac{u^2+v^2}{2} - gy\Big)\, da\, db, \tag{2.24}$$

with (u,v) given by (2.21). The canonical Hamiltonian system resulting from (2.24) is

$$\dot{\mathbf{X}} = \frac{\delta\mathcal{H}}{\delta\mathbf{\Xi}}, \qquad \dot{\mathbf{\Xi}} = -\frac{\delta\mathcal{H}}{\delta\mathbf{X}}, \tag{2.25}$$

where

$$\mathbf{X} = (x,y), \qquad \mathbf{\Xi} = (\xi_1,\xi_2).$$

In view of (2.21), $\tilde{\mathcal{H}}$ is convex in λ so that the minimum in (2.24) will exist. Moreover, $\dfrac{\delta\tilde{\mathcal{H}}}{\delta\lambda} = 0$, relation which is equivalent to the incompressibility condition (2.2) - see [26] for the detailed computations. The expression (2.25) of the governing equations for water waves as a canonical Hamiltonian system is due to Maddocks and Pego [26].

To recast the water wave problem as a Hamiltonian system in terms of scalar functions (α,β) instead of the vector functions $(\mathbf{X},\mathbf{\Lambda})$, one can proceed as follows. Associated with the fluid velocity (u,v) there is a stream function ψ, determined up to a constant by

$$\psi_x = -v, \qquad \psi_y = u. \tag{2.26}$$

Introduce now the scalar function $\theta(t,x,y)$ by requiring

$$\theta_t = \dot{\lambda}, \tag{2.27a}$$

$$\theta = 0 \quad \text{on} \quad y = \eta(t,x), \tag{2.27b}$$

and determine the positive function $\alpha(t,x,y) > 0$ by solving the first-order linear partial differential equation

$$\alpha_x(\psi_x - \theta_y) + \alpha_y(\psi_y + \theta_x) + \alpha\,\omega = 0, \tag{2.28}$$

with initial data $\alpha \equiv 1$ on the curve $y = \eta(t,x)$. Here $\omega = -\psi_{xx} - \psi_{yy}$ in view of (2.6) and (2.26). Note that to ensure the existence and uniqueness of a solution to (2.28) by the method of characteristics, it suffices that

$$\begin{vmatrix} 1 & \psi_x - \theta_y \\ \eta_x & \psi_y + \theta_x \end{vmatrix} \neq 0 \quad \text{on} \quad y = \eta(t,x)$$

cf. [18]. This determinant can be explicitely computed as

$$\psi_y + \theta_x - (\psi_x - \theta_y)\eta_x = u + \theta_x + v\eta_x + \theta_y\eta_x = u + v\eta_x = u + \eta_t\eta_x + u\eta_x^2$$

if we take into account (2.4), (2.26) and the fact that (2.27b) ensures $\theta_x + \theta_y \eta_x = 0$ on $y = \eta(t,x)$. Viewing the above expression as a polynomial in η_x, we see that if

$$\eta_t^2 < 4u^2 \quad \text{on} \quad y = \eta(t,x), \tag{2.29}$$

then the method of characteristics gives the unique solution of (2.28). Defining

$$\beta(t,x,y) = \int_0^x \frac{u(t,s,y) + \theta_x(t,s,y)}{\alpha(t,s,y)}\, ds + \int_{-d}^y \frac{v(t,0,s) + \theta_y(t,0,s)}{\alpha(t,0,s)}\, ds,$$

we deduce that

$$u = -\theta_x + \alpha\beta_x, \tag{2.30a}$$
$$v = -\theta_y + \alpha\beta_y. \tag{2.30b}$$

Note that this decomposition is trivial in the irrotational case: if ϕ is the velocity potential of the irrotational flow (defined up to a constant by the requirements $\phi_x = u$, $\phi_y = v$), we can take $\beta = \phi + \theta$ and $\alpha \equiv 1$. To our knowledge, Clebsch-type representations of the form (2.30) were previously used only in the context of water flows without a free boundary (see e.g. the discussion in [30]).

An explicit calculation confirms that (2.30) yields

$$\dot{u} = \partial_x\Big(-\theta_t + \alpha\beta_t + \frac{u^2+v^2}{2}\Big) - \alpha_x\dot{\beta} + \beta_x\dot{\alpha}, \tag{2.31a}$$
$$\dot{v} = \partial_y\Big(-\theta_t + \alpha\beta_t + \frac{u^2+v^2}{2}\Big) - \alpha_y\dot{\beta} + \beta_y\dot{\alpha}. \tag{2.31b}$$

On the other hand, from (2.9), (2.15) and (2.27) we infer that

$$\theta_{tx} = -\dot{u}, \qquad \theta_{ty} = -\dot{v} - g. \tag{2.32}$$

Denoting

$$H = \alpha\beta_t + \frac{u^2+v^2}{2} + gy, \tag{2.33}$$

we obtain from (2.31) and (2.32) that

$$H_x = \alpha_x\dot{\beta} - \beta_x\dot{\alpha}, \tag{2.34a}$$
$$H_y = \alpha_y\dot{\beta} - \beta_y\dot{\alpha}. \tag{2.34b}$$

Hence $H = H(t,\alpha,\beta)$ by the implicit function theorem. By the chain rule,

$$H_x = \frac{\partial H}{\partial \alpha}\,\alpha_x + \frac{\partial H}{\partial \beta}\,\beta_x,$$

$$H_y = \frac{\partial H}{\partial \alpha}\,\alpha_y + \frac{\partial H}{\partial \beta}\,\beta_y.$$

In combination with (2.34), the above relations yield

$$\alpha_x \left(-\dot\beta + \frac{\partial H}{\partial \alpha}\right) + \beta_x \left(\dot\alpha + \frac{\partial H}{\partial \beta}\right) = 0, \tag{2.35a}$$

$$\alpha_y \left(-\dot\beta + \frac{\partial H}{\partial \alpha}\right) + \beta_y \left(\dot\alpha + \frac{\partial H}{\partial \beta}\right) = 0. \tag{2.35b}$$

The assumption of non-vanishing vorticity ensures that the linear system (2.35) has a non-zero determinant since $\alpha_y\beta_x - \alpha_x\beta_y = -\omega$ in view of (2.30). Therefore its only solution is the zero solution, so that

$$\dot\alpha = -\frac{\partial H}{\partial \beta}, \qquad \dot\beta = \frac{\partial H}{\partial \alpha}, \tag{2.36}$$

with H given by (2.33). This is the Hamiltonian form of the water wave problem with non-vanishing vorticity in terms of conjugate variables which are scalar functions, provided that the free surface satisfies (2.29). Note that for waves that are not near breaking one has $u \neq 0$ on the free surface [21] so that (2.29) is a smallness condition on the time-variation of the free surface profile.

3 Example

In this section we will compute the canonically conjugated variables (α, β) for the Hamiltonian formulation of the water wave problem in the case of a flow with constant vorticity $\omega \neq 0$ and flat surface $y = 0$ over the bed $y = -d$. The velocity field is given by

$$u = -\omega(y + d + 1), \qquad v = 0,$$

while

$$P = -gy$$

is the pressure. Clearly (2.29) holds. We first find

$$\theta = -gty.$$

Therefore (2.28) becomes

$$gt\,\alpha_x - \omega(y + d + 1)\,\alpha_y + \omega\,\alpha = 0.$$

The unique solution with data $\alpha \equiv 1$ on the non-characteristic curve $y = 0$ is

$$\alpha = \frac{y + d + 1}{d + 1}.$$

This yields

$$\beta = -\omega(d + 1)\,x - g(d + 1)t\,\ln(y + d + 1).$$

Therefore

$$\begin{aligned} H &= -g(y + d + 1)\,\ln(y + d + 1) + \frac{\omega^2(y + d + 1)^2}{2} + gy \\ &= -g(d + 1)\alpha\,\ln[\alpha(d + 1)] + \frac{\omega^2(d + 1)^2\alpha^2}{2} + g(d + 1)(\alpha - 1). \end{aligned}$$

We compute now

$$\begin{aligned}
\frac{\partial H}{\partial \alpha} &= -g(d+1)\,\ln[\alpha(d+1)] + \omega^2(d+1)^2\alpha, \\
\frac{\partial H}{\partial \beta} &= 0,
\end{aligned}$$

and

$$\begin{aligned}
\dot{\alpha} &= \alpha_t + u\alpha_x + v\alpha_y = 0, \\
\dot{\beta} &= \beta_t + u\beta_x + v\beta_y = -g(d+1)\,\ln[\alpha(d+1)] + \omega^2(d+1)^2\alpha,
\end{aligned}$$

which completes the checking of (2.36).

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