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Periods of the Goldfish Many-Body Problem

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Γηράσκω δ'αἰεί πολλὰ διδασκόμενος
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Abstract

Calogero's goldfish N-body problem describes the motion of N point particles subject to mutual interaction with velocity-dependent forces under the action of a constant magnetic field transverse to the plane of motion. When all coupling constants are equal to one, the model has the property that for generic initial data, all motions of the system are periodic. In this paper we investigate which are the possible periods of the system for fixed N, and we show that there exist initial data that realize each of these possible periods. We then discuss the asymptotic behaviour of the maximal period for large particle number N.

1 Introduction

In his book [1] F. Calogero explains different techniques to construct solvable many-body problems. We very briefly review some of the techniques involved in the derivation of the equations of motion of the goldfish many-body problem:

1. The evolution of the zeros of a polynomial the coefficients of which evolve in time can be seen as a dynamical system of interacting point particles [1, 3]. Even if the coefficients of the polynomial evolve in a simple (linear) manner, the zeros might have a complicated evolution law due to the highly nonlinear relations between the zeros and the coefficients. Yet the dynamical system obtained in this way is solvable by construction.

2. The evolution of an N^{th} order polynomial over the complex will lead to an N-body problem in the plane.

3. The evolution of any system in real time t can be substituted by an evolution in a complex variable τ, which is a periodic function of t. The evolution in real time t corresponds to travelling on a closed contour on the complex τ plane. The analytic structure of the solutions of the system in τ translate into periodicity properties of the solutions of the equivalent problem in real time t. This trick can be applied to modify a wide class of equations in such a way that the modified equations feature many periodic solutions [8].

In the rest of this Section we review some of the more relevant results on the goldfish many-body problem. Most of these results can be found in [9].

The goldfish N-body problem in the plane is given by the following equations of motion:

$$\ddot{\mathbf{r}}_i = \omega \hat{\mathbf{k}} \wedge \dot{\mathbf{r}}_i + 2 \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{a_{ij}}{r_{ij}^2} \left[\dot{\mathbf{r}}_i (\dot{\mathbf{r}}_j \cdot \mathbf{r}_{ij}) + \dot{\mathbf{r}}_j (\dot{\mathbf{r}}_i \cdot \mathbf{r}_{ij}) - \mathbf{r}_{ij} (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j) \right], \tag{1.1}$$

where $\mathbf{r}_i \equiv \mathbf{r}_i(t)$ denotes the position in the plane of the ith particle, which for notational convenience we imagine immersed in ordinary three-dimensional space, so that $\mathbf{r}_i \equiv (x_i, y_i, 0)$; $\hat{\mathbf{k}}$ is the unit three-vector orthogonal to that plane, $\hat{\mathbf{k}} \equiv (0, 0, 1)$, so that $\hat{\mathbf{k}} \wedge \mathbf{r}_n \equiv (-y_n, x_n, 0)$, and

$$r_{ij}^2 = (\mathbf{r_i} - \mathbf{r_j}) \cdot (\mathbf{r_i} - \mathbf{r_j})$$

is the distance squared between two particles. The model features pairwise velocity dependent forces that decrease when the particles are far apart. For simplicity we assume that the coupling constants a_{ij} are all real and that $\omega > 0$ is a positive constant to which the fundamental period

$$T = \frac{2\pi}{\omega} \tag{1.2}$$

can be associated.

This N-body problem in the plane is invariant under translations, rotations and changes of scale. Moreover, when the two-body velocity-dependent forces are absent, the model has a simple physical interpretation: it describes the motion of N point charges under the action of a constant magnetic field orthogonal to the plane (a cyclotron). It is obvious that every particle performs a uniform circular motion in this case. Maybe less obvious is the fact that, when the interactions are present, there exist a region R in phase space having the same dimension as the full phase space such that every trajectory originating in R is periodic. Later we see that when $a_{ij} = 1$, every orbit is periodic for *generic* initial conditions (excluding a set of null measure in phase space).

It is convenient to write the Newtonian equations (1.1) as a system of complex ordinary differential equations (ODEs) via the natural identification

$$\mathbf{r}_n \equiv (x_n, y_n, 0) \Leftrightarrow z_n \equiv x_n + \mathrm{i} y_n \ ,$$

whereby the real Newtonian equations of motion in the plane (1.1) become the following equations describing the motion of N points $z_i \equiv z_i(t)$ in the complex z-plane:

$$\ddot{z}_i = \mathrm{i}\omega \dot{z}_i + 2 \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} \frac{\dot{z}_i \dot{z}_j}{z_i - z_j}. \tag{1.3}$$

We recall that, if the coupling constants a_{ij} depend symmetrically on their two indices, the system (1.3) is Hamiltonian (see [1], [2]) and can be derived in the standard manner from the Hamiltonian

$$H(\mathbf{z},\mathbf{p}) = \sum_{n=1}^{N}\left[\frac{\mathrm{i}\omega}{c}z_n + e^{c\,p_n}\prod_{m=1,m\neq n}^{N}(z_n - z_m)^{-a_{nm}}\right] \quad , \tag{1.4}$$

where $\mathbf{z}$ denotes the N-vector $\mathbf{z} \equiv (z_1, z_2, ..., z_N)$, and c is an arbitrary (nonvanishing) constant, which does not appear in the equations of motion. Moreover, it is easy to see that the center of mass,

$$Z(t) = N^{-1}\sum_{i=1}^{N} z_n(t) \quad , \tag{1.5}$$

moves periodically with period T on a circular trajectory in the complex z-plane:

$$Z(t) = Z(0) + \dot{Z}(0)\frac{e^{(\mathrm{i}\omega t)} - 1}{\mathrm{i}\omega} \quad . \tag{1.6}$$

If we perform the following change of independent variable

$$\tau = \frac{\mathrm{e}^{\mathrm{i}\omega t} - 1}{\mathrm{i}\omega}, \qquad \zeta_i(\tau) = z_i(t), \tag{1.7}$$

the equations (1.3) can be rewritten as

$$\zeta_i'' = 2\sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}\frac{\zeta_i'\zeta_j'}{\zeta_i - \zeta_j}. \tag{1.8}$$

Note that as the real variable t (the *physical* time) varies from 0 to T, the (complex) variable τ goes from $\tau = 0$ back to $\tau = 0$ by travelling counter-clockwise on a circular contour C on the upper half complex τ-plane with its center at i/ω and radius $1/\omega$. The relations among the initial data for (1.3) and (1.8) are

$$z_i(0) = \zeta_i(0) \ , \qquad \dot{z}_i(0) = \zeta_i'(0) \quad . \tag{1.9}$$

The advantage of making the change of variables (1.7) is that the analyticity properties of the solutions of (1.8) as functions of the complex variable τ are directly translated into periodicity properties of the solutions of the *physical* system (1.1), as expressed by the following theorem:

Theorem 1. *If a solution* $\big(\zeta_1(\tau), \ldots, \zeta_N(\tau)\big)$ *of the system* (1.8) *is a* holomorphic *or* meromorphic *function of* τ *both inside and on the circular contour* C*, then the corresponding solution* $\big(\mathbf{r}_1(t), \ldots, \mathbf{r}_N(t)\big)$ *of the system* (1.1) *is* nonsingular *and* completely periodic *in real time* t*, with period* T*. Moreover, if the only singularities of* $\zeta_i(\tau)$ *inside the disk enclosed by* C *are a finite number of algebraic branch points, then the corresponding solution of* (1.1) *is again* completely periodic *with period an integer multiple of* T*.*

A special case of the equations of motion (1.1) corresponds to all coupling constants being equal to one ($a_{ij} = 1$). In this case the model is integrable, indeed exactly solvable as we see in the following Section.

The main purpose of this approach is that the powerful machinery of complex analysis can be used to derive results about the periodicity of the solutions of many-body problems of which (1.1) is just a simple example. For other results of this type see also [5, 8, 11], among others. The understanding of the transition to chaos within this framework is also the object of current research [10].

2 Possible periods of the goldfish many-body problem

As it is carefully described by F. Calogero in his book [1], one technique to construct solvable many-body problems is to look at the evolution of the zeros of a polynomial whose coefficients evolve in a known manner. A very simple (linear) evolution rule for the coefficients generally produces a complicated (nonlinear) evolution for the zeros by virtue of the highly nonlinear relation between the zeros and the coefficients of a polynomial. More precisely consider the following monic polynomial of degree N with τ-dependent coefficients

$$P(\zeta,\tau) = \zeta^N + \sum_{j=1}^{N} c_j(\tau)\,\zeta^{N-j} = \prod_{i=1}^{N}[\zeta - \zeta_i(\tau)]. \tag{2.1}$$

For instance, if $P(\zeta,\tau)$ is made to satisfy $P_{\tau\tau} = 0$, the zeros and coefficients evolve according to

$$c_i'' = 0\,, \tag{2.2}$$

$$\zeta_i'' = 2\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\zeta_i'\zeta_j'}{\zeta_i - \zeta_j}. \tag{2.3}$$

We can thus see that the evolution in τ of the coefficients is trivial, while the evolution of the zeros is governed precisely by the equations (1.8) with all coupling constants a_{ij} equal to one. It is worth to note that the equations (2.3) have also been analyzed independently by Prosen in the context of quantum chaos and random gaussian polynomials [23] and are a particular case of a larger class of integrable systems derived by Ruisjenaars and Schneider [24].

In the context of this work, the equations (2.3) can be interpreted as the Newtonian equations of motion of a system of interacting particles moving in the plane as explained in the previous Section. More complicated (yet solvable) many-body problems in the plane can be obtained by imposing other partial differential equations (PDEs) on the polynomial $P(\zeta,\tau)$. The largest family of PDEs leading to a system of second order linear coupled ODEs for the evolution of the coefficients has been explored in [1], while nonlinear evolution of the coefficients has also been treated in [13].

We analyze the periodicity of the solutions of (1.1) when $a_{ij} = 1$. We first observe that in this case the explicit solution $\{z_1(t), \dots, z_N(t)\}$ of (1.3) corresponding to the initial data $\{z_i(0), \dot{z}_i(0)\}$ can be obtained by solving the following polynomial equation in z

$$\sum_{i=1}^{N} \frac{\dot{z}_i(0)}{z - z_i(0)} = \frac{\mathrm{i}\omega}{\mathrm{e}^{\mathrm{i}\omega t} - 1}\,. \tag{2.4}$$

Yet the best way to understand the periodicity is to realize that $z_i(t) = \zeta_i(\tau)$ are the zeros of a polynomial whose coefficients $c_j(\tau)$ are periodic functions of t (since they are linear functions of τ, and τ is a periodic function of t). After one period, the coefficients of the polynomial go back to their previous values, the set of zeros is periodic with period T, but the zeros might have exchanged their position. More specifically,

$$\{z_1(t+T), z_2(t+T), \ldots, z_N(t+T)\} = \{z_{\pi(1)}(t), z_{\pi(2)}(t), \ldots, z_{\pi(N)}(t)\}, \tag{2.5}$$

where $\pi \in S_N$ is an element of the symmetric group of N elements. Every permutation $\pi \in S_N$ can be decomposed as a product of disjoint cycles, each cycle containing the particles that are exchanging their positions. The period of the solution corresponds to the *order* of the permutation, i.e. the least integer q such that $\pi^q = \mathrm{id}$. For fixed N the period of the solution of (1.8) is therefore given by

$$\left\{\mathrm{lcm}\,(\lambda_1, \ldots, \lambda_s) \,:\, \lambda_1 + \cdots + \lambda_s = N\right\} \tag{2.6}$$

for some partition $\lambda \equiv \{\lambda_1, \ldots, \lambda_s\}$ of N. The maximum of this quantity,

$$G(N) = \max_{\lambda \vdash N} \{\mathrm{lcm}\,(\lambda)\}\,, \tag{2.7}$$

over all partitions of N is sometimes called the Landau function [15] in the literature. As an example all partitions of $N = 7$ can be found in Table 1 below, where it is clear that $G(7) = 12$. For a certain particle number N, we denote by $\mathbb{T}(N)$ the set of all possible

Table 1. Orders of a permutation of 7 elements

Partition	lcm	Partition	lcm
$\{7\}$	7	$\{1,1,1,4\}$	4
$\{1,6\}$	6	$\{1,1,2,3\}$	6
$\{2,5\}$	10	$\{1,2,2,2\}$	2
$\{3,4\}$	12	$\{1,1,1,1,3\}$	3
$\{1,1,5\}$	5	$\{1,1,1,2,2\}$	2
$\{1,2,4\}$	4	$\{1,1,1,1,1,2\}$	2
$\{1,3,3\}$	3	$\{1,1,1,1,1,1,1\}$	1
$\{2,2,3\}$	6		

periods [1], which clearly includes all numbers from 1 to N. The first few values of $\mathbb{T}(N)$ have been collected in Table 2. These are all the possible periods for a fixed N, but which of these periods is actually exhibited by the system depends on the choice of initial data $\{z_i(0), \dot{z}_i(0)\}$ and in general it is not easy to predict *a priori.* We turn then to the following

[1] Of course we are assuming here that $\omega = 2\pi$ so that the fundamental period T is unity.

Table 2. Possible periods for the first few N

N	$\mathbb{T}(N)$	N	$\mathbb{T}(N)$
1	1	7	1-7, 10, 12
2	1-2	8	1-8, 10, 12, 15
3	1-3	9	1-9, 10, 12, 14, 15, 20
4	1-4	10	1-10, 12, 14, 15, 20, 21, 30
5	1-5, 6	11	1-11, 12, 14, 15, 18, 20, 21, 24, 28, 30
6	1-6	12	1-12, 14, 15, 18, 20, 21, 24, 28, 30, 35, 42, 60

Question: Do initial data $\{z_i(0), \dot{z}_i(0)\}$ exist such that the solution $\{z_1(t), \ldots, z_N(t)\}$ of the system (1.3) with $a_{ij} = 1$ has every possible period in $\mathbb{T}(N)$?

In the rest of the Section we argue that this is indeed the case. To this purpose we first show that (2.3) admits a period N solution. Indeed, by inserting the following ansatz into (2.3)

$$\zeta_j(\tau) = A + B_j(\tau - \tau_b)^{\Gamma}, \qquad j = 1, \ldots, N, \tag{2.8}$$

it can be seen [1, 7] that the system admits the similarity solution (2.8) provided that

$$\Gamma = 1/N, \qquad B_j = B\, \mathrm{e}^{2\pi \mathrm{i} \frac{j}{N}}, \qquad j = 1, \ldots, N. \tag{2.9}$$

This special similarity solution corresponds to placing all particles on the vertices of a regular N-gon and the only singularity occurs at $\tau = \tau_b$ where all particles collide simultaneously. If the branch point τ_b sits outside the circle C in the complex τ-plane with centre at i/ω and radius $1/\omega$ then the period of this solution is one (see Fig.1a) as entailed by Theorem 1. On the contrary, if the initial conditions are such that τ_b sits inside C, then the solution has period N (see Fig.1b) as it will visit the N-sheeted Riemann surface associated to the N^{th} root. In this motion the j^{th} particle takes the position of the $(j+1)^{\text{th}}$ particle after every fundamental period. Note from (2.8), (2.9) and (1.9) that, given initial data $\{z_i(0), z_i'(0)\}$, the branch point occurs at

$$\tau_b = -\frac{z_i(0)}{N z_i'(0)} \tag{2.10}$$

so that it is always possible to choose initial data such that τ_b falls inside the circle C and the corresponding solution has period N.

The next step comes by noting that, when two groups of particles are very far apart, their motions can be analyzed independently of each other. Without loss of generality, we assume that the first $i = 1, \ldots, M$ particles belong to the first group while the rest belong to the second group. The equations of motions are

$$\zeta_i'' = 2 \sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{\zeta_i' \zeta_j'}{\zeta_i - \zeta_j} + 2 \sum_{\substack{j=M+1 \\ j \neq i}}^{N} \frac{\zeta_i' \zeta_j'}{\zeta_i - \zeta_j}, \qquad i = 1, \ldots, M\,, \tag{2.11}$$

$$\zeta_i'' = 2 \sum_{\substack{j=M+1 \\ j \neq i}}^{N} \frac{\zeta_i' \zeta_j'}{\zeta_i - \zeta_j} + 2 \sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{\zeta_i' \zeta_j'}{\zeta_i - \zeta_j}, \qquad i = M+1, \ldots, N\,. \tag{2.12}$$

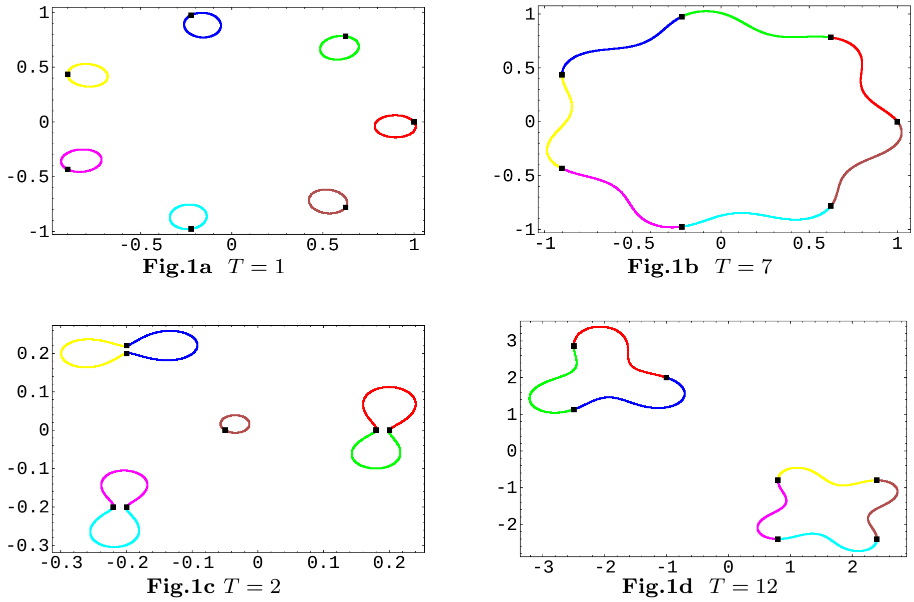

Figure 1. A few different periodic motions for $N = 7$

If generic initial conditions are chosen (i.e. such that no collisions occur at finite time), the velocities are bounded for all time from 0 to $G(N)T$, say $\max|\zeta_i'(\tau)| < K$. Now we choose the initial position of the particles such that the two groups are a distance D apart :

$$\zeta_i(0) = w_i, \qquad i = 1, \ldots, M, \tag{2.13}$$

$$\zeta_i(0) = D + w_i, \quad i = M+1, \ldots, N, \tag{2.14}$$

with $|D| \gg |w_i|$. It is clear that in the limit of D going to infinity the second terms in (2.11) and (2.12) become negligible with respect to the first terms, and the system effectively decouples. The period of the system for these initial conditions is clearly the least common multiple of the periods of the two subgroups. If we keep in mind that a system of N particles has a period N solution, the above argument can be applied iteratively to show that initial conditions exist such that every single period in $\mathbb{T}(N)$ is realized.

3 Asymptotic behavior of the maximal period for large N

It was shown in the previous section that the maximal period of the periodic solutions of (1.1) is given by the Landau function $G(N)$ defined in (2.7). In this Section we will discuss some properties of $G(N)$ and we analyze its asymptotic behavior for large N. The first few values of $G(N)$ together with the corresponding prime factors are shown in Table 3 (Grantham [14] has computed $G(N)$ up to $N = 500\,000$). From the first few values it is already possible to observe the unruly behavior of $G(N)$.

Table 3. First few values of $G(N)$

N	$G(N)$	Prime factors of $G(N)$	N	$G(N)$	Prime factors of $G(N)$	N	$G(N)$	Prime factors of $G(N)$
1	1	1	11	30	$2\cdot3\cdot5$	21	420	$2^2\cdot3\cdot5\cdot7$
2	2	2	12	60	$2^2\cdot3\cdot5$	22	420	$2^2\cdot3\cdot5\cdot7$
3	3	3	13	60	$2^2\cdot3\cdot5$	23	840	$2^3\cdot3\cdot5\cdot7$
4	4	2^2	14	84	$2^2\cdot3\cdot7$	24	840	$2^3\cdot3\cdot5\cdot7$
5	6	$2\cdot3$	15	105	$3\cdot5\cdot7$	25	1260	$2^2\cdot3^2\cdot5\cdot7$
6	6	$2\cdot3$	16	140	$2^2\cdot5\cdot7$	26	1260	$2^2\cdot3^2\cdot5\cdot7$
7	12	$2^2\cdot3$	17	210	$2\cdot3\cdot5\cdot7$	27	1540	$2^2\cdot5\cdot7\cdot11$
8	15	$3\cdot5$	18	210	$2\cdot3\cdot5\cdot7$	28	2310	$2\cdot3\cdot5\cdot7\cdot11$
9	20	$2^2\cdot5$	19	420	$2^2\cdot3\cdot5\cdot7$	29	2520	$2^3\cdot3^2\cdot5\cdot7$
10	30	$2\cdot3\cdot5$	20	420	$2^2\cdot3\cdot5\cdot7$	30	4620	$2^2\cdot3\cdot5\cdot7\cdot11$

No explicit expression of $G(N)$ as a function of N is known, yet results on the asymptotic behaviour of $G(N)$ for large N are known as far back as the early 1900s. This asymptotic behaviour (see Fig. 2) is given by

$$\log G(N) = (N \log N)^{1/2} + \frac{N^{1/2} \log\log N}{2\,(\log N)^{1/2}} + O\left(\sqrt{\frac{N}{\log N}}\right) \quad . \tag{3.1}$$

The first term of this formula was proved by Landau in his *Handbuch* [15], while the subsequent terms of the asymptotic behaviour were proved later by Shah [25]. Since then there has been a number of papers devoted to the study of this function (see, for instance, [16–20]). A particularly interesting result obtained by Erdös and Turán states that very few permutations of N elements have orders as large as $G(N)$, *most* of them (in a sense made precise in [12]) having orders whose logarithm grows like $(\log N)^2/2$.

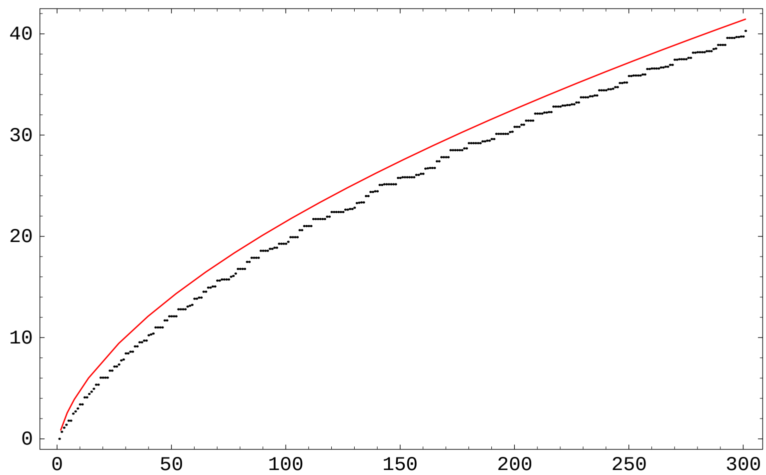

Figure 2. The functions $\log G(N)$ and $\sqrt{N \log N}$ for N up to 301

In the rest of the Section we present a somewhat different proof of Landau's result, namely that

$$\log G(N) \sim \sqrt{N \log N} \qquad \text{for large } N. \tag{3.2}$$

To this purpose, we firstly introduce two functions defined on the set of primes. Let $P(N)$ be the prime number such that the sum of all primes less than $P(N)$ is not greater than N, but the sum of all primes up to and including $P(N)$ is greater than N. Next we define $F(N)$ to be the product of primes strictly less than $P(N)$. For example, for $N = 36$, we have $2 + 3 + 5 + 7 + 11 = 28$ and $2 + 3 + 5 + 7 + 11 + 13 = 41$ so that $P(36) = 13$ and $F(36) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$.

Since $G(N)$ is clearly a non-decreasing function of N, it follows that

$$F(N) \leq G(N) . \tag{3.3}$$

In order to obtain an upper bound for $G(N)$ we need two technical lemmas [18],

Lemma 1 (Shah). *Let $q_1 < ... < q_s$ be all the primes dividing $G(N)$. Then*

$$\sum_{j=1}^{s} \log q_j < 2 + \log F(N) + \log P(N) .$$

Lemma 2. *Let q be a prime and e an integer greater than 1. If q^e divides $G(N)$, then*

$$q^e \leq 2P(N) \quad \text{and} \quad q \leq \sqrt{2P(N)} .$$

Now let

$$G(N) = \prod_{j=1}^{s} q_j^{e_j}$$

be the prime factorization of $G(N)$, which can be decomposed as

$$\log G(N) = \sum_{j \text{ s.t. } e_j = 1} \log q_j + \sum_{j \text{ s.t. } e_j > 1} e_j \log q_j , \tag{3.4}$$

the first subsum corresponding to the prime factors that appear only once and the second corresponding to the factors for which $e_j > 1$. By Lemma 1 the first subsum in (3.4) is at most $2 + \log F(N) + \log P(N)$; while by Lemma 2 it follows that each term in the second subsum in (3.4) is lesser than $\log 2P(N)$ and there are at most $\sqrt{2P(N)}$ of them. With (3.3), this entails

$$\log F(N) \leq \log G(N) \leq 2 + \log F(N) + \log P(N) + \sqrt{2P(N)}\,[\log 2P(N)] . \tag{3.5}$$

To obtain the asymptotic behaviour of $G(N)$ it suffices now to understand the behaviour of $F(N)$ and $P(N)$ for large N.

Incidentally one might expect that, for those N that are the sum of the first s consecutive primes, the value of $G(N)$ is just the product of these primes, namely that $G(N) = F(N)$ for $N = \sum_{j=1}^{s} p_j$. However, contrary to our intuition, this statement happens to be false: the first such N for which $G(N) \neq F(N)$ occurs at $N = 100$

$$\begin{aligned} F(100) &= \prod_{j=1}^{9} p_j = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 = 223\,092\,870 , \\ G(100) &= G(97) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = 232\,792\,560 . \end{aligned} \tag{3.6}$$

To proceed we introduce the functions

$$S(x) = \sum_{p \leq x} p \quad \text{and} \quad \theta(x) = \sum_{p \leq x} \log p \,, \tag{3.7}$$

where the sums are taken over all consecutive primes p less or equal to the positive real variable x. The asymptotic behavior for large x of these two functions

$$S(x) \sim \frac{x^2}{2 \log x} \quad \text{and} \quad \theta(x) \sim x \tag{3.8}$$

is actually an equivalent form of the Prime Number Theorem (see, for instance, [22]). Now, since $\log F(N) = \theta(P(N) - 1)$ and $P(N) \sim P(N) - 1$ for large N, the second of (3.8) implies that $\log F(N) \sim P(N)$, and thus it suffices to show that $P(N) \sim \sqrt{N \log N}$ for large N. By the definition of $P(N)$ and $S(x)$, (see (3.7)), we have that

$$S(P(N) - 1) \leq N < S(P(N)) \,.$$

Clearly the first of (3.8) implies that for large x we have $S(x-1) \sim S(x)$ and

$$\frac{P(N)^2}{2 \log P(N)} \sim N \,. \tag{3.9}$$

We proceed by *reductio ad absurdum*: suppose that $P(N)$ is not asymptotic to $\sqrt{N \log N}$. Then there is a positive number ϵ such that for infinitely many values of N one of the following two inequalities holds:

$$P(N) \leq (1 - \epsilon)\sqrt{N \log N} \quad \text{or} \quad P(N) \geq (1 + \epsilon)\sqrt{N \log N} \;. \tag{3.10}$$

Since $x^2/(2 \log x)$ is an increasing function for $x > \sqrt{e}$, the first of (3.10) entails

$$\frac{P(N)^2}{2\, N \log P(N)} \leq \frac{(1-\epsilon)^2 (\log N)}{\log N + \log \log N + 2 \log (1 - \epsilon)} \,. \tag{3.11}$$

As N approaches infinity, the right hand side of (3.11) approaches $(1-\epsilon)^2$, while by (3.9)) the left hand side approaches unity. It follows that the first inequality of (3.10) cannot hold for infinitely many N. The same argument applies to the second inequality in (3.10), we conclude that $P(N) \sim \sqrt{N \log N}$ and therefore $\log F(N) \sim \sqrt{N \log N}$. By (3.5) this implies in turn the desired result (3.2). ■

4 Final Remarks

The results on the asymptotic behaviour of the highest period derived in the previous Section are not only applicable to the goldfish many-body problem (2.3), but also to many of the dynamical systems considered in [1], where the particle positions are the zeros of a polynomial the coefficients of which evolve periodically in time. We have shown (by a rather physical argument) that for the goldfish there are initial conditions such that every possible period is realized. To be able to predict for each initial condition what is the corresponding period is not an easy problem. In order to tackle this problem a global analysis of the topology of the Riemann surface associated to the solutions of the complex systems of ODEs is needed. The various periods in $\mathbb{T}(N)$ correspond to all the topologically different closed contours on this Riemann surface. This approach is the subject of current research.

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References

[1] Calogero F, *Classical many-body problems amenable to exact treatments*, Lecture Notes in Physics Monograph **m 66**, Springer, Berlin, 2001.

[2] Calogero F, The "Neatest" Many-Body Problem Amenable to Exact Treatments (a "Goldfish"?), *Physica* **D 152/153** (2001), 78–84.

[3] Calogero F, Motion of Poles and Zeros of Special Solutions of Nonlinear and Linear Partial Differential Equations, and Related 'Solvable' Many-Body Problems, *Nuovo Cimento* **B43** (1978), 177–241.

[4] Calogero F, A class of integrable Hamiltonian systems whose solutions are (perhaps) all completely periodic, *J. Math. Phys.* **38** (1997), 5711-5719.

[5] Calogero F, A complex deformation of the classical gravitational many-body problem that features a lot of completely periodic motions, *J. Phys. A: Math. Gen.* **35** (2002), 3619-3627.

[6] Calogero F, Solvable Three-Body Problem and Painlevé Conjectures, *Theor. Math. Phys.* **133** (2002), 1443–1452; Erratum **134** (2003), 139.

[7] Calogero F, Tricks of the Trade: Relating and Deriving Solvable and Integrable Dynamical Systems, in Calogero–Moser–Sutherland Models, Editors: van Diejen J F and Vinet L, Proceedings of the Workshop on Calogero–Moser–Sutherland Models held in Montreal, 10–15 March 1997, CRM Series in Mathematical Physics, Springer, 2000, 93–116.

[8] Calogero F and Francoise J-P, Periodic motions galore: how to modify nonlinear evolution equations so that they feature a lot of periodic solutions, *J. Nonlin. Math. Phys.* **9** (2002), 99-125.

[9] Calogero F, Françoise J-P and Sommacal M, Periodic solutions of a many-rotator problem in the plane. II. Analysis of various motions, *J. Nonlinear Math. Phys.* **10** (2003) 157–214.

[10] Calogero F, Gómez-Ullate D, Santini P M and Sommacal M, Chaos as travel on a Riemann surface, work in progress.

[11] Calogero F and Inozemtsev V I, Nonlinear harmonic oscillators, *J. Phys. A: Math. Gen.* **35** (2002) 10365-10375.

[12] Erdös P and Turán P, On Some Problems of Statistical Group Theory, Z fur Wahrscheinlichkeitstheorie und verw. Gebiete, **18** (1965), 151-163.

[13] Gómez-Ullate D, Hone A N W and Sommacal M, New many-body problems in the plane with periodic solutions, *New J. Phys.* **6** (2004), 1–23.

[14] Grantham J, The Largest Prime Dividing the Maximal Order of an Element of S_n, Math. Comput. **64** (1995), 407-410.

[15] Landau E, Über die Maximalordnung der Permutationen gegebenes Grades, *Archiv der Math. und Phys.* (1903), 92-103.
Handbuch der Lehre von der Verteilung der primzahlen, 2nd Ed. Chelsea, New York, 1953.

[16] Massias J, Majoration explicite de l'ordre maximum d'un element du group symtrique, Ann. Fac. Sci. Toulouse Math. (5), **6** (1984) no. 3-4, 269-281.

[17] Massias J, Nicolas J and Robin G, Effective Bounds for the Maximal Order of an Element in the Symmetric Group, Math. Comput. **53** (1989), 665-678.

[18] Miller W, The Maximum Order of an Element of a Finite Symmetric Group, The American Mathematical Monthly, **94** (1987), 497-506.

[19] Nathanson M B, On the Greatest Order of an Element of the Symmetric Group, The American Mathematical Monthly, **79** (1972), 500-501.

[20] Nicolas J L, Sur l'ordre maximum d'un élément dans le group S_n des permutations, Acta Arithmetica XIV (1968) 315-325.

[21] Nicolas J L, Calcul de l'ordre maximum d'un élément du groupe symétrique, Rev. Francaise Informat. Recherche Operationelle, **3** (1969), 43-50.

[22] Niven I and Zuckerman H S, An Introduction to the Theory of Numbers, 4th ed., Wiley, New York, 1980.

[23] Prosen T, Parametric statistics of zeros of Husimi representations of quantum chaotic eigenstates and random polynomials, J. Phys. A, **29** (1996) 5429–5440.

[24] Ruijsenaars S N M and Schneider H, A New Class of Integrable Systems and its relation to solitons, Ann. Phys. (NY) **170** (1986) 370-405.

[25] Shah S M, An inequality for the arithmetical function $g(x)$, *J. Ind. Math. Soc.* **3** (1938), 316–318.