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The Time Periodic Solution of the Burgers Equation on the Half-Line and an Application to Steady Streaming

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Abstract

The phenomenon of steady streaming, or acoustic streaming, is an important physical phenomenon studied extensively in the literature. Its mathematical formulation involves the Navier-Stokes equations, and due to the complexity of these equations is usually studied heuristically using formal perturbation expansions. It turns out that the Burgers equation formulated on the half line provides a simple model of the above phenomenon. The physical situation corresponds to the solution of the Dirichlet problem on the half-line, which decays as $x \to \infty$ and which is *time periodic*. We show that the Dirichlet problem, where the usual prescription of the initial condition is now replaced by the requirement of the time periodicity, yields a well posed problem. Furthermore, we show that the solution of this problem tends to the "inner" and "outer" solutions obtained by the perturbation expansions.

1 Introduction

The phenomenon of "steady streaming", or "acoustic streaming" as it is sometimes called, dates back to the 19th century with the work of Kundt on the circulations of air in tubes (see Rayleigh 1883, Rosenblat 1959). There are related phenomena of circulations in the flows under water waves with a free surface (Longuet-Higgins 1953, 1960; Hunt and Johns 1963). Following Rayleigh, Schlichting (1932) did both experimental and theoretical work on a particular case of the flow induced by a circular cylinder oscillating along a diameter.

The nature of the phenomenon, which is common to all the papers mentioned above, may be explained as follows. If the flow of fluid (liquid or gas) is periodic in time but with zero mean, and is parallel to a rigid surface, then the governing equations ensure that a thin periodic boundary layer is formed in the neighborhood of that surface. If the time frequency is ω-radians per second and ν is the kinematic viscosity then the thickness of this layer is of order $(\nu/w)^{1/2}$. This boundary layer is known as a Stokes layer. A case of particular simplicity is that in which the flow is parallel to a flat rigid surface, whose velocity is $U_w \cos wt$ parallel to itself, t being the time and U_w a constant. Then the velocity in the fluid neighboring the surface is

$$u = U_w e^{-\eta} \cos(\omega t - \eta), \tag{1.1}$$

where

$$\eta = y(w/2\nu)^{1/2}, \tag{1.2}$$

y being the coordinate normal to the surface. This is the prototype Stokes layer (Stokes 1851), see also (Stuart 1963, Benney 1964). If a Galilean transformation is imposed such that the surface is at rest but the fluid has velocity $U \cos wt$ parallel to that surface as y and η tend to infinity, then the velocity in the fluid is

$$U[\cos wt - e^{-\eta} \cos(wt - \eta)]. \tag{1.3}$$

In the above description, the fluid motion is intrinsically linear so that the simple form of the Stokes layer emerges exactly: the nonlinear terms are identically zero for (1.1) and (1.3).

The situation is quite different, however, if the velocity (as y and η tend to infinity) has the form

$$U(\xi) \cos wt, \tag{1.4}$$

where ξ is the coordinate parallel to the surface: the latter may be curved and its curvature is neglected if the Stokes layer thickness $(\nu/w)^{1/2}$ is small compared with a typical length d: thus ν/wd^2 is supposed to be small. The nonlinear term in the Navier-Stokes equation, namely $u \cdot \nabla u$, where u is the vector velocity, is not zero however. It has terms of two types: (i) there are terms proportional to $\exp(2iwt)$ and to $\exp(-2iwt)$ which generate corresponding flow components; the equations that govern these components do have solutions which tend to zero at the edge of the Stokes layer as is required, (ii) a mean term is also generated, which drives the steady streaming; we can refer to this mean term, which is an effect of rectification, as a Reynolds stress (or, rather a derivative of the Reynolds stress). An equation can be obtained for the mean flow generated by this Reynolds stress but, somewhat paradoxically, it is not possible to obtain a solution whose component of velocity that is parallel to the surface, tends to zero as the edge of the Stokes layer is approached. Rather the best that can be achieved is to ensure that the solution is finite for that component of velocity.

Indeed it is found that, as the edge of the Stokes layer is approached, the velocity component approaches

$$\frac{-3}{4w} U(\xi) \frac{dU(\xi)}{d\xi}. \tag{1.5}$$

Since ξ has a typical length d, this velocity component has scale U_0^2/wd, where U_0 is the scale of $U(\xi)$. Now in problems of the Navier-Stokes equations, the famous Reynolds number plays a significant role; it is the product of a velocity and a distance, divided by the kinematic viscosity. In the present case we have

$$R_s = (U_0^2/wd)(d/\nu) = U_0^2/w\nu. \tag{1.6}$$

This parameter is known as the steady-streaming Reynolds number. Stuart (1963, 1966) showed the importance of this concept for the calculation of the flow forced by (1.5) outside the Stokes layer.

In the papers of Rayleigh (1883) and Schlichting (1932), it is implicit that the parameter R_s is small, and the calculation of the flow outside the Stokes layer is performed on that basis. Even so, Schlichting's paper makes it clear that he was aware that his theory for small values of R_s is inappropriate as an explanation of his experimental work, for which $R_s = 250$.

Stuart (1963, 1966) showed how an "outer-boundary-layer theory", which is valid for large values of R_s, can be used to obtain a solution for which (1.5) applies at the edge of the Stokes layer but which tends to zero as the distance from the surface tends to infinity. Implicit in that work is the idea that this "outer" (or "steady-streaming') boundary layer is much thicker than the Stokes layer, indeed by a factor $R_s^{1/2}$. Following Stuart's work, Riley (1998) has pursued problems of this type in much detail.

A problem can be posed that shows many of the characteristics that are outlined above, but without the complications of the full Navier-Stokes equations. The relevant equation is a Burgers type equation for $u(x,t)$, namely

$$u_t + \beta(u - k\beta)u_x = \frac{1}{2}u_{xx}, \tag{1.7}$$

where β is a small parameter which plays a role analogous to $R_s^{-1/4}$ above (in the sense that β also multiplies the nonlinear term), R_s being large and k is $O(1)$. The boundary conditions are

$$\begin{aligned} x = 0: &\quad u(0,t) = \cos t, \\ x \to \infty: &\quad u(\infty, t) \to 0, \end{aligned} \tag{1.8}$$

and

$$u(x,t) = u(x, t + 2\pi).$$

The Cole-Hopf transformation

$$u - k\beta = -\theta_x/\beta\theta \tag{1.9}$$

gives

$$\theta_t - \frac{1}{2}\theta_{xx} = 0, \tag{1.10}$$

with the boundary conditions

$$\begin{aligned} x = 0: &\quad \theta_x(0,t) + \beta(\cos t - k\beta)\theta(0,t) = 0, \\ x \to \infty: &\quad \theta_x(\infty,t)/\theta(\infty,t) = k\beta^2, \\ &\theta_x(x,t)/\theta(x,t) = \theta_x(x,t+2\pi)/\theta(x,t+2\pi). \end{aligned} \tag{1.11}$$

The equation (1.10) is simply stated, but the boundary and periodicity conditions are complicated. We return to a discussion of this problem in later sections.

In the meantime we note that a formal solution to (1.7), (1.8) can be obtained by expanding u in a power series in β

$$u = u_0(x,t) + \beta u_1(x,t) + \beta^2 u_2(x,t) + \cdots,$$

and it is quickly found that

$$u_0 = e^{-x}\cos(t-x). \tag{1.12}$$

At order β, the u_1 terms involving e^{2it} and e^{-2it} give no difficulty, but the steady (or rectified) term has to satisfy

$$u_{1xx} = -e^{-2x};$$

the solution which is bounded as $x \to \infty$ and which is zero at $x = 0$ is

$$u_1 = \frac{1}{4}(1 - e^{-2x}). \tag{1.13}$$

We note that as $x \to \infty$, $u \to \frac{1}{4}\beta$ plus higher order terms. A re-scaling of (1.7) for the steady part of the solution with

$$u(x) = \beta v(x), \quad x = z/2\beta^2$$

yields

$$(v-k)v_z = v_{zz}, \tag{1.14}$$

with the boundary conditions

$$z \to 0: \quad v \to 1/4,$$
$$z \to \infty: \quad v \to 0. \tag{1.15}$$

The solution of (1.14) subject to (1.15) is

$$v = \frac{2k}{1+(8k-1)e^{kz}}. \tag{1.16}$$

Thus

$$u = \frac{2\beta k}{1+(8k-1)e^{2\beta^2 kx}} \tag{1.17}$$

We note that the length scale of this "outer" region is $(2\beta^2)^{-1}$ times the scale of the "inner" region of (1.12) and (1.13). Also $z \to 0$ in (1.15) corresponds to $x \to \infty$ in (1.13), in the sense of $\beta \to 0$ with x fixed and $\beta \to 0$ with z fixed giving an equivalence.

This heuristic argument will be justified in later sections by solving explicitly Burgers equation (1.7) with the conditions (1.8). We note that the mathematical novelty of this problem is that it is posed on the half-line and that it requires periodicity in t. We recall that the usual Dirichlet problem for the Burgers equation on the half-line was analysed by Calogero and De Lillo. In their formulation, in addition to the Dirichlet boundary condition, one also specifies the initial condition $u(x, 0)$. In the present situation, instead of specifying $u(x, 0)$, one requires periodicity.

In section 3 we will analyse the time periodic solution of the Burgers equation on the half-line with two different boundary conditions at the origin: we specify either (a) the integral of $u(x,t)$, or (b) $u(0,t)$. The physical problem corresponds to the case (b), however we have also included case (a), because for this case the coefficients A_n of the associated Fourier series (given by the representation (3.3)) can be computed explicitly. For case (b) the coefficients A_n satisfy a *second* order difference equation, see equation (3.4b). We will show that this equation has a *unique* solution by imposing the condition that $A_n \to 0$ as $n \to \infty$ (which is needed for the convergence of the series). Although we cannot give the explicit form of A_n in general, we will compute A_n in the case that β is small. In this case we will show that the associated series converges and that the solution $u(x,t)$ tends to the "inner" and "outer" solutions obtained by the perturbation expansion, see equations (5.15) and (5.16).

2 From Burgers to the heat equation

Proposition 2.1. Let β and c be positive constants. Let the real-valued function $u(x,t)$ solve

$$u_t = \frac{1}{2}u_{xx} - \beta u u_x + c u_x, \quad 0 < x < \infty, \quad t > 0, \tag{2.1a}$$

$$u \to 0 \text{ as } x \to \infty, \tag{2.1b}$$

$$u(x,t) \text{ is } 2\pi \text{ periodic in } t, \tag{2.1c}$$

and one of the following boundary conditions:

$$\int_0^\infty u(x,t)dx = \cos t, \quad t > 0, \tag{2.2a}$$

or

$$u(0,t) = \cos t, \quad t > 0. \tag{2.2b}$$

Define $\varphi(x,t)$ by

$$\varphi(x,t) = e^{-\beta \int_\infty^x u(\xi,t)d\xi} - 1. \tag{2.3}$$

Then $\varphi(x,t)$ is a real-valued function and solves

$$\varphi_t = \frac{1}{2}\varphi_{xx} + c\varphi_x, \quad 0 < x < \infty, \quad t > 0, \tag{2.4a}$$

$$\varphi \to 0 \text{ as } x \to \infty, \tag{2.4b}$$

$$\varphi(x,t) \text{ is } 2\pi \text{ periodic in } t, \tag{2.4c}$$

and satisfies one of the following boundary conditions, respectively:

$$\varphi(0,t) = e^{\beta \cos t} - 1, \quad t > 0, \tag{2.5a}$$

or

$$\varphi_x(0,t) + \beta \cos t \varphi(0,t) + \beta \cos t = 0, \quad t > 0. \tag{2.5b}$$

For the physical application $c = \beta^2 k$.

Proof The function φ is well defined and $\varphi \to 0$ as $x \to \infty$.

If E is defined by

$$E \doteqdot \exp\left[-\beta \int_\infty^x u(\xi,t)d\xi\right],$$

then

$$\varphi_x = -\beta u E,$$

$$\varphi_{xx} = -\beta u_x E + \beta^2 u^2 E,$$

$$\varphi_t = -\beta \int_\infty^x u_t d\xi,$$

thus

$$\varphi_t - \frac{1}{2}\varphi_{xx} - c\varphi_x = \left(\int_\infty^x u_t d\xi - \frac{1}{2}u_x + \frac{1}{2}\beta u^2 - cu\right) E\beta = 0.$$

Also

$$\varphi(x, t + 2\pi) = e^{-\beta \int_\infty^x u(\xi, t+2\pi) d\xi} - 1 = \varphi(x, t).$$

Equation (2.2a) implies

$$\varphi(0, t) = e^{\beta \cos t} - 1.$$

Furthermore,

$$\varphi_x(0, t) = -\beta u(0, t) \exp\left[-\beta \int_\infty^0 u(\xi, t) d\xi\right],$$

$$\varphi(0, t) = \exp\left[-\beta \int_\infty^0 u(\xi, t) d\xi\right] - 1,$$

hence

$$\varphi_x(0, t) = -\beta u(0, t) \left[\varphi(0, t) + 1\right],$$

which implies (2.5b).

3 The solution for $u(x, t)$

Proposition 3.1. Assume that the positive constants β and c, are such that λ_n defined by

$$\lambda_n \doteqdot -c + (c^2 + 2in)^{\frac{1}{2}}, \quad n = 1, 2, \cdots \tag{3.1}$$

has 1 value with $\mathrm{Re}\lambda_n < 0$. Then the solution of (2.1)-(2.2) is given by

$$u(x, t) = -\frac{1}{\beta} \frac{\varphi_x(x, t)}{1 + \varphi(x, t)}, \tag{3.2}$$

where

$$\varphi(x, t) = \frac{1}{2\pi} \left\{ A_0 e^{\lambda_0 x} + \sum_1^\infty \left(A_n e^{\lambda_n x + int} + \bar{A}_n e^{-\bar{\lambda}_n x - int} \right) \right\}, \tag{3.3}$$

and $\lambda_0, \{A_j\}_0^\infty$ are defined as follows:

$$\lambda_0 = -2c;$$

in the case of the boundary condition (2.2a)

$$A_n = \int_0^{2\pi} e^{-int} \left[e^{\beta \cos t} - 1\right] dt, \tag{3.4a}$$

whereas in the case of the boundary condition (2.2b)

$$2\lambda_n A_n + \beta(A_{n+1} + A_{n-1}) + 2\pi\beta\delta_{n,1} = 0, \quad n = 0, 1, 2, \cdots . \tag{3.4b}$$

Proof. Equation (2.3) implies equation (3.2). Thus the problem reduces to solving an initial-boundary value problem for the heat equation (2.4)-(2.5).

Let

$$\hat{\varphi}(x,n)=\int_0^{2\pi}\varphi(x,t)e^{-int}dt, \tag{3.5}$$

$$\varphi(x,t)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\hat{\varphi}(x,n)e^{int}=\frac{1}{2\pi}\left\{\hat{\varphi}(x,0)+\sum_{1}^{\infty}\left(\hat{\varphi}(x,n)e^{int}+\hat{\varphi}(x,-n)e^{-int}\right)\right\}.$$

Equation (3.5) and reality imply

$$\hat{\varphi}(x,-n)=\overline{\hat{\varphi}(x,n)}$$

.

Thus

$$\varphi(x,t)=\frac{1}{2\pi}\left\{\hat{\varphi}(x,0)+\sum_{1}^{\infty}(\hat{\varphi}(x,n)e^{int}+\overline{\hat{\varphi}(x,n)}e^{-int})\right\}. \tag{3.6}$$

Equations (3.5) and (2.4a) imply

$$\hat{\varphi}_{xx}+2c\hat{\varphi}_x=2\int_0^{2\pi}\varphi_t(x,t)e^{-int}dt=2\left[\varphi(x,t)e^{-int}\Big|_0^{2\pi}+in\hat{\varphi}\right].$$

Thus using periodicity, we find

$$\hat{\varphi}_{xx}+2c\hat{\varphi}_x-2in\hat{\varphi}=0. \tag{3.7}$$

$n=0$**:**

$$\hat{\varphi}=c_1+A_0e^{\lambda_0 x},\quad \lambda_0=-2c;$$

the boundness requirement as $x\to\infty$ implies $c_1=0$. Thus

$$\hat{\varphi}(x,0)=A_0e^{\lambda_0 x}. \tag{3.8a}$$

$n=1,2,\cdots$**:**

$$\hat{\varphi}(x,n)=A_ne^{\lambda_n x},\quad \lambda_n^2+2c\lambda_n-2in=0. \tag{3.8b}$$

Substituting (3.8) into (3.6) we find (3.3). If the boundary condition (2.2a) is valid, then

$$A_n=\hat{\varphi}(0,n)=\int_0^{2\pi}\varphi(0,t)e^{-int}dt,\quad n=0,1,2,\cdots,$$

which is equation (3.4a).

If the boundary condition (2.2b) is valid, then multiplying equation (2.5b) by e^{-int} and integrating from 0 to 2π we find

$$\int_0^{2\pi}e^{-int}\left[2\varphi_x(0,t)+\beta(e^{it}+e^{-it})\varphi(0,t)+\beta(e^{it}+e^{-it})\right]=0,\quad n=0,1,\cdots.$$

Thus

$$2\hat{\varphi}_x(0,n)+\beta[\hat{\varphi}(0,n+1)+\hat{\varphi}(0,n-1)]+2\pi\beta\delta_{n,1}=0, n=0,1,\cdots.$$

Using

$$\hat{\varphi}_x(0,n)=\lambda_nA_n,$$

the above equation becomes (3.4b).

4 The boundary value problem (2.1)-(2.2a) with $c = \beta^2 k$ as $\beta \to 0$

Let

$$c = \beta^2 k. \tag{4.1}$$

The solution of this problem is given by (3.2) and (3.3) where A_n is defined by (3.4a). The integral (3.4a) can be computed explicitly. For simplicity we only compute it as $\beta \to 0$:

$$\begin{aligned} A_n &= \int_0^{2\pi} e^{-int}\left[\beta\cos t + \frac{\beta^2}{2!}(\cos t)^2 + O(\beta^3)\right] dt, \quad n = 0, 1, \cdots \\ &= \oint_{|z|=1} \frac{1}{iz^{n+1}}\left[\frac{\beta}{2}(z+\frac{1}{z}) + \frac{\beta^2}{8}\left(z^2 + \frac{1}{z^2} + 2\right) + O(\beta^3)\right] dz \\ &= \frac{\beta}{2i}\oint_{|z|=1}\frac{1}{z^{n+1}}\left(z+\frac{1}{z}\right)dz + \frac{\beta^2}{8i}\oint_{|z|=1}\frac{1}{z^{n+1}}\left(z^2+2+\frac{1}{z^2}\right)dz + O(\beta^3). \end{aligned}$$

Thus

$$A_n = 2i\pi\left[\frac{\beta}{2i}\delta_{n,1} + \frac{\beta^2}{8i}\left(2\delta_{n,0} + \delta_{n,2}\right)\right] + O(\beta^3).$$

Hence

$$A_1 = \pi\beta + O(\beta^3), \quad A_0 = \frac{\beta^2\pi}{2} + O(\beta^3), \quad A_2 = \frac{\pi\beta^2}{4} + O(\beta^3).$$

Also

$$\lambda_n = -\sqrt{2}e^{\frac{i\pi}{4}}\sqrt{n} - k\beta^2 + O(\frac{\beta^4}{\sqrt{n}}).$$

Therefore equation (3.3) implies

$$\phi(x,t) = \frac{\beta^2}{4}e^{-2\beta^2 kx} + \frac{\beta}{2}e^{-\beta^2 kx}F(x,t) + \frac{\beta^2}{8}e^{-\beta^2 kx}G(x,t) + O(\beta^3 E) \tag{4.2}$$

where

$$F(x,t) = e^{-(1+i)x+it} + e^{-(1-i)x-it},$$

$$G(x,t) = e^{-\sqrt{2}(1+i)x+2it} + e^{-\sqrt{2}(1-i)x-2it}, \tag{4.3}$$

and $E(x,t)$ is an exponential function such that $E \to 0$ exponentially fast as $x \to 0$ and $E(x/\beta^2, t) \to 0$ exponentially fast as $\beta \to 0$.

Using the definition of $u(x,t)$ in terms of $\phi(x,t)$,i.e. equation (3.2), we find

$$u = -\frac{-\frac{\beta^3 k}{2}e^{-2\beta^2 kx} + \frac{1}{2}e^{-\beta^2 kx}F_x + \frac{\beta}{8}e^{-\beta^2 kx}G_x + O(\beta^2 E)}{1 + \frac{\beta^2}{4}e^{-2\beta^2 kx} + \frac{\beta}{2}e^{-\beta^2 kx}F + O(\beta^2 E)}. \tag{4.4}$$

$1.x = O(\beta^{-2})$

In this case $E \to 0$ as $\beta \to 0$. Thus u tends exponentially fast to

$$u \sim \frac{2\beta^3 k}{\beta^2 + 4e^{2\beta^2 kx}} \tag{4.5}$$

2. $x = O(1)$
In this case $\exp(-2\beta^2 kx) = 1 + O(\beta^2)$, thus

$$u = -[-\frac{\beta^3 k}{2} + \frac{F_x}{2} + \frac{\beta}{8}G_x + O(\beta^3 E)][1 - \frac{\beta}{2}F - \frac{\beta^2}{4} + O(\beta^2 E)],$$

or,

$$u = \frac{\beta^3 k}{2} - \frac{F_x}{2} + (FF_x - \frac{G_x}{4})\frac{\beta}{2} + O(\beta^2 E). \tag{4.6}$$

Equations (4.5) and (4.6) define the outer and inner solutions respectively. The inner and outer limits of these solutions are obtained by letting $x \to 0$ and $x \to \infty$ in equations (4.5) and (4.6) respectively. Using the fact that $E \to 0$ as $x \to \infty$, it follows that both these limits equal $\beta^3 k/2$.

5 The analysis of (2.1)-(2.2b) with $c = \beta^2 k$, $k \neq 1/8$, as $\beta \to 0$

The solution of the linear homogeneous equation (3.4b), with the requirement that

$$A_n \to 0 \quad \text{as} \quad n \to \infty \tag{5.1}$$

is unique. This solution is given by

$$A_n = -2\pi\beta F_1 + (-1)^n A_0 F_1 \cdots F_n, \tag{5.2}$$

where F_n is defined by

$$F_n = \cfrac{1}{\frac{2\lambda_n}{\beta} - \cfrac{1}{\frac{2\lambda_{n+1}}{\beta} - \cfrac{1}{\frac{2\lambda_{n+2}}{\beta}\cdots}}} \tag{5.3}$$

and the real constant A_0 is determined from

$$-4cA_0 + \beta(A_1 + \bar{A}_1) = 0. \tag{5.4}$$

Indeed, the definition F_n implies

$$F_n = \frac{1}{\frac{2\lambda_n}{\beta} - F_{n+1}}.$$

Solving this equation for $2\lambda_n/\beta$ we find

$$\frac{2\lambda_n}{\beta} = F_{n+1} + \frac{1}{F_n}.$$

Substituting this expression into equation (3.4b) we obtain

$$A_{n+1} + F_{n+1}A_n + \frac{1}{F_n}(A_n + F_n A_{n-1}) = -2\pi\beta\delta_{n,1}, \quad n = 0, 1, 2, \cdots \tag{5.5}$$

Letting

$$G_n \doteqdot A_n + F_n A_{n-1}, \tag{5.6}$$

equation (5.5) becomes

$$G_{n+1} + \frac{1}{F_n} G_n = -2\pi\beta\delta_{n,1}. \tag{5.7}$$

Since $F_n \sim \frac{\beta}{2\lambda_n}$, $n \to \infty$, it follows that there does **not** exist a homogeneous solution of equation (5.7) that decays as $n \to \infty$. Thus $G_n = -2\pi\beta F_1 \delta_{n,1}$, and equation (5.6) yields

$$A_n + F_n A_{n-1} = -2\pi\beta F_1 \delta_{n,1}.$$

The unique solution of this equation is given by (5.2). Evaluating equation (3.4b) at $n = 0$ and using $A_{-1} = \bar{A}_1$ we find (5.4).

Equation (5.2) implies that

$$A_n \sim (-1)^n A_0 \left(\frac{\beta}{2}\right)^n \prod_1^n \frac{1}{\lambda_j}, \quad n \to \infty, \tag{5.8}$$

thus the series (3.3) converges.

The asymptotic behavior as $\beta \to 0$

The definition of λ_n (equation (3.1)) together with the requirement that Re $\lambda_n < 0$, imply that if $c = \beta^2 k$, then

$$\lambda_n = -\sqrt{n}(1 + i) + O(\beta^2). \tag{5.9}$$

Equations (5.2), (5.3) yield

$$A_0 = a_0 + O(\beta), A_1 = \beta a_1 + O(\beta^2), A_2 = \beta^2 a_2 + O(\beta^3), \cdots . \tag{5.10}$$

Equation (3.4b) with $n = 0, 1$, becomes

$$n = 0: \quad A_1 + \bar{A}_1 - 4k\beta A_0 = 0 \tag{5.11}$$

$$n = 1: \quad 2\lambda_1 A_1 + \beta(A_2 + A_0) + 2\pi\beta = 0. \tag{5.12}$$

Substituting equations (5.10) into (5.11), (5.12), we find

$$a_1 + \bar{a}_1 = 4ka_0 \tag{5.13a}$$

$$-2(1 + i)a_1 + a_0 + 2\pi = 0. \tag{5.13b}$$

Equation (5.13b) yields

$$a_1 = \frac{(a_0 + 2\pi)(1 - i)}{4}. \tag{5.14a}$$

Substituting this expression in equation (5.13a) and using that a_0 is real, we find

$$a_0 = \frac{2\pi}{8k - 1}. \tag{5.14b}$$

Substituting (5.14b) into the expression for $\varphi(x, t)$ (equation (3.3)) we obtain

$$\phi(x, t) = \frac{e^{-2\beta^2 kx}}{8k - 1} + \frac{2k}{8k - 1} e^{-\beta^2 kx} F(x, t)\beta + O(\beta^2 E), \tag{5.15}$$

where

$$F(x,t) = (1-i)e^{-(1+i)x+it} + (1+i)e^{-(1-i)x-it}, \tag{5.16}$$

and $E(x,t)$ is an exponential function such that $E \to 0$ exponentially fast as $x \to \infty$ and $E(x/\beta^2, t) \to 0$ exponentially fast as $\beta \to 0$.

Using the definition of $u(x,t)$ in terms of $\phi(x,t)$,i.e. equation (3.2), we find

$$u = \frac{2\beta k e^{-2\beta^2 kx} - 2ke^{-\beta^2 kx}F_x + O(\beta^2 E)}{8k - 1 + e^{-2\beta^2 kx} + O(\beta E)}. \tag{5.17}$$

1.$x = O(\beta^{-2})$

In this case $E \to 0$ as $\beta \to 0$. Thus u tends exponentially fast to

$$u \sim \frac{2\beta k}{1 + (8k-1)e^{2\beta^2 kx}}. \tag{5.18}$$

2.$x = O(1)$

In this case $\exp(-2\beta^2 kx) = 1 + O(\beta^2)$, thus

$$u = \frac{1}{8k}[2\beta k - 2kF_x + O(\beta E)].$$

Using the definition of F, i.e. equation (5.16), to compute F_x, we find

$$u(x,t) = \frac{1}{2}[e^{-(1+i)x+it} + e^{-(1-i)x-it}] + \frac{\beta}{4} + O(\beta E). \tag{5.19}$$

Equations (5.18) and (5.19), which define the outer and the inner solutions respectively agree with equations (1.17) and (1.12). The inner and the outer limits of these solutions are obtained by letting $x \to 0$ and $x \to \infty$ in equations (5.18) and (5.19) respectively. Using the fact that $E \to 0$ as $x \to \infty$, it follows that both these limits equal $\beta/4$.

Appendix

The Analysis of equation (3.4b)

Let A_n satisfy the linear homogeneous difference equation

$$A_{n+1} + \frac{2\Lambda_n}{\beta}A_n + A_{n-1} = 0, \quad n = 2, 3, \cdots \tag{A.1}$$

where β is a constant and $\Lambda_n \to \infty$ monotonically as $n \to \infty$, for example $\Lambda_n = \lambda_n$, where λ_n is defined by equation (3.1). We shall show that

$$A_n \sim \left(-\frac{\beta}{2}\right)^{n-1}\left[c_1 \prod_1^{n-1} \frac{\Lambda_j}{\Lambda_{j-1}^2} + c_2 \prod_1^{n-1} \Lambda_j\right], \quad n \to \infty. \tag{A.2}$$

Indeed, we first make the change of variables

$$A_n = B_n \prod_{j=1}^{n-1} \Lambda_j^2. \tag{A.3}$$

Thus equation (A.1) becomes

$$B_{n+1}+\frac{2}{\beta\Lambda_n}B_n+\frac{B_{n-1}}{\Lambda_{n-1}^2\Lambda_n^2}=0,\quad n=2,3,\cdots \qquad (A.4)$$

We write this equation in matrix form,

$$\Psi_n=V_n\Psi_{n-1}, V_n=\begin{pmatrix}0 & 1\\ -\frac{1}{\Lambda_{n-1}^2\Lambda_n^2} & -\frac{2}{\beta\Lambda_n}\end{pmatrix},\quad \Psi_n=\begin{pmatrix}B_n\\ B_{n+1}\end{pmatrix}. \qquad (A.5)$$

The eigenvalues and the eigenvectors of the matrix V_n are

$$\mu_n^{\pm}=-\frac{1}{\beta\Lambda_n}\pm\sqrt{\frac{1}{\beta^2\Lambda_n^2}-\frac{1}{\Lambda_n^2\Lambda_{n-1}^2}},\quad \begin{pmatrix}1\\ \mu_n^{\pm}\end{pmatrix}. \qquad (A.6)$$

Using the gauge transformation

$$\Psi_n=T_n\Phi_n,\quad T_n=\begin{pmatrix}1 & 1\\ \mu_n^+ & \mu_n^-\end{pmatrix}, \qquad (A.7)$$

we find

$$\Phi_n=M_nT_n^{-1}T_{n-1}\Phi_{n-1},\quad M_n=\begin{pmatrix}\mu_n^+ & 0\\ 0 & \mu_n^-\end{pmatrix}. \qquad (A.8)$$

Equation (A.6a) implies

$$\mu_n^+=-\frac{\beta}{2\Lambda_n\Lambda_{n-1}^2}\left(1+O(\frac{1}{\Lambda_n^2})\right),$$

$$\mu_n^-=-\frac{2}{\beta\Lambda_n}\left(1+O(\frac{1}{\Lambda_n^2})\right). \qquad (A.9)$$

These estimates and the explicit form of T_n imply that

$$T_n^{-1}T_{n-1}=I+O(\frac{1}{\Lambda_n^3}),$$

Thus the WKB approximation of equation (A.8a) implies that Φ_n can be aproximated by $\Phi_n^+=(f_n^+,f_n^-)^T$, where f_n^+,f_n^- satisfy the equations

$$f_n^+=\mu_n^+f_{n-1},\quad f_n^-=\mu_n^-f_{n-1}.$$

We recall that the general solution of the first order difference equation $F_{n+1}=g_nF_n$, is given by

$$F_n=g_1\dots g_{n-1}.$$

Thus to the leading order as $n\to\infty$,

$$f_n^+=\prod_1^{n-1}\left(-\frac{\beta}{2\Lambda_j\Lambda_{j-1}^2}\right)=(-\frac{\beta}{2})^{n-1}\prod_1^{n-1}\frac{1}{\Lambda_j\Lambda_{j-1}^2},$$

$$f_n^-=\prod_1^{n-1}\left(-\frac{\beta}{2\Lambda_j}\right)=(-\frac{\beta}{2})^{n-1}\prod_1^{n-1}\frac{1}{\Lambda_j}.$$

Hence using (A.7) and (A.3) we find (A.2).

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