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Time Discretization of F. Calogero's "Goldfish" System

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Abstract

Time-discretized versions of F. Calogero's rational and hyperbolic "goldfish" systems are presented, and their exact solutions are given.

1 Introduction

The central role played in the modern mathematical physics by the multi-particle systems of the Calogero-Moser type is widely accepted. They include the original Calogero-Moser (CM) models [1, 6], and a widely ramified tree of their generalizations. The original CM models (rational, hyperbolic, or elliptic) are governed by Newtonian equations of motion

$$\ddot{x}_k = -\sum_{j\neq k} \mathcal{V}'(x_k - x_j), \tag{1.1}$$

with the potential $\mathcal{V}$ of the respective type:

$$\mathcal{V}(x) = -\gamma^2/x^2\,, \quad \text{or} \quad \mathcal{V}(x) = -\gamma^2/\sinh^2(x)\,, \quad \text{or} \quad \mathcal{V}(x) = -\gamma^2\wp(x)\,. \tag{1.2}$$

The parameter γ is usually supposed to be pure imaginary in order to assure the repulsive nature of pairwise interactions at small distances. One can consider rational, resp. hyperbolic model as a particular case of the elliptic one when both or one of the periods of the elliptic functions are sent to infinity.

A relativistic generalization of the Calogero-Moser models constitute the Ruijsenaars-Schneider (RS) models [7]. They are governed by Newtonian equations of motion,

$$\ddot{x}_k = \sum_{j\neq k} \dot{x}_k\dot{x}_j\mathcal{W}(x_k - x_j), \tag{1.3}$$

where

$$\mathcal{W}(x)=\frac{2\gamma^2}{x(\gamma^2-x^2)},\quad\text{or}\quad \mathcal{W}(x)=\frac{2\sinh^2(\gamma)\coth(x)}{\sinh^2(\gamma)-\sinh^2(x)},\quad\text{or}\quad \mathcal{W}(x)=\frac{\wp'(x)}{\wp(\gamma)-\wp(x)}. \tag{1.4}$$

To list some of generalizations of the CM type models, going in various directions, we mention: (i) CM and RS models with *external potentials*; (ii) *spin* CM and RS models with additional internal degrees of freedom; (iii) CM and RS models associated to *root systems*; (iv) *quantum-mechanical* CM and RS models; and so on. Various generalizations can be combined almost freely among themselves. This gives a huge tree of integrable models.

In the present paper we are dealing with a curious limiting case of the RS models, which however was discovered by Calogero [2] in 1978, well before the full RS systems were found by Ruijsenaars and Schneider. These systems correspond to the limit $\gamma\to\infty$, which makes sense in the rational and hyperbolic cases only. In this limit the first and the second functions in (1.4) turn into

$$\mathcal{W}(x)=\frac{2}{x},\qquad\text{resp.}\qquad \mathcal{W}(x)=2\coth(x), \tag{1.5}$$

so that Newtonian equations (1.3) become

$$\ddot{x}_k=\sum_{j\neq k}\frac{2\dot{x}_k\dot{x}_j}{x_k-x_j},\qquad\text{resp.}\qquad \ddot{x}_k=\sum_{j\neq k}2\dot{x}_k\dot{x}_j\coth(x_k-x_j). \tag{1.6}$$

Their discoverer attributes to them the status of a "goldfish", the "neatest many–body problem amenable to exact treatments", see [3, 4]. Indeed, the corresponding equations (1.3) are, in a sense, degenerate, and can be solved in a very neat and explicit form: $\xi=x_k(t)$, $k=1,2,\ldots,N$, are the N roots of the equation

$$\sum_{j=1}^{N}\frac{\dot{x}_j(0)}{\xi-x_j(0)}=\frac{1}{t},\qquad\text{resp.}\qquad \sum_{j=1}^{N}\dot{x}_j(0)\coth(\xi-x_j(0))=v\coth(vt), \tag{1.7}$$

where $v=\sum_{k=1}^{N}\dot{x}_k$ is an integral of motion.

Our main goal in this paper is to present the time discretization of these "goldfish" models, which share the outstanding properties with their continuous time counterparts. These discrete time Newtonian equations of motion read:

$$\frac{\widetilde{x}_k-x_k}{x_k-\underset{\sim}{x}_k}=\prod_{j\neq k}\frac{x_k-\underset{\sim}{x}_j}{x_k-\widetilde{x}_j},\qquad\text{resp.}\qquad \frac{\sinh(\widetilde{x}_k-x_k)}{\sinh(x_k-\underset{\sim}{x}_k)}=\prod_{j\neq k}\frac{\sinh(x_k-\underset{\sim}{x}_j)}{\sinh(x_k-\widetilde{x}_j)}. \tag{1.8}$$

These are second order difference equations for the functions $x:h\mathbb{Z}\to\mathbb{R}^N$, where we write x_k for $x_k(t)$, $\widetilde{x}_k$ for $x_k(t+h)$, and $\underset{\sim}{x}_k$ for $x_k(t-h)$, respectively. One immediately sees that these equations approximate eqs. (1.6) in the limit $h\to 0$, if one assumes that the functions x are interpolated by some smooth functions $x:\mathbb{R}\to\mathbb{R}^N$ at the points $t\in h\mathbb{Z}$, so that

$$\widetilde{x}_k=x_k+h\dot{x}_k+\tfrac{h^2}{2}\ddot{x}_k+o(h^2),\qquad \underset{\sim}{x}_k=x_k-h\dot{x}_k+\tfrac{h^2}{2}\ddot{x}_k+o(h^2).$$

Indeed, the left-hand and the right-hand sides of eqs. (1.8) approximate

$$1+\frac{h\ddot{x}_k}{\dot{x}_k}+o(h) \qquad \text{and} \qquad 1+h\sum_{j\neq k}\dot{x}_j\mathcal{W}(x_k-x_j)+o(h),$$

respectively. Eqs. (1.8) are Lagrangian, i.e., they admit the variational formulation: solutions x are critical points of some action functional $\sum_{t\in h\mathbb{Z}}\Lambda(x(t+h),x(t))$. The corresponding Euler–Lagrange equations have the form

$$\frac{\partial}{\partial x_k}\big(\Lambda(\widetilde{x},x)+\Lambda(x,\underset{\sim}{x})\big)=0,$$

and can be put down as

$$p_k=-\partial\Lambda(\widetilde{x},x)/\partial x_k\,, \qquad \widetilde{p}_k=\partial\Lambda(\widetilde{x},x)/\partial\widetilde{x}_k\,. \tag{1.9}$$

The latter formulas determine a symplectic map $(x,p)\mapsto(\widetilde{x},\widetilde{p})$. See [8] for a general theory of discrete time Lagrangian mechanics and for the corresponding references.

We will present in Sect. 4, 6 the explicit solutions for these discrete time systems. It should be mentioned that we do not study the qualitative behaviour of these solutions, in particular, we do not worry about the collisions and the continuation of solutions beyond collisions. This might be a topic of a separate publication. For the reader's convenience, we reproduce the explicit solutions of the continuous time systems in our notations in Sect. 3, 5. Our derivation is somewhat different from the original one by F. Calogero, which was based on the study of certain linear differential equations with partial derivatives.

2 Hamiltonian formulation of the goldfish models

The Newtonian equations of motion (1.3), written in the first-order form,

$$\dot{x}_k=b_k\,, \qquad \dot{b}_k=\sum_{j\neq k}b_kb_j\mathcal{W}(x_k-x_j)\,, \tag{2.1}$$

can be easily given a Hamiltonian formulation. Namely, these models are Hamiltonian with respect to the following Poisson brackets:

$$\{x_k,x_j\}=0, \quad \{b_k,x_j\}=b_k\delta_{kj}\,, \quad \{b_k,b_j\}=-(1-\delta_{kj})b_kb_j\mathcal{W}(x_k-x_j)\,, \tag{2.2}$$

and with the Hamilton function

$$H(x,b)=\sum_{k=1}^{N}b_k\,. \tag{2.3}$$

The above brackets can be realized by the canonically conjugate variables (x_k,p_k) with the Poisson brackets

$$\{x_k,x_j\}=0, \quad \{p_k,x_j\}=\delta_{kj}\,, \quad \{p_k,p_j\}=0, \tag{2.4}$$

by means of the formulas

$$b_k=e^{p_k}\prod_{j\neq k}(x_k-x_j)^{-1}, \qquad \text{resp.} \qquad b_k=e^{p_k}\prod_{j\neq k}(\sinh(x_k-x_j))^{-1}. \tag{2.5}$$

3 Explicit solution of the rational system

Theorem 1. *The solution of the initial value problem for the equation* (1.3) *with* $\mathcal{W}(x) = 2/x$, *i.e.*

$$\ddot{x}_k = \sum_{j\neq k} \frac{2\dot{x}_k \dot{x}_j}{x_k - x_j}\,, \tag{3.1}$$

with the initial data

$$x_k(0), \quad \dot{x}_k(0) = b_k(0), \tag{3.2}$$

is given by the following statement: $\xi = x_k(t)$, $k = 1, 2, \ldots, N$, *are the* N *roots of the equation*

$$\sum_{j=1}^{N} \frac{b_j(0)}{\xi - x_j(0)} = \frac{1}{t}\,. \tag{3.3}$$

Proof. Define the quantities $x_k = x_k(t)$ for $k = 1, 2, \ldots, N$ as the N roots of the equation (3.3). Hence, they are the roots of the following monic polynomial of degree N:

$$p(\xi; t) = \prod_{k=1}^{N} (\xi - x_k(t)) = \prod_{k=1}^{N} (\xi - x_k(0)) \left(1 - t \sum_{j=1}^{N} \frac{\dot{x}_j(0)}{\xi - x_j(0)} \right). \tag{3.4}$$

Expanding for small t, we see that they satisfy the desired initial conditions (3.2). By differentiating with respect to t, we find the following identity:

$$p_t(\xi; t) = \prod_{k=1}^{N} (\xi - x_k(t)) \sum_{j=1}^{N} \frac{-\dot{x}_j(t)}{\xi - x_j(t)} = \prod_{k=1}^{N} (\xi - x_k(0)) \sum_{j=1}^{N} \frac{-\dot{x}_j(0)}{\xi - x_j(0)}\,. \tag{3.5}$$

Thus, the following quantity is an integral of motion (does not depend on t):

$$P(\xi) = \prod_{k=1}^{N} (\xi - x_k) \sum_{j=1}^{N} \frac{\dot{x}_j}{\xi - x_j}\,. \tag{3.6}$$

Differentiating (3.5) with respect to t once more, we find:

$$\begin{aligned} 0 = p_{tt}(\xi; t) &= p(\xi; t) \left(-\sum_{k=1}^{N} \frac{\ddot{x}_k}{\xi - x_k} + {\sum_{k,j}}' \frac{\dot{x}_k \dot{x}_j}{(\xi - x_k)(\xi - x_j)} \right) \\ &= p(\xi; t) \sum_{k=1}^{N} \frac{1}{\xi - x_k} \Big(-\ddot{x}_k + \sum_{j\neq k} \frac{2\dot{x}_k \dot{x}_j}{x_k - x_j} \Big). \end{aligned}$$

This shows that x_k satisfy the differential equations (3.1), which finishes the proof. ■

Corollary 1. *Coefficients of the polynomial*

$$P(\xi) = \sum_{k=1}^{N} b_k \prod_{j\neq k} (\xi - x_j) = \sum_{n=0}^{N-1} (-1)^n I_n \xi^{N-1-n} \tag{3.7}$$

where $b_k = \dot{x}_k$, *are integrals of motion of the system* (3.1). *In particular,*

$$I_0 = H = \sum_{k=1}^{N} b_k\,, \qquad I_1 = \sum_{k=1}^{N} b_k \sum_{j\neq k} x_j\,, \qquad I_2 = \sum_{k=1}^{N} b_k \sum_{i\neq j\neq k\neq i} x_i x_j\,.$$

Corollary 2. *A companion statement to that of Theorem 1 holds:* $\xi = x_k(0)$, *for* $k = 1, 2, \ldots, N$, *are the* N *roots of the equation*

$$\sum_{j=1}^{N} \frac{\dot{x}_j(t)}{\xi - x_j(t)} = -\frac{1}{t}\,. \tag{3.8}$$

Proof. We can rewrite (3.4) as

$$p(\xi; t) = p(\xi; 0) - tP(\xi) \quad \Rightarrow \quad p(\xi; 0) = p(\xi; t) + tP(\xi)\,.$$

Using independence of $P(\xi)$ on t, we put this as

$$p(\xi; 0) = \prod_{i=1}^{N}(\xi - x_i(0)) = \prod_{i=1}^{N}(\xi - x_i(t))\left(1 + t\sum_{j=1}^{N}\frac{\dot{x}_j(t)}{\xi - x_j(t)}\right). \tag{3.9}$$

This proves the Corollary. ∎

Corollary 3. *We have:*

$$b_j(0) = \dot{x}_j(0) \quad = \quad -\frac{1}{t}\cdot\frac{\prod_{i=1}^{N}(x_j(0) - x_i(t))}{\prod_{i\neq j}(x_j(0) - x_i(0))}\,, \tag{3.10}$$

$$b_j(t) = \dot{x}_j(t) \quad = \quad \frac{1}{t}\cdot\frac{\prod_{i=1}^{N}(x_j(t) - x_i(0))}{\prod_{i\neq j}(x_j(t) - x_i(t))}\,. \tag{3.11}$$

Proof. This follows from the statements concerning (3.3), (3.8), by Lemma 1. ∎

4 Discretization of the rational system

Now we are in a position to discuss the discrete time analogs of the above results. The corresponding formulas mimic (3.10), (3.11).

Theorem 2. *Consider the symplectic map* $(x, p) \mapsto (\widetilde{x}, \widetilde{p})$ *given by the following Lagrangian equations:*

$$e^{p_k} \quad = \quad -h^{-1}\prod_{i=1}^{N}(x_k - \widetilde{x}_i)\,, \tag{4.1}$$

$$e^{\widetilde{p}_k} \quad = \quad h^{-1}\prod_{i=1}^{N}(\widetilde{x}_k - x_i)\,, \tag{4.2}$$

with the Lagrange function

$$\Lambda(\widetilde{x}, x) = \sum_{k,j=1}^{N}\varphi(\widetilde{x}_k - x_j)\,, \quad \textit{where} \quad \varphi(\xi) = \int_0^{\xi}\log(\eta)d\eta\,, \tag{4.3}$$

and with the corresponding Newtonian equations of motion

$$\frac{\widetilde{x}_k - x_k}{x_k - \underset{\sim}{x}_k} = \prod_{i\neq k} \frac{(x_k - \underset{\sim}{x}_i)}{(x_k - \widetilde{x}_i)}\,. \tag{4.4}$$

In terms of the variables (x,b) equations (4.1), (4.2) read as

$$b_k = -h^{-1} \frac{\prod_{i=1}^N (x_k - \widetilde{x}_i)}{\prod_{i\neq k}(x_k - x_i)}\,, \tag{4.5}$$

$$\widetilde{b}_k = h^{-1} \frac{\prod_{i=1}^N (\widetilde{x}_k - x_i)}{\prod_{i\neq k}(\widetilde{x}_k - \widetilde{x}_i)}\,. \tag{4.6}$$

The solutions $x_k = x_k(nh)$, $k = 1,2,\ldots,N$, of this discrete time system are the N roots of the equation

$$\sum_{j=1}^N \frac{b_j(0)}{\xi - x_j(0)} = \frac{1}{nh}\,. \tag{4.7}$$

Proof is based on the following lemma on the Lagrange interpolation (whose proof is put in Sect. 7):

Lemma 1. *Let $\xi_1,\ldots,\xi_M,\eta_1,\ldots,\eta_M$ be $2M$ pairwise distinct numbers. The system of M linear equations*

$$\sum_{j=1}^M \frac{y_j}{\eta_k - \xi_j} = -\alpha\,, \qquad k = 1,\ldots,M, \tag{4.8}$$

has a unique solution:

$$y_j = \alpha \frac{\prod_{i=1}^M (\xi_j - \eta_i)}{\prod_{i\neq j}(\xi_j - \xi_i)}\,, \qquad j = 1,\ldots,M. \tag{4.9}$$

According to Lemma 1, defining equations (4.5), (4.6) can be re-formulated in the following fashion: the quantities $\widetilde{x}_k$ are the N roots of the equation $1 - h\sum_{j=1}^N b_j/(\xi - x_j) = 0$, while the quantities x_k are the N roots of the equation $1 + h\sum_{j=1}^N \widetilde{b}_j/(\xi - \widetilde{x}_j) = 0$. The first claim may be re-formulated as the polynomial identity

$$\widetilde{p}(\xi) = \prod_{k=1}^N (\xi - \widetilde{x}_k) = \prod_{k=1}^N (\xi - x_k)\left(1 - h\sum_{j=1}^N \frac{b_j}{\xi - x_j}\right), \tag{4.10}$$

while the second one – as the polynomial identity

$$p(\xi) = \prod_{k=1}^N (\xi - x_k) = \prod_{k=1}^N (\xi - \widetilde{x}_k)\left(1 + h\sum_{j=1}^N \frac{\widetilde{b}_j}{\xi - \widetilde{x}_j}\right). \tag{4.11}$$

As a consequence of these two identities, we have:

$$\widetilde{p}(\xi) - p(\xi) = \prod_{k=1}^N (\xi - x_k)\sum_{j=1}^N \frac{-hb_j}{\xi - x_j} = \prod_{k=1}^N (\xi - \widetilde{x}_k)\sum_{j=1}^N \frac{-h\widetilde{b}_j}{\xi - \widetilde{x}_j}. \tag{4.12}$$

Therefore, as in the continuous time case, we conclude that the polynomial

$$P(\xi) = \prod_{k=1}^{N} (\xi - x_k) \sum_{j=1}^{N} \frac{b_j}{\xi - x_j} \tag{4.13}$$

is an integral of motion, i.e. does not depend on the (discrete) time. From the relation $p(\xi; t+h) - p(\xi; t) = -hP(\xi)$ we immediately derive:

$$p(\xi; nh) = p(\xi; 0) - nhP(\xi) = \prod_{k=1}^{N} (\xi - x_k(0)) \left(1 - nh \sum_{j=1}^{N} \frac{b_j(0)}{\xi - x_j(0)}\right), \tag{4.14}$$

which proves the Theorem. ■

Corollary 4. *Solutions $x_k = x_k(nh)$ of the discrete time system of Theorem 2 are interpolated by the solutions $x_k = x_k(t)$ of the continuous time system of Theorem 1 at $t = nh$.*

5 Explicit solution of the hyperbolic system

Theorem 3. *The solution of the initial value problem for the equation* (1.3) *with $\mathcal{W}(x) = 2\coth(x)$, i.e.*

$$\ddot{x}_k = \sum_{j \neq k} 2\dot{x}_k \dot{x}_j \coth(x_k - x_j)\,, \tag{5.1}$$

with the initial data

$$x_k(0), \quad \dot{x}_k(0) = b_k(0), \tag{5.2}$$

is given by the following formula: $\xi = x_k(t)$, $k = 1, 2, \ldots, N$, are the N roots of the equation

$$\sum_{j=1}^{N} b_j(0) \coth(\xi - x_j(0)) = v \coth(vt)\,, \tag{5.3}$$

where

$$v = \sum_{k=1}^{N} b_k = \sum_{k=1}^{N} \dot{x}_k \tag{5.4}$$

is an integral of motion.

Proof is based on the following lemma on the hyperbolic Lagrange interpolation, whose proof is given in Sect. 7:

Lemma 2. *Let $\xi_1, \ldots, \xi_M, \eta_1, \ldots, \eta_M$ be $2M$ pair-wise distinct numbers. The system of M linear equations*

$$\sum_{j=1}^{M} y_j \coth(\eta_k - \xi_j) = -\alpha\,, \qquad k = 1, \ldots, M, \tag{5.5}$$

has a unique solution:

$$y_j = \frac{\alpha}{\cosh(\Delta)} \cdot \frac{\prod_{i=1}^{M} \sinh(\xi_j - \eta_i)}{\prod_{i \neq j} \sinh(\xi_j - \xi_i)}, \qquad j = 1, \dots, M, \tag{5.6}$$

with

$$\sum_{j=1}^{M} y_j = \alpha \tanh(\Delta), \tag{5.7}$$

where

$$\Delta = \sum_{j=1}^{M} (\xi_j - \eta_j). \tag{5.8}$$

Return to the proof of Theorem 3. For the given initial data $x_k(0)$, $b_k(0)$, with

$$\sum_{j=1}^{N} b_k(0) = v,$$

define the quantities $x_k = x_k(t)$, $k = 1, 2, \dots, N$, as the N roots of the equation (5.3). It is easy to see that these quantities have the initial values (5.2). Since, by Lemma 2, these quantities satisfy the identity

$$\sum_{j=1}^{N} b_k(0) = v \coth(vt) \tanh\Big(\sum_{j=1}^{N} (x_j(t) - x_j(0)) \Big),$$

we conclude that

$$\sum_{j=1}^{N} x_j(t) - \sum_{j=1}^{N} x_j(0) = vt. \tag{5.9}$$

Equation (5.3) may be alternatively written as

$$\sum_{j=1}^{N} \frac{b_j(0) e^{2x_j(0)}}{e^{2\xi} - e^{2x_j(0)}} = \frac{v}{e^{2tv} - 1}, \tag{5.10}$$

upon subtracting $\sum_{j=1}^{N} b_j(0)/2 = v/2$ from both parts. According to formula (5.10), the quantities e^{2x_k} are the roots of the following monic polynomial in $e^{2\xi}$ of degree N:

$$\begin{aligned} p(\xi; t) &= \prod_{k=1}^{N} \Big(e^{2\xi} - e^{2x_k(t)} \Big) \\ &= \prod_{k=1}^{N} \Big(e^{2\xi} - e^{2x_k(0)} \Big) \left(1 - \frac{e^{2tv} - 1}{v} \sum_{j=1}^{N} \frac{b_j(0) e^{2x_j(0)}}{e^{2\xi} - e^{2x_j(0)}} \right). \end{aligned} \tag{5.11}$$

Introduce the following polynomial, which is, up to the constant factor -2, the t–derivative of $p(\xi;t)$:

$$P(\xi;t) = \prod_{k=1}^{N}\left(e^{2\xi} - e^{2x_k(t)}\right)\sum_{j=1}^{N}\frac{\dot{x}_j(t)e^{2x_j(t)}}{e^{2\xi} - e^{2x_j(t)}}\,. \tag{5.12}$$

¿From (5.11) there follows the identity:

$$P(\xi;t) = e^{2vt}P(\xi;0)\,. \tag{5.13}$$

Therefore, $P_t(\xi;t) - 2vP(\xi;t) = 0$. Writing this in length, we find:

$$\begin{aligned} 0 &= \sum_{k=1}^{N}\frac{(\ddot{x}_k + 2\dot{x}_k^2 - 2v\dot{x}_k)e^{2x_k}}{e^{2\xi} - e^{2x_k}} - \sum_{k,j}{}' \frac{2\dot{x}_k\dot{x}_j e^{2x_k+2x_j}}{\left(e^{2\xi} - e^{2x_k}\right)\left(e^{2\xi} - e^{2x_j}\right)} \\ &= \sum_{k=1}^{N}\frac{e^{2x_k}}{e^{2\xi} - e^{2x_k}}\left(\ddot{x}_k + 2\dot{x}_k^2 - 2v\dot{x}_k - \sum_{j\neq k}\frac{4\dot{x}_k\dot{x}_j e^{2x_j}}{e^{2x_k} - e^{2x_j}}\right). \end{aligned}$$

Taking into account that

$$2v\dot{x}_k - 2\dot{x}_k^2 = 2\sum_{j\neq k}\dot{x}_k\dot{x}_j\,,$$

we derive for all k:

$$\ddot{x}_k = 2\sum_{j\neq k}\dot{x}_k\dot{x}_j + \sum_{j\neq k}\frac{4\dot{x}_k\dot{x}_j e^{2x_j}}{e^{2x_k} - e^{2x_j}} = 2\sum_{j\neq k}\dot{x}_k\dot{x}_j\coth(x_k - x_j)\,,$$

and the proof is complete. ■

Corollary 5. *Coefficients of the exponential polynomial*

$$\sum_{k=1}^{N} b_k\prod_{j\neq k}\left(1 - e^{-2x_j+2\xi}\right) = \sum_{n=0}^{N-1}(-1)^n I_n e^{2n\xi} \tag{5.14}$$

where $b_k = \dot{x}_k$, are integrals of motion of the system (3.1)*. In particular,*

$$I_0 = H = \sum_{k=1}^{N} b_k\,, \qquad I_1 = \sum_{k=1}^{N} b_k\sum_{j\neq k}e^{-2x_j}\,, \qquad I_2 = \sum_{k=1}^{N} b_k\sum_{i\neq j\neq k\neq i}e^{-2x_i-2x_j}\,.$$

Proof. Taking (5.9) into account, we see that the quantity

$$\Big(\prod_{k=1}^{N}e^{-2x_k(t)}\Big)P(\xi;t)$$

is an integral of motion (does not depend on t). ■

Corollary 6. *A companion statement to that of Theorem 3 holds: $\xi = x_k(0)$, for $k = 1, 2, \ldots, N$, are the N roots of the equation*

$$\sum_{j=1}^{N} \dot{x}_j(t) \coth(\xi - x_j(t)) = -v \coth(vt) \,. \tag{5.15}$$

Proof. With the help of (5.13) we can rewrite (5.11) as

$$p(\xi;t) = p(\xi;0) - \frac{e^{2vt}-1}{v} P(\xi;0) \quad \Rightarrow \quad p(\xi;0) = p(\xi;t) + \frac{1-e^{-2vt}}{v} P(\xi;t) \,.$$

In other words,

$$p(\xi;0) = p(\xi;t) \left(1 + \frac{1-e^{-2vt}}{v} \sum_{j=1}^{N} \frac{\dot{x}_j(t) e^{2x_j(t)}}{e^{2\xi} - e^{2x_j(t)}} \right) . \tag{5.16}$$

Now the equation

$$\sum_{j=1}^{N} \frac{\dot{x}_j(t) e^{2x_j(t)}}{e^{2\xi} - e^{2x_j(t)}} = -\frac{v}{1-e^{-2vt}} \tag{5.17}$$

is equivalent to (5.15). ■

Corollary 7. *We have:*

$$b_j(0) = \dot{x}_j(0) \quad = \quad -\frac{v}{\sinh(vt)} \cdot \frac{\prod_{i=1}^{N} \sinh(x_j(0) - x_i(t))}{\prod_{i \neq j} \sinh(x_j(0) - x_i(0))} \,, \tag{5.18}$$

$$b_j(t) = \dot{x}_j(t) \quad = \quad \frac{v}{\sinh(vt)} \cdot \frac{\prod_{i=1}^{N} \sinh(x_j(t) - x_i(0))}{\prod_{i \neq j} \sinh(x_j(t) - x_i(t))} \,. \tag{5.19}$$

Proof. This follows from Theorem 3 and Corollary 6 by Lemma 2. ■

6 Discretization of the hyperbolic system

We introduce now the integrable discrete time analog of the system (5.1). The formulas are tailored after (5.18), (5.19).

Theorem 4. *For an arbitrary function $\phi(h) = 1 + O(h)$, consider the symplectic map $(x,p) \mapsto (\widetilde{x}, \widetilde{p})$ given by the following Lagrangian equations:*

$$e^{p_k} \quad = \quad -\frac{\phi(h)}{h} \prod_{i=1}^{N} \sinh(x_k - \widetilde{x}_i) \,, \tag{6.1}$$

$$e^{\widetilde{p}_k} \quad = \quad \frac{\phi(h)}{h} \prod_{i=1}^{N} \sinh(\widetilde{x}_k - x_i) \,, \tag{6.2}$$

with the Lagrange function

$$\Lambda(\widetilde{x},x)=\sum_{k,j=1}^{N}\varphi(\widetilde{x}_k-x_j)\,,\quad where\quad \varphi(\xi)=\int_0^{\xi}\log\sinh(\eta)d\eta\,, \tag{6.3}$$

and the Newtonian equations of motion

$$\frac{\sinh(\widetilde{x}_k-x_k)}{\sinh(x_k-\underset{\sim}{x}_k)}=\prod_{i\neq k}\frac{\sinh(x_k-\underset{\sim}{x}_i)}{\sinh(x_k-\widetilde{x}_i)}\,. \tag{6.4}$$

In terms of the variables (x_k,b_k) *the map (6.1), (6.2) is written as*

$$b_k \;=\; -\frac{\phi(h)}{h}\frac{\prod_{i=1}^{N}\sinh(x_k-\widetilde{x}_i)}{\prod_{i\neq k}\sinh(x_k-x_i)}\,, \tag{6.5}$$

$$\widetilde{b}_k \;=\; \frac{\phi(h)}{h}\frac{\prod_{i=1}^{N}\sinh(\widetilde{x}_k-x_i)}{\prod_{i\neq k}\sinh(\widetilde{x}_k-\widetilde{x}_i)}\,. \tag{6.6}$$

Then we have:

$$\sum_{j=1}^{N}(\widetilde{x}_j-x_j)=\Delta=hv(1+O(h))\,, \tag{6.7}$$

where $\Delta=\Delta(v,h)$ *is defined by the equation*

$$\sinh(\Delta)=\frac{hv}{\phi(h)}\,, \tag{6.8}$$

and $v=\sum_{k=1}^{N}b_k=\sum_{k=1}^{N}\widetilde{b}_k$ *is an integral of motion. The solutions* $x_k=x_k(nh)$, $k=1,2,\ldots,N$, *of this discrete time system are the* N *roots of the equation*

$$\sum_{j=1}^{N}b_j(0)\coth(\xi-x_j(0))=v\coth(n\Delta)\,. \tag{6.9}$$

Proof. The formulas (6.7), (6.8) follow directly from Lemma 2. This Lemma also allows us to re-formulate the defining equations (6.5), (6.6) as follows: $\xi=\widetilde{x}_k$ are the N roots of the equation $\sum_{j=1}^{N}b_j\coth(\xi-x_j)=v\coth(\Delta)$, while $\xi=x_k$ are the N roots of the equation $\sum_{j=1}^{N}\widetilde{b}_j\coth(\xi-\widetilde{x}_j)=-v\coth(\Delta)$. The first claim may be re-formulated as the polynomial identity

$$\widetilde{p}(\xi)=\prod_{k=1}^{N}\left(e^{2\xi}-e^{2\widetilde{x}_k}\right)=\prod_{k=1}^{N}\left(e^{2\xi}-e^{2x_k}\right)\left(1-\frac{e^{2\Delta}-1}{v}\sum_{j=1}^{N}\frac{b_je^{2x_j}}{e^{2\xi}-e^{2x_j}}\right), \tag{6.10}$$

while the second one – as the polynomial identity

$$p(\xi)=\prod_{k=1}^{N}\left(e^{2\xi}-e^{2x_k}\right)=\prod_{k=1}^{N}\left(e^{2\xi}-e^{2\widetilde{x}_k}\right)\left(1+\frac{1-e^{-2\Delta}}{v}\sum_{j=1}^{N}\frac{\widetilde{b}_je^{2\widetilde{x}_j}}{e^{2\xi}-e^{2\widetilde{x}_j}}\right). \tag{6.11}$$

Using notation (5.12), we rewrite these two identities as

$$\widetilde{p}(\xi)-p(\xi)=-\frac{e^{2\Delta}-1}{v}\,P(\xi)=-\frac{1-e^{-2\Delta}}{v}\,\widetilde{P}(\xi). \tag{6.12}$$

So, we conclude

$$\widetilde{P}(\xi)=e^{2\Delta}P(\xi)\quad\Rightarrow\quad P(\xi;nh)=e^{2n\Delta}P(\xi;0)\,. \tag{6.13}$$

Putting this into (6.12), we derive:

$$\begin{aligned} p(\xi;nh) &= p(\xi;0)-\frac{e^{2\Delta}-1}{v}\sum_{m=0}^{n-1}P(\xi;mh)=p(\xi;0)-\frac{e^{2n\Delta}-1}{v}\,P(\xi;0)\\ &= p(\xi;0)\left(1-\frac{e^{2n\Delta}-1}{v}\sum_{j=1}^{N}\frac{b_j(0)e^{2x_j(0)}}{e^{2\xi}-e^{2x_j(0)}}\right). \end{aligned} \tag{6.14}$$

which proves the Theorem. ■

Corollary 8. *If $\phi(h)=hv/\sinh(hv)$ (thus depending on initial conditions), so that $\Delta(v,h)=hv$, then the discrete time solutions are interpolated by the solutions of the continuous time equation* (5.1).

7 Proof of Lemmas 1,2

Proof of Lemma 1. Consider the rational function

$$\varphi(\xi)=\alpha+\sum_{j=1}^{M}\frac{y_j}{\xi-\xi_j}\,.$$

It has M poles ξ_j and M zeros η_j, therefore, taking into account its value on the infinity, we find:

$$\varphi(\xi)=\alpha\frac{\prod_{i=1}^{M}(\xi-\eta_i)}{\prod_{i=1}^{M}(\xi-\xi_i)}\,.$$

Now formula (4.8) follows from $\lim_{\xi\to\xi_j}(\xi-\xi_j)\varphi(\xi)$. ■

Proof of Lemma 2. Introduce the function

$$\varphi(\xi)=\alpha+\sum_{j=1}^{M}y_j\coth(\xi-\xi_j)\,. \tag{7.1}$$

Clearly, we have:

$$\varphi(\xi)=\beta+\sum_{j=1}^{M}\frac{2y_je^{2\xi_j}}{e^{2\xi}-e^{2\xi_j}}\,, \tag{7.2}$$

where

$$\beta = \alpha + \sum_{j=1}^{M} y_j \,. \tag{7.3}$$

So, $\varphi(\xi)$ is a rational function of $e^{2\xi}$; it has M poles at $e^{2\xi} = e^{2\xi_j}$, and M zeros at $e^{2\xi} = e^{2\eta_j}$. Taking into account its behaviour at $\xi \to \infty$, we find:

$$\varphi(\xi) = \beta \, \frac{\prod_{i=1}^{M} \left(e^{2\xi} - e^{2\eta_i} \right)}{\prod_{i=1}^{M} \left(e^{2\xi} - e^{2\xi_i} \right)} = \beta \, e^{-\Delta} \, \frac{\prod_{i=1}^{M} \sinh(\xi - \eta_i)}{\prod_{i=1}^{M} \sinh(\xi - \xi_i)} \,. \tag{7.4}$$

Combining (7.2), (7.4), we see that

$$2 y_j e^{2\xi_j} = \lim_{\xi \to \xi_j} \varphi(\xi) \Big(e^{2\xi} - e^{2\xi_j} \Big) = \beta \, \frac{\prod_{i=1}^{M} \left(e^{2\xi_j} - e^{2\eta_i} \right)}{\prod_{i \neq j} \left(e^{2\xi_j} - e^{2\xi_i} \right)} \tag{7.5}$$

or

$$y_j = \beta \, e^{-\Delta} \cdot \frac{\prod_{i=1}^{M} \sinh(\xi_j - \eta_i)}{\prod_{i \neq j} \sinh(\xi_j - \xi_i)} \,. \tag{7.6}$$

It remains to determine the value of β. To this aim, compare the formulas (7.2), (7.4) at $\xi \to -\infty$:

$$\beta - 2 \sum_{j=1}^{M} y_j = \beta \, e^{-2\Delta} \quad \Rightarrow \quad \sum_{j=1}^{M} y_j = \frac{\beta}{2} \, (1 - e^{-2\Delta}) \,. \tag{7.7}$$

Comparing this with (7.3), we see that

$$\alpha = \frac{\beta}{2} \, (1 + e^{-2\Delta}) \quad \Leftrightarrow \quad \beta = \frac{2\alpha}{1 + e^{-2\Delta}} \,. \tag{7.8}$$

This expression, being plugged into (7.6), yields (5.6). ■

8 Conclusions

It should be mentioned that the rational and the hyperbolic "goldfish" systems also admit an elliptic generalization [5]. In this paper, Calogero and Francoise demonstrated how to solve explicitly the initial value problem for the differential equation

$$\ddot{x}_k = 2 \sum_{j \neq k} \dot{x}_k \dot{x}_j \zeta(x_k - x_j), \tag{8.1}$$

for the case when the integral of motion $\sum_{k=1}^{N} b_k = \sum_{k=1}^{N} \dot{x}_k$ vanishes. The construction is as follows: set

$$1 - \tau \sum_{k=1}^{N} b_k(0) \zeta(\xi - x_k(0)) = e^{-\varphi(\tau)} \frac{\prod_{k=1}^{N} \sigma(\xi - x_k(\tau))}{\prod_{k=1}^{N} \sigma(\xi - x_k(0))} \,, \tag{8.2}$$

so that $x_k(\tau)$ are the zeros of the left–hand side, while $\exp(-\varphi(\tau))$ is the normalizing factor, which is completely determined by the initial data. Then it can be shown that the functions $x_k(\tau)$ satisfy the following differential equations:

$$x_k'' - 2\varphi' x_k' = 2\sum_{j\neq k} x_k' x_j' \zeta(x_k - x_j), \tag{8.3}$$

while the function $\varphi(\tau)$ (which can be considered determined, once the initial data are known) satisfies the differential equation

$$\varphi'' - (\varphi')^2 = \frac{1}{2}\sum_{k,j=1}^{N} x_k' x_j' \gamma(x_k - x_j),$$

where the function $\gamma(\xi) = \zeta'(\xi) + \zeta^2(\xi)$, regular at the origin, is introduced. Performing the change of variables $\tau \mapsto t$, $dt/d\tau = \exp(-2\varphi)$, we see that (8.1) is satisfied. It turns out that this construction admits, to a large extent, a time discretization. Namely, introduce the function $\varphi(\tau)$ as before. Consider the symplectic map $(x, b) \mapsto (\widetilde{x}, \widetilde{b})$ defined by the following Lagrangian equations of motion:

$$\begin{aligned} b_k &= -\frac{\exp(\varphi - \widetilde{\varphi})}{h} \cdot \frac{\prod_{i=1}^{N} \sigma(x_k - \widetilde{x}_i)}{\prod_{i\neq k} \sigma(x_k - x_i)}, \\ \widetilde{b}_k &= \frac{\exp(\widetilde{\varphi} - \varphi)}{h} \cdot \frac{\prod_{i=1}^{N} \sigma(\widetilde{x}_k - x_i)}{\prod_{i\neq k} \sigma(\widetilde{x}_k - \widetilde{x}_i)}, \end{aligned}$$

with the corresponding Newtonian equations of motion

$$\frac{\sigma(\widetilde{x}_k - x_k)}{\sigma(x_k - \underset{\sim}{x}_k)} = \exp(\widetilde{\varphi} - \underset{\sim}{\varphi}) \cdot \prod_{i\neq k} \frac{\sigma(x_k - \underset{\sim}{x}_i)}{\sigma(x_k - \widetilde{x}_i)}. \tag{8.4}$$

(Here the tilde and the undertilde denote the $\pm h$ shift in the discrete time τ.) Then the functions $x_k(\tau)$ for $\tau = nh$ are given by (8.2), i.e. they are the zeros of the left–hand side of (8.2). The difference equation (8.4) is, obviously, a discretization of (8.3). Unfortunately, in the discrete time context there lacks the concept of the change of independent variable, which does not allow us to find a discretization for (8.1).

References

[1] F. Calogero. Exactly solvable one-dimensional many-body problems. *Lett. Nuovo Cim.*, 1975, **13**, 411–416.

[2] F. Calogero. Motion of poles and zeros of special solutions of nonlinear and linear partial differential equations, and related "solvable" many-body problems. *Nuovo Cim. B*, 1978, **43**, 177–241.

[3] F. Calogero. The neatest many-body problem amenable to exact treatments (a "goldfish"?). *Phys. D*, 2001, **152/153**, 78–84.

[4] F. Calogero. Classical many-body problems amenable to exact treatments. Berlin etc: Springer, 2001.

[5] F. Calogero, J.-P. Francoise. A novel solvable many-body problem with elliptic interactions. *Internat. Math. Res. Notices*, 2000, no. 15, 775–786.

[6] J. Moser. Three integrable Hamiltonian systems connected with isospectral deformations. *Adv. Math.*, 1975, **16**, 197–220.

[7] S.N.M. Ruijsenaars, H. Schneider. A new class of integrable systems and its relation to solitons. *Ann. Phys.*, 1986, **170**, 370–405.

[8] Yu.B. Suris. *The problem of integrable discretization: Hamiltonian approach*. Basel etc.: Birkhäuser, 2003.