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Integrable and Nonintegrable Initial Boundary Value Problems for Soliton Equations [1]

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Abstract

It is well-known that the basic difficulty in studying the initial boundary value problems for linear and nonlinear PDEs is the presence, in any method of solution, of unknown boundary values. In the first part of this paper we review two spectral methods in which the above difficulty is faced in different ways. In the first method one uses the analyticity properties of the x-scattering matrix $S(k,t)$ to replace the unknown boundary values by elements of the scattering matrix itself, thus obtaining a closed integro-differential evolution equation for $S(k,t)$. In the second method one uses the analyticity properties of $S(k,t)$ to eliminate the unknown boundary values by a suitable projection, obtaining a nonlinear Riemann Hilbert problem for $S(k,t)$. The second approach allows also to identify in a natural way a known subclass of boundary conditions which gives rise to a spectral formalism based on linear operations (and therefore called "integrable boundary conditions"). In the last part of the paper we present a new method to identify a whole hierarchy of integrable boundary conditions.

1 Introduction

Initial Boundary Value (IBV) problems for partial differential equations play an important role in applications to Physics and, in general, to the Natural Sciences.

Since the discovery of the inverse scattering (spectral) transform method to solve the IBV problem on the infinite line with vanishing boundary conditions for a class of distinguished nonlinear evolution equations, like the Korteweg de Vries (KdV), nonlinear Schrödinger (NLS) and sine Gordon (SG) equations (see, f.i., [1]), several attempts have been made to extend this method to the case of more complicated IBV problems, in which,

[1]We AD and PMS were first students and then collaborators and friends of Francesco Calogero. In this special occasion we thank him for all we learned from him with our best wishes.

f.i., Dirichlet, or Neumann, or Robin boundary conditions are prescribed on the semi - infinite line or on the segment. It is well-known that the basic difficulty associated with these problems is that the evolution equation of the traditional x - scattering matrix $S(k,t)$, as given by the Lax equations, cannot be directly integrated in most of the cases, because its coefficients depend on unknown boundary data.

Different approaches to the study of IBV problems for soliton equations have been developed so far. In [2], a nonlinear analogue of the sine transform has been introduced. In [3] an "elbow scattering" in the (x,t)- plane is presented to deal with the semi-line problem for KdV, leading to Gel'fand - Levitan - Marchenko formulations. In [4, 5] a different approach, based on a simultaneous x-t spectral transform, has been introduced and rigorously developed in [6], to solve IBV problems for soliton equations. This method allows for a rigorous asymptotics [7] and captures in a natural way the known cases of linearizable IBV problems. Some integrable boundary conditions for soliton equations and their connection with symmetries and Bäcklund Transformations (BTs) have been investigated in [8] and in references therein quoted. In some non-generic cases of soliton equations corresponding to singular dispersion relations, like the stimulated Raman scattering (SRS) equations and the SG equation in light-cone coordinates, the evolution equation of the scattering matrix does not contain unknown boundary data and a more standard treatment can be developed (see, f.i., [4],[9, 10]). A different approach, based on the Kac-Moody representation for the SG equation in laboratory coordinates, was used in [11] to solve the Robin problem on the semi-line; it uses critically the finite-speed character of the equation. We finally remark that IBV problems for C - integrable equations have been considered, for instance, in [12].

As it was already explained, the basic difficulty in studying the IBV problems for linear and nonlinear PDEs is the presence, in any formalism of solution, of unknown boundary values.

In [4, 5, 6] such a problem is faced in the philosophy of constructing a relation among all boundary values, with the goal of expressing the unknown ones in terms of the given ones. In the linear case, this relation, often called "global relation" [5], involves only the boundary values and therefore it provides an effective solution of the problem (see also [13] and [14]). In the nonlinear case, instead, this relation takes the form of a set of nonlinear and nonlocal equations involving also the unknown scattering matrix (see also [15, 16]).

In our research we have explored different strategies to deal with unknown boundary values and, in the first part of this paper, we review two spectral methods, illustrated on the initial - boundary value problems on the semiline for the celebrated NLS equation. In the first method [17] one uses the analyticity properties of the x-scattering matrix $S(k,t)$ to replace the unknown boundary values by elements of the scattering matrix itself, thus obtaining a closed nonlinear integro-differential evolution equation for $S(k,t)$. This method works in both the semiline and the segment cases. In the second method [18] one uses the analyticity properties of the $S(k,t)$ to eliminate the unknown boundary values by a suitable projection, obtaining a nonlinear Riemann Hilbert problem for $S(k,t)$. The elimination strategy can also be formulated in a different, but closely related way, consisting in the construction of a suitable extension of the profile, from the semiline to the whole line, which allows one to reformulate the original IBV problem as a Cauchy problem on the line with prescribed discontinuities [18]. This last strategy has been successfully applied also to the NLS on the segment [19]; in this case i) the extension of the profile is periodic, ii) the associated spectral transform makes essential use of the finite gap (algebro-geometric) formalism [20], and iii) the time evolution of the countable number of algebro-geometric spectral data is described by a nonlinear dynamical system with algebraic right-hand side.

The elimination formalism allows also to identify in a natural way a known subclass of Robin boundary data [21, 22] which give rise to a spectral formalism consisting of linear operations. We refer to this type of boundary conditions as "integrable boundary conditions". In the last part of the paper we present a method, based on the Bäcklund Transformations (BTs), which allows one to identify systematically a whole hierarchy of integrable boundary conditions.

More precisely, the paper is organized as follows. In section §2 we pose the IBV problem on the semiline for the NLS equation, we summarize the associated (and classical) Direct and Inverse Spectral Transforms in the x-variable and we derive the t-evolution of the scattering matrix. In section §3 we describe the first method, replacing the unknown BVs by their spectral representations in terms of $S(k,t)$. In section §4 we describe the second method, eliminating from the formalism the unknown BVs by a suitable projection. We also briefly describe the alternative point of view corresponding to the construction of a suitable extension of the profile to the whole line. In section §5 we present our method, based on the BTs, which allows one to identify systematically a whole hierarchy of integrable boundary conditions.

2 Direct and Inverse Spectral Transforms for the NLS equation on the semiline

Our approaches are illustrated on the prototype example of the NLS equation

$$iq_t + q_{xx} + 2c|q|^2 q = 0, \quad q = q(x,t) \in \mathbb{C}, \tag{2.1}$$

where c is an arbitrary real parameter, which describes the amplitude modulation of a wave packet in a strongly dispersive and weakly nonlinear medium, but applies as well to most of the known examples of dispersive soliton equations in $1+1$ dimensions, like the KdV and the modified KdV equations. When applied to the SG equation in light cone coordinates, the spectral matrix used in our approach satisfies a linear evolution equation not containing unknown boundary values, and so no problem arises.

Besides the initial value $q(x,0) = q_0(x)$, the boundary value which uniquely specifies the solution is

$$f(t) = a_1 v(t) + a_2 w(t), \qquad t \geq 0 \tag{2.2}$$

(the Robin BV problem), where we have set

$$v(t) = q(0,t), \qquad w(t) = q_x(0,t). \tag{2.3}$$

Here a_1 and a_2 are given real constants and, if $a_2 = 0$ ($a_1 = 0$) this is the Dirichlet (Neumann) BV problem. Thus the problem is that of constructing the solution $q(x,t)$ of (2.1) when the initial value $q_0(x)$ and the boundary value $f(t)$ are given functions in an appropriate functional space (we may assume that they are complex valued functions which rapidly decay as $x \to \infty$ and $t \to \infty$).

To solve the above problem we make essential use of the fact that the NLS equation (2.1) is the integrability condition of the following system of linear 2×2 matrix equations (the well-known Lax pair) [23]:

$$\Psi_x = (ik\sigma_3 + Q)\Psi, \qquad \Psi_t = (2ik^2\sigma_3 + \tilde{Q})\Psi + \Psi\Gamma \tag{2.4}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$, Γ is an arbitrary x-independent matrix and

$$Q(x,t) = \begin{pmatrix} 0 & -c\bar{q}(x,t) \\ q(x,t) & 0 \end{pmatrix}, \qquad \tilde{Q}(x,t) = 2kQ - i\sigma_3 Q_x + iQ^2\sigma_3, \tag{2.5}$$

where here and below a superimposed bar indicates complex conjugation.

As in the tradition of the spectral transform method, the Jost solutions $\Psi_+(x,t,k)$ and $\Psi_-(x,t,k)$ of (2.4a) are defined by the conditions:

$$\Psi_-(0,t,k) = I, \quad \Psi_+(x,t,k)e^{-ikx\sigma_3} \to I, \ x \to \infty, \tag{2.6}$$

and the scattering matrix $S(k,t)$ is introduced by the relation:

$$\Psi_+(x,t,k) = \Psi_-(x,t,k)S(k,t). \tag{2.7}$$

It is well-known that the Jost solutions and the scattering matrix have unit determinant; it is also well-known that the symmetry satified by Q: $Q^\dagger = -\mathcal{C}Q\mathcal{C}^{-1}$, where $Q^\dagger$ is the Hermitian conjugated of Q and $\mathcal{C} = \mathrm{diag}(1, c)$, implies the following symmetry relations:

$$(\Psi_\pm^\dagger(x,t,k))^{-1} = \mathcal{C}\Psi_\pm(x,t,\bar{k})\mathcal{C}^{-1}, \quad (S^\dagger(k,t))^{-1} = \mathcal{C}S(\bar{k},t)\mathcal{C}^{-1} \tag{2.8}$$

corresponding to the following structures:

$$\Psi_\pm(x,t,k) = \begin{pmatrix} \psi_{11}(x,t,k) & -c\bar{\psi}_{21}(x,t,\bar{k}) \\ \psi_{21}(x,t,k) & \bar{\psi}_{11}(x,t,\bar{k}) \end{pmatrix}_\pm . \tag{2.9}$$

As a consequence, it is convenient to parametrize the matrix S by introducing the two functions $\alpha(k,t)$ and $\beta(k,t)$ according to the definition

$$S(k,t) = \begin{pmatrix} \alpha(k,t) & -c\bar{\beta}(\bar{k},t) \\ \beta(k,t) & \bar{\alpha}(\bar{k},t) \end{pmatrix}. \tag{2.10}$$

It is standard to show that the first column of $N := \Psi_-(x,t,k)e^{-ikx\sigma_3}$ is analytic in the lower half k–plane (LHP) with asymptotics:

$$\begin{pmatrix} N_{11}(x,t,k) \\ N_{21}(x,t,k) \end{pmatrix} = \begin{pmatrix} 1 + O(k^{-1}) \\ \frac{q(x,t)}{2ik} + O(k^{-2}) \end{pmatrix}, \tag{2.11}$$

while the first column of $M := \Psi_+(x,t,k)e^{-ikx\sigma_3}$ is analytic in the upper half k–plane (UHP) with the same asymptotics (2.11). The analyticity properties and the asymptotics of the second columns follow from equations (2.9). The scattering matrix $S(k,t) =$

$M(0,t,k)$ shares the analyticity properties of M and its asymptotics are written down below in some detail for future use:

$$\begin{aligned} \alpha(k,t) &= 1 + \tfrac{c}{2ik}\int_0^\infty dx|q|^2(x,t) + O(k^{-2}), \quad \mathrm{Im}k \geq 0, \\ \beta(k,t) &= \tfrac{1}{2ik}v(t) - \tfrac{1}{(2ik)^2}\check{w}(t) + O(k^{-3}), \quad \mathrm{Im}k \geq 0, \end{aligned} \tag{2.12}$$

where $\check{w}(t) = w(t) - cv(t)\int_0^\infty dx|q|^2(x,t)$.

Direct and Inverse Problems do not differ substantially from the case of NLS on the line and we refer to the classical literature [23, 1] for details. Here we just summarize, for the sake of completeness, their essential features.

The Direct Problem is the mapping from the initial condition $q(x,0) = q_0(x)$ to the elements $\alpha(k,0), \beta(k,0)$ of the scattering matrix at $t = 0$ or, more precisely, to the following spectral data:

$$q(x,0) = q_0(x) \qquad \Rightarrow \qquad \{\rho(k,0), k_j(0), \rho_j(0), j = 1,..,n\} \tag{2.13}$$

where $\rho(k,0) = \beta(k,0)/\alpha(k,0)$, $k_j(0), j = 1,..,n$ are the zeros (if any, and assumed to be simple) of $\alpha(k,0)$ in the upper half k plane and $\rho_j = \beta(k_j)/\alpha'(k_j)$.

The Inverse Problem is the mapping from the evoluted elements $\alpha(k,t), \beta(k,t)$ of the scattering matrix (or, more precisely, from the evoluted spectral data) to the NLS field $q(x,t)$:

$$\{\rho(k,t), k_j(t), \rho_j(t), j = 1,..,n(t)\} \qquad \Rightarrow \qquad q(x,t) \tag{2.14}$$

It can be achieved using the above analyticity properties of the Jost eigenfunctions, which allow one to interprete the scattering equation (2.7) as a Riemann Hilbert (RH) problem on the real k axis. This RH problem can be formulated as a system of linear integral equations:

$$\begin{aligned} &\begin{pmatrix} N_{11}(x,t,k) \\ N_{21}(x,t,k) \end{pmatrix} - \tfrac{1}{2\pi i}\int_{-\infty}^{\infty} \tfrac{dk'}{k'-(k-i0)}\rho(k',t)e^{-2ik'x} \begin{pmatrix} -c\overline{N}_{21}(x,t,k') \\ \overline{N}_{11}(x,t,k') \end{pmatrix} - \\ &\sum_{j=0}^{n(t)} \tfrac{\rho_j(t)}{k-k_j(t)} e^{-2ik_j(t)x} \begin{pmatrix} -c\overline{N}_{21}(x,t,\bar{k}_j) \\ \overline{N}_{11}(x,t,\bar{k}_j) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \tag{2.15}$$

which allows one to reconstruct, from the scattering data, the eigenfunction $N(x,t,k)$, and then $q(x,t) = 2i\lim_{k\to\infty}(kN_{21}(x,t,k)), \mathrm{Im}k < 0$, which follows from (2.11).

This is the right moment to stress two important features of the present formulation as contrasted with the infinite line case. First, the spectral data (2.14) are not independent from each other since $\rho(k)$ is meromorphic in the UHP. Second, as we shall see in the following section, α evolves in time in a way that depends crucially on the boundary conditions; therefore the number n of its zeroes in the UHP as well as their position will depend on time: $n = n(t)$, $k_j = k_j(t)$.

Let us now look at the intermediate step, the time evolution. Here the real crux of the spectral method appears in the evolution equation of the scattering matrix, see (2.4) and (2.7),

$$S_t = 2ik^2[\sigma_3, S] + Z(k,t)S, \tag{2.16}$$

since the matrix $Z(k,t)$ has a separate dependence on both the boundary data $v(t)$ and $w(t)$ according to the following expressions

$$\begin{aligned} Z(k,t) &= 2kV(t) - i\sigma_3 W(t) + iV^2(t)\sigma_3, \\ V(t) &= Q(0,t), \quad W(t) = Q_x(0,t). \end{aligned} \tag{2.17}$$

As a consequence, the evolution equation (2.16) cannot be immediately integrated to yield the scattering matrix $S(k,t)$, whose knowledge is essential to reconstruct $Q(x,t)$ via the solution of the inverse problem. In the following two sections we propose two different ways to close the system and get $S(k,t)$.

3 Replacing the unknown boundary values by the scattering matrix

This basic difficulty can be overcome using the analyticity properties of α and β, which allow one to express the unknown boundary data in terms of α, β, thus obtaining the desired closed evolution equation [17].

Indeed, the analyticity properties of α and β in the UHP lead to the Cauchy formulas:

$$\begin{pmatrix} \alpha(k,t)-1 \\ \beta(k,t) \end{pmatrix} = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{dk'}{k'-k} \begin{pmatrix} \alpha(k',t)-1 \\ \beta(k',t) \end{pmatrix}, \qquad \mathrm{Im} k = 0. \tag{3.1}$$

Considering the $|k| >> 1$ asymptotics of (3.1) and using equations (2.12), we obtain the desired spectral representation of the coefficients of the asymptotic expansions of α and β:

$$\begin{aligned} c\int_0^\infty dx|q|^2(x,t) &= -\tfrac{2}{\pi i}\int_{-\infty}^{\infty} dk[\alpha_+(k,t)-1], \qquad v(t) = -\tfrac{2}{\pi}\int_{-\infty}^{\infty} dk\beta_+(k,t), \\ w(t) - cv(t)\int_0^\infty dx|q|^2(x,t) &= \tfrac{4i}{\pi}\int_{-\infty}^{\infty} dk[\beta_-(k,t) + \tfrac{iv(t)}{2}], \end{aligned} \tag{3.2}$$

where $\alpha_\pm(k,t)$ ($\beta_\pm(k,t)$) are related to the even and odd parts of $\alpha(k,t)$ ($\beta(k,t)$):

$$\begin{aligned} \alpha_+(k,t) &= (\alpha(k,t)+\alpha(-k,t))/2 \;, \quad \alpha_-(k,t) = k(\alpha(k,t)-\alpha(-k,t))/2, \\ \beta_+(k,t) &= (\beta(k,t)+\beta(-k,t))/2 \;, \quad \beta_-(k,t) = k(\beta(k,t)-\beta(-k,t))/2\,. \end{aligned} \tag{3.3}$$

Therefore the t - evolution of the scattering data corresponding to the initial-boundary value problem on the semiline is given by the system of equations (2.16), namely:

$$\begin{aligned} \alpha_{+t} &= -ic|v|^2\alpha_+ - 2c\bar{v}\beta_- + ic\bar{w}\beta_+ \;, \quad \beta_{+t} = -i(4k^2 - c|v|^2)\beta_+ + 2v\alpha_- + iw\alpha_+, \\ \alpha_{-t} &= -ic|v|^2\alpha_- - 2ck^2\bar{v}\beta_+ + ic\bar{w}\beta_- \;, \quad \beta_{-t} = -i(4k^2 - c|v|^2)\beta_- + 2k^2v\alpha_+ + iw\alpha_-\,, \end{aligned} \tag{3.4}$$

where the unknown boundary data are replaced, respectively, by the following spectral representations.

1: For the Dirichlet problem, one replaces the unknown boundary value $w(t)$ by:

$$w(t) = -\frac{2}{\pi i}v(t)\int_{-\infty}^{\infty} dk[\alpha_+(k,t)-1] + \frac{4i}{\pi}\int_{-\infty}^{\infty} dk[\beta_-(k,t) + \frac{iv(t)}{2}]. \tag{3.5}$$

2: For the Neumann problem, one replaces the unknown boundary value $v(t)$ by:

$$v(t) = -\frac{2}{\pi}\int_{-\infty}^{\infty} dk\beta_+(k,t). \tag{3.6}$$

3: For the general Robin problem, one replaces the unknown boundary values $v(t), w(t)$ by:

$$v(t) = -\frac{2}{\pi}\int_{-\infty}^{\infty} dk\beta_+(k,t), \qquad w(t) = \frac{1}{a_2}f(t) + \frac{2a_1}{\pi a_2}\int_{-\infty}^{\infty} dk\beta_+(k,t). \tag{3.7}$$

Of course the initial conditions $\alpha_\pm(k,0)$, $\beta_\pm(k,0)$ for the system (3.4) are obtained by solving the direct problem. As a result of the replacements (3.5-3.7), the evolution of the scattering matrix $S(k,t)$ is obtained by solving a closed system of nonlinear integro-differential equations.

Of course, this method of solution applies also to the *linear* Schrödinger equation. In analogy with this case, it is possible to prove that the solutions $\alpha(k,t)$, $\beta(k,t)$ of the above nonlinear integro-differential evolution equations exist unique. This is in full agreement with PDE theory, in which the boundary values (3.5), (3.6) and (3.7) are necessary and sufficient to obtain one and only one solution $q(x,t)$ with given initial condition. It is interesting to remark that, if we replaced not only the unknown boundary conditions, but also the assigned ones by their spectral representations, then the nonlinear evolutions would loose uniqueness and the solutions would depend on arbitrary functions of time (which could therefore be interpreted as the "illegitimately suppressed" boundary data). This method works well also in the case of the IBV problem on the segment [17].

4 Eliminating the unknown boundary values

To eliminate the unknown boundary values, we make use of the simple parity properties of $Z(k,t)$ as a function of k (see (2.17)) [18]. Rewriting (2.16) in the form

$$(S(k,t)e^{2ik^2\sigma_3 t})_t(S(k,t)e^{2ik^2\sigma_3 t})^{-1} = 2ik^2\sigma_3 + Z(k,t), \quad k \in \mathcal{R}, \tag{4.1}$$

and applying to it the linear operator Π

$$\Pi G(k) := A(k)G(k) - G(-k)A(k), \quad A(k) := a_1 I + 2ika_2\sigma_3, \tag{4.2}$$

we obtain, after some manipulations, the following evolution

$$\begin{gathered}\tilde{S}_t(k,t) = 2ik^2[\sigma_3, \tilde{S}(k,t)] + 4kA^{-1}(k)S^{-1}(-k,t)F(t)S(k,t), \\ F(t) := a_1V(t) + a_2W(t) = \begin{pmatrix} 0 & -c\bar{f}(t) \\ f(t) & 0 \end{pmatrix}.\end{gathered} \tag{4.3}$$

which does not contain the unknown boundaries, where the matrix $\tilde{S}(k,t)$, defined by

$$\tilde{S}(k,t) = A^{-1}(k)S^{-1}(-k,t)A(k)S(k,t), \tag{4.4}$$

has unit determinant and its asymptotic value as $|k| \to \infty$ is the unit matrix:

$$\det \tilde{\mathrm{S}}(\mathrm{k},\mathrm{t}) = 1, \quad \tilde{\mathrm{S}}(\mathrm{k},\mathrm{t}) = \mathrm{I} + \mathrm{O}(\mathrm{k}^{-1}). \tag{4.5}$$

Though this is an important step, it does not yield the solution of our problem since the unknown scattering matrix $S(k,t)$ still appears in the evolution (4.3). Thus one has to find the way to relate $S(k,t)$ and $\tilde{S}(k,t)$ to each other. The relation $S(k,t) \to \tilde{S}(k,t)$ is of course trivial as it is given by the definition (4.4) itself. This relation yields the initial value $\tilde{S}(k,0)$ for the integration of the evolution equation (4.3), i.e. $Q(x,0) \to \Psi(x,0,k) \to S(k,0) = \Psi(0,0,k) \to \tilde{S}(k,0)$. As for the inverse relation, $\tilde{S}(k,t) \to S(k,t)$, one has instead to set up a RH problem, which finally leads to a Cauchy-type integral equation. Starting with rewriting (4.4) in the form $A(k)S(k,t) = S(-k,t)A(k)\tilde{S}(k,t)$, and noting that the first column of $S(k,t)$ in (2.10) is analytic in the UHP with the asymptotic behaviour (2.12), one can write down two coupled integral equations for $\alpha(k,t)$ and $\beta(k,t)$ in terms of $\tilde{S}(k,t)$ by going through the standard RH problem technique. By assuming, just for the sake of simplicity, that no poles occur in the UHP, and by substituting $\tilde{S}(k,t)$ with its expression obtained by formally integrating the evolution equation (4.3), one finally ends up with the two coupled nonlinear integral equations

$$\begin{aligned} \alpha(k,t) &= 1 + \tfrac{c}{2\pi i} \int_{-\infty}^{\infty} \tfrac{dk'}{k'-(k+i0)} e^{-4ik'^2 t} h(k',t)\bar{\beta}(k',t), \\ \beta(k,t) &= -\tfrac{1}{2\pi i} \int_{-\infty}^{\infty} \tfrac{dk'}{k'-(k+i0)} e^{-4ik'^2 t} h(k',t)\bar{\alpha}(k',t), \end{aligned} \tag{4.6}$$

whose nonlinearity is due to the fact that the function $h(k,t)$ depends itself on the unknowns $\alpha(k,t)$ and $\beta(k,t)$:

$$h(k,t) = \frac{A(k) + B(k,t)}{C(k) + D(k,t)}, \tag{4.7}$$

where:

$$\begin{aligned} &A(k) := a(k)\alpha_0(k)\beta_0(-k) - a(-k)\alpha_0(-k)\beta_0(k), \\ &B(k,t) := -4k \int_0^t dt' e^{4ik^2 t'} [f(t')\alpha(k,t')\alpha(-k,t') + c\bar{f}(t')\beta(k,t')\beta(-k,t')], \\ &C(k) := a(-k)\alpha_0(-k)\bar{\alpha}_0(k) + ca(k)\beta_0(-k)\bar{\beta}_0(k); \\ &D(k,t) := -4kc \int_0^t dt' [f(t')\alpha(-k,t')\bar{\beta}(k,t') - \bar{f}(t')\bar{\alpha}(k,t')\beta(-k,t')], \\ &a(k) = a_1 + 2ika_2 \end{aligned} \tag{4.8}$$

and where $\alpha_0(k)$ and $\beta_0(k)$ are, respectively, the known initial values $\alpha(k,0)$ and $\beta(k,0)$.

We observe that this formulation naturally singles out the linearizable IBV problems [21, 22], these being those for which the boundary value $f(t)$, see (2.2), vanishes: $f(t) = 0$. Indeed, in this case, the kernel function $h(k,t)$ (4.8) does not depend on the unknowns $\alpha(k,t)$ and $\beta(k,t)$, and the equations (4.6) become linear. This formulation is also very useful when $f(t)$ is small in some norm. If, f.i., $\sup f < \epsilon << 1$, then we can easily construct the following asymptotic series in powers of ϵ:

$$\begin{pmatrix} \alpha(k,t) \\ \beta(k,t) \end{pmatrix} = \begin{pmatrix} \alpha^{(0)}(k,t) \\ \beta^{(0)}(k,t) \end{pmatrix} + \epsilon \begin{pmatrix} \alpha^{(1)}(k,t) \\ \beta^{(1)}(k,t) \end{pmatrix} + O(\epsilon^2), \tag{4.9}$$

where the leading term $(\alpha^{(0)}(k,t),\beta^{(0)}(k,t))^T$ satisfies the *linear* integral equation

$$\begin{aligned}\alpha^{(0)}(k,t) &= \frac{c}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-(k+i0)}e^{-4ik'^2t}\frac{A(k')}{C(k')}\bar\beta^{(0)}(k',t)+1,\\ \beta^{(0)}(k,t) &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-(k+i0)}e^{-4ik'^2t}\frac{A(k')}{C(k')}\bar\alpha^{(0)}(k',t).\end{aligned} \tag{4.10}$$

Once $(\alpha^{(0)}(k,t),\beta^{(0)}(k,t))^T$ is known, the correction $(\alpha^{(1)}(k,t),\beta^{(1)}(k,t))^T$ satisfies the *linear* forced integral equation

$$\begin{aligned}\alpha^{(1)}(k,t) &= \frac{c}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-(k+i0)}e^{-4ik'^2t}\frac{A(k')}{C(k')}\bar\beta^{(1)}(k',t)+\\ &\frac{c}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-(k+i0)}e^{-4ik'^2t}h_1(k',t)\bar\beta^{(0)}(k',t),\\ \beta^{(1)}(k,t) &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-(k+i0)}e^{-4ik'^2t}\frac{A(k')}{C(k')}\bar\alpha^{(1)}(k',t)+\\ &\frac{c}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-(k+i0)}e^{-4ik'^2t}h_1(k',t)\bar\alpha^{(0)}(k',t),\end{aligned} \tag{4.11}$$

where

$$h_1(k,t) := \frac{B_0(k,t)C(k)-A(k)D_0(k,t)}{C^2(k)} \tag{4.12}$$

and $B_0(k,t), D_0(k,t)$ are obtained from $B(k,t), D(k,t)$ by replacing $\alpha(k,t)$ and $\beta(k,t)$ with $\alpha^{(0)}(k,t)$ and $\beta^{(0)}(k,t)$.

As a side remark, we also note that setting $c = 0$ eliminates the nonlinearity in all the formulae given above, so that (2.1) becomes the linear Schrödinger equation and our equations (4.6), (4.7) and (4.8) yield $\alpha(k,t) = 1$ while $\beta(k,t)$ coincides with the usual explicit expression of the Fourier transform of the solution $q(x,t)$.

Finally, we deem of interest to report here also an approach to the IBV problem for the NLS equation (2.1) which is different from the one given above, and yet equivalent as it eventually leads to the same equations (4.6), (4.7) and (4.8). The main feature of this approach is that the IBV problem is reformulated on the whole line, i.e. for $x \in (-\infty,\infty)$, and that the matrix $\tilde S(k,t)$, as defined by (4.4), acquires now a spectral meaning within the standard direct and inverse problem associated with the Lax equation, say the first ODE in (2.4). The price one pays to arrive at this more familiar formulation is that the nonlinear PDE one has to solve now is the NLS equation with a inhomogeneous source term rather than the NLS equation (2.1). This approach is briefly sketched here with two limitations which are merely dictated by the sake of simplicity; namely we confine our treatment to the Dirichelet and Neumann BV problems and, second, we assume that the spectral data at any time $t \geq 0$ have no discrete spectrum component.

The starting observation is that, if $q(x,t)$ is a solution of the NLS equation (2.1) for $x \in (0,\infty)$ and $t \geq 0$, then the function

$$\tilde q(x,t) = q(x,t)H(x) - \eta q(-x,t)H(-x), \quad \eta = \pm 1, \tag{4.13}$$

as defined for any real value of x, satisfies the PDE

$$i\tilde q_t + \tilde q_{xx} + 2c|\tilde q|^2\tilde q = (1+\eta)v(t)\delta'(x) + (1-\eta)w(t)\delta(x). \tag{4.14}$$

where $H(x)$ is the step function ($H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x < 0$), $\delta(x)$ is the Dirac delta distribution, $\delta'(x)$ is its derivative and $v,\ w$ are defined in (2.3). Obviously, $\eta = 1$ ($\eta = -1$) is the appropriate choice when one deals with the Dirichelet (Neumann) IBV problem.

As implied by the spectral method based on the Lax equations, it is convenient to rewrite (4.13) and (4.14) in matrix form by introducing the 2×2 off-diagonal matrix

$$\tilde{Q}(x,t) = Q(x,t)H(x) - \eta Q(-x,t)H(-x), \quad \eta = \pm 1, \tag{4.15}$$

and the PDE

$$i\tilde{Q}_t - \sigma_3(\tilde{Q}_{xx} - 2\tilde{Q}^3) = \Sigma(x,t) \tag{4.16}$$

which is, of course, equivalent to (4.14) if the source term is (see (2.17))

$$\Sigma(x,t) = -\sigma_3[(1+\eta)V(t)\delta'(x) + (1-\eta)W(t)\delta(x)]. \tag{4.17}$$

The spectral approach to the equation (4.16) is based on the spectral equation

$$\tilde{\Psi}_x = (ik\sigma_3 + \tilde{Q}(x,t))\tilde{\Psi}, \qquad \tilde{\Psi} = \tilde{\Psi}(x,t,k), \tag{4.18}$$

and it is standard. The Jost solution $\tilde{\Psi}$ is defined by the asymptotic condition (2.6), $\tilde{\Psi}\exp(-ikx\sigma_3) \to I,\ x \to \infty$, which readly provides its expression in terms of the solution $\Psi_+(x,t,k)$ introduced by (2.6) on the semiline,

$$\tilde{\Psi}(x,t,k) = \Psi_+(x,t,k)H(x) + E\Psi_+(-x,t,-k)E\tilde{S}(k,t)H(-x), \tag{4.19}$$

where $E = $diag $(1,\eta)$ and

$$\tilde{S}(k,t) = ES^{-1}(-k,t)ES(k,t), \tag{4.20}$$

is precisely the scattering matrix which is defined in the usual way, namely

$$\tilde{\Psi}(x,t,k) \to e^{ikx\sigma_3}\tilde{S}(k,t), \quad x \to -\infty. \tag{4.21}$$

At this point we note that this scattering matrix $\tilde{S}(k,t)$ coincides with the matrix (4.4) with $a_2 = 0$ in the Dirichelet case ($\eta = 1$) and with $a_1 = 0$ in the Neumann case ($\eta = -1$).

It is common expedient now to introduce also the other Jost solution of (4.18),

$$\tilde{\Phi}(x,t,k) = \tilde{\Psi}(x,t,k)\tilde{S}^{-1}(k,t), \tag{4.22}$$

and to take into account the identity

$$\tilde{S}_t + 2ik^2[\tilde{S},\sigma_3] = i\int_{-\infty}^{\infty} dx\tilde{\Phi}^{-1}(x,t,k)[i\tilde{Q}_t - \sigma_3(\tilde{Q}_{xx} - 2\tilde{Q}^3)]\tilde{\Psi}(x,t,k), \tag{4.23}$$

which, together with the inhomogeneous PDE (4.16), entails the evolution equation for the scattering matrix,

$$\tilde{S}_t(k,t) = 2ik^2[\tilde{S}(k,t),\sigma_3] + i\int_{-\infty}^{\infty} dx\tilde{\Phi}^{-1}(x,t,k)\Sigma(x,t)\tilde{\Psi}(x,t,k). \tag{4.24}$$

It is now easy to show that inserting in the integral in the RHS of this equation the expressions (4.22), (4.19) and (4.17) yields precisely the evolution equation (4.3) for the Dirichelet and Neumann IBV problems, say

$$\tilde{S}_t(k,t) = 2ik^2[\sigma_3, \tilde{S}(k,t)] + 2k(1+\eta)S^{-1}(-k,t)V(t)S(k,t) - i(1-\eta)\sigma_3 S^{-1}(-k,t)W(t)S(k,t). \tag{4.25}$$

We end this section remarking that a good side of the present approach is that one may take advantage of the more traditional inverse scattering (spectral) technique on the whole line. In particular, to investigate the large time behaviour of the solution $q(x,t)$ of the IBV problem, since the asymptotic expression, if the boundary value rapidly vanishes as $t \to \infty$, are readly at hand in the usual spectral theory on the whole line.

5 Higher order integrable boundary conditions

In this section we show the way to introduce IBVPs for the NLS equation (2.1) that can be integrated by spectral means via linear operations only. The associated boundary conditions at $x = 0$ are expressed by one relation involving not only the boundary values $v(t)$ and $w(t)$, but also their derivatives with respect to t. An IBVP$_n$ is an IBVP where the boundary condition at $x = 0$ involves derivatives of $v(t)$ and $w(t)$ up to order n. Thus the condition (2.2) (together with the initial condition $q(x,0) = q_0(x)$) provides a simple instance of an IBVP$_0$. The general strategy to obtain these higher order IBVPs makes use of BTs whose action on an IBVP$_{(n-1)}$ yields an IBVP$_n$. Since two solutions of the NLS equation which are related to each other by a BT satisfy a system of two coupled differential nonlinear equations (see below), the connection formulae between two IBVPs are awfully complicate and barely tractable. Therefore we limit our discussion here to an IBVP$_1$, and, moreover, explicit formulae are given only in the particular case in which the starting IBVP$_0$ is the Dirichelet problem.

First we very shortly review definitions and properties of a BT, and we collect those formulae that are specially tailored to our purposes. The formulae and propositions reported below are all well-known and available in the literature (see f.i. [24, 25, 26]). As BTs are local, at this stage we do not necessary have to deal with boundary conditions. If $q^{(0)}(x,t)$ is a solution of the NLS equation (2.1), then a second solution $q(x,t)$ is related to it by a BT if it satisfies the system of first-order PDEs

$$q_x - q_x^{(0)} = -2is(q - q^{(0)}) + 2pz(q + q^{(0)}) \ , \tag{5.1}$$

$$q_t - q_t^{(0)} = 4psz(q + q^{(0)}) + 2ipz(q_x + q_x^{(0)}) + i(q - q^{(0)})[c|q|^2 + c|q^{(0)}|^2 - 4s^2] \, , \tag{5.2}$$

where s and p are two given *real* constants and $z(x,t)$ is the assumedly real function

$$z = \sqrt{1 - \frac{c}{4p^2}|q - q^{(0)}|^2} \ . \tag{5.3}$$

This system of PDEs, (5.1) and (5.2), may be conveniently referred to as Partial Differential Backlund Transformation (PDBT) so as to distinguish it from an alternative, and equivalent, formulation of the same BT, which may be reffered to as Spectral Backlund

Transformation (SBT) because its expression follows from the Lax pair (2.4) associated with the NLS equation (2.1).

The SBT takes the following explicit expression:

$$q(x,t) = q^{(0)}(x,t) - 4p\frac{\zeta_2^{(0)}(x,t)\overline{\zeta_1^{(0)}}(x,t)}{|\zeta_1^{(0)}(x,t)|^2 + c|\zeta_2^{(0)}(x,t)|^2} \ , \tag{5.4}$$

where the complex functions $\zeta_1^{(0)}(x,t)$ and $\zeta_2^{(0)}(x,t)$ are the components of the vector solution

$$\zeta^{(0)}(x,t) = \begin{pmatrix} \zeta_1^{(0)}(x,t) \\ \zeta_2^{(0)}(x,t) \end{pmatrix} \tag{5.5}$$

of the pair of Lax equations (see (2.4))

$$\zeta_x^{(0)} = (i\bar{a}\sigma_3 + Q^{(0)})\zeta^{(0)} \ , \tag{5.6}$$

$$\zeta_t^{(0)} = (2i\bar{a}^2\sigma_3 + 2\bar{a}Q^{(0)} - i\sigma_3 Q_x^{(0)} + i\sigma_3 Q^{(0)2})\zeta^{(0)} \ . \tag{5.7}$$

Of course, the constants s and p which appear in (5.4) are the same constants as in the PDBT, (5.1) and (5.2), and the complex constant a is the spectral parameter associated with the BT (see (5.6) and (5.7)),

$$a = s + ip \ ; \tag{5.8}$$

moreover, the 2×2 matrix $Q^{(0)}(x,t)$ is the off-diagonal matrix (2.5) with $q(x,t)$ replaced by $q^{(0)}(x,t)$. Here it should be emphasized that the chain of linear operations $q^{(0)} \to \zeta^{(0)} \to q$ linearizes the PDBT, namely the system of nonlinear PDEs (5.1) and (5.2). It should be also pointed out that the BT we are considering here depends also on an additional complex parameter, say γ. This parameter enters in the PDBT formulation as a constant of integration when integrating the two PDEs (5.1) and (5.2), or, equivalently, in the SBT when solving the Lax pair of ODEs (5.6) and (5.7) with the condition

$$\zeta^{(0)}(0,0) = \begin{pmatrix} \gamma \\ 1 \end{pmatrix} \ , \tag{5.9}$$

which then defines the constant γ.

Let us consider now, for future use, the inversion of this BT. As for the PDBT, this requires solving the same PDEs (5.1) and (5.2) for $q^{(0)}$ when q is given. This is formally due to the fact the the transformation $p \to -p, q^{(0)} \leftrightarrow q$ leaves the system of equations (5.1) and (5.2) unchanged. The inversion of the SBT (5.4) is merely $q^{(0)} = q + 4p\zeta_2^{(0)}\overline{\zeta_1^{(0)}}/(|\zeta_1^{(0)}|^2 + c|\zeta_2^{(0)}|^2)$ which however, for symmetry argument, takes also the symmetric form

$$q^{(0)}(x,t) = q(x,t) - 4p\frac{\zeta_2(x,t)\overline{\zeta_1}(x,t)}{|\zeta_1(x,t)|^2 + c|\zeta_2(x,t)|^2} \ , \tag{5.10}$$

Indeed this formula obtains by replacing p with $-p$ and by noticing that the vector $\zeta(x,t)$ given by the symmetric expressions

$$\zeta(x,t) = \frac{\zeta^{(0)}(x,t)}{|\zeta_1^{(0)}(x,t)|^2 + c|\zeta_2^{(0)}(x,t)|^2} \ , \ \zeta^{(0)}(x,t) = \frac{\zeta(x,t)}{|\zeta_1(x,t)|^2 + c|\zeta_2(x,t)|^2} \tag{5.11}$$

satisfies the Lax equations (see (2.5))

$$\zeta_x = (ia\sigma_3 + Q)\zeta \ , \tag{5.12}$$

$$\zeta_t = (2ia^2\sigma_3 + 2aQ - i\sigma_3 Q_x + i\sigma_3 Q^2)\zeta \ . \tag{5.13}$$

with the condition

$$\zeta(0,0) = \frac{1}{c+|\gamma|^2}\begin{pmatrix}\gamma \\ 1\end{pmatrix} \ . \tag{5.14}$$

Let us consider now the case in which $q^{(0)}(x,t)$ satisfies an IBVP_0, for instance let this solution of the NLS equation be uniquely specified by the initial condition $q^{(0)}(x,0) = q_0^{(0)}(x)$ and by the Robin boundary condition (see(2.2))

$$f^{(0)}(t) = a_1 v^{(0)}(t) + a_2 w^{(0)}(t) \tag{5.15}$$

with obvious notation, see (2.3). And let us then find out the IBVP which is solved by the solution $q(x,t)$ of the NLS which obtains via the BT reported above. The initial data $q(x,0) = q_0(x)$ is clearly obtained by solving half BT, namely the ODE (5.1), or, equivalently, by applying its spectral version SBT, (5.4) together with the ODE (5.6), at $t=0$. As for the boundary $x=0$, the transformation is less straightforward. In fact on this boundary the system of equations

$$\left\{\begin{array}{c} a_1 v^{(0)} + a_2 w^{(0)} = f^{(0)} \\ w - w^{(0)} = -2is(v - v^{(0)}) + 2pz_0(v + v^{(0)}) \\ v_t - v_t^{(0)} = 4psz_0(v+v^{(0)}) + 2ipz_0(w + w^{(0)}) + i(v - v^{(0)})(c|v|^2 + c|v^{(0)}|^2 - 4s^2) \end{array}\right. \tag{5.16}$$

all hold, the first being the Robin condition (5.15), while the second and third equations, together with the definition

$$z_0(t) = \sqrt{1 - \frac{c}{4p^2}|v(t) - v^{(0)}(t)|^2} \tag{5.17}$$

are the x-part and, respectively, the t-part of the PDBT at $x=0$, compare (5.1) and (5.2) with (2.3). In order to extract from these equations the boundary condition satisfied by the solution $q(x,t)$ one has to eliminate the functions $v^{(0)}(t)$ and $w^{(0)}(t)$ so as to end up with a relation involving only $v(t), w(t), v_t(t)$ and $w_t(t)$. Before proceeding further, let us pause to derive this way the boundary condition which obtains if we deal instead with the *linear* Schrödinger equation rather than with the NLS equation. This result easily follows by setting $c=0$ in all formulae given above, and reads

$$a_1 v_t + a_2 w_t - f_t^{(0)} + 4i|a|^2(a_1 v + a_2 w - f^{(0)}) = 4a_2(pv_t - s|a|^2 v) + 4ipa_1 w \ , \tag{5.18}$$

which is, as expected, a linear combination of v, w, v_t and w_t (i.e. a first order generalization of the Robin problem). In the nonlinear case the elimination of $v^{(0)}(t)$ and $w^{(0)}(t)$ is not an easy task as it requires finding the roots of a fourth degree polynomial, because of the presence of the square root (5.17). Only for the Dirichlet problem, $a_1 = 1, a_2 = 0$, the boundary condition for the solution $q(x,t)$, takes the following simple explicit form:

$$\begin{array}{c} v_t + iv(4|a|^2 - c|v - f^{(0)}|^2 - c|v|^2 - c|f^{(0))}|^2) - 4ipuw = \\ f_t^{(0)} + if^{(0)}[4(s-ipu)^2 - c|v|^2 - c|f^{(0)}|^2] \ , \ \ u = \sqrt{1 - \frac{c}{4p^2}|v - f^{(0)}|^2} \, . \end{array} \tag{5.19}$$

Because of the particular case we are considering, this condition does not involve the time derivative of $w(t)$, and it specifies, together with the initial condition, an IBVP_1 which may be considered a first order extension of the Dirichlet problem.

Next we turn our attention to the method of constructing the solution of the IBVP_1 characterized by the initial condition $q(x,0) = q_0(x)$ and by the boundary condition (5.19). Note that the coefficients entering this condition are not the most general ones and are ment to identify the spectral parameter $a = s + ip$, see (5.8), which is connected to the BT one has to apply to solve this problem. Of course this method relies on the possibility to map this IBVP into the standard Dirichelet problem by means of a BT. First we map the function $q_0(x)$ into the function $q_0^{(0)}(x)$ via the formula (5.10) at $t = 0$. Next we solve the Dirichelet problem with this initial data and with the boundary condition $v^{(0)}(t) = f^{(0)}(t)$. Once the solution $q^{(0)}(x,t)$ of this IBVP has been found, the solution $q(x,t)$ is obtained via the SBT formula (5.4). The validity of this construction is implied by the results reported above. Moreover, the integrability of this Dirichlet type first order IBVP is guaranteed by the condition that also the associated Dirichlet problem is integrable, namely $f^{(0)}(t) = 0$ (see section 4). Therefore we conclude that the IBVP_1:

$$\begin{cases} q(x,0) = q_0(x) \\ v_t(t) + 4i|a|^2 v(t) - 2ic|v(t)|^2 v(t) - 4ipw(t)\sqrt{1 - \frac{c}{4p^2}|v(t)|^2} = 0 \end{cases} \tag{5.20}$$

is also integrable.

We end this section by showing that, by applying once more a BT to the first order Dirichlet IBVP (5.20), one obtains an IBVP_2. The arguments are only sketched as, in general, explicit formulas are out of reach. Again the starting equations are the boundary condition at $x = 0$ (5.20) and the PDBT at $x = 0$ (with (5.17)):

$$\begin{cases} v_t^{(0)}(t) + 4i|a|^2 v^{(0)}(t) - 2ic|v^{(0)}(t)|^2 v^{(0)}(t) - 4ipw^{(0)}(t)\sqrt{1 - \frac{c}{4p^2}|v^{(0)}(t)|^2} = 0 \\ w - w^{(0)} = -2is(v - v^{(0)}) + 2pz_0(v + v^{(0)}) \\ v_t - v_t^{(0)} = 4psz_0(v + v^{(0)}) + 2ipz_0(w + w^{(0)}) + i(v - v^{(0)})(c|v|^2 + c|v^{(0)}|^2 - 4s^2)\,. \end{cases} \tag{5.21}$$

Now we solve the second equation for $w^{(0)}$ and substitute this expression of $w^{(0)}$ in the sum of the first and third equations, so to eliminate $v_t^{(0)}$. This way we obtain an equation of the form $v_t = F(v, w, v^{(0)})$, where F is an explicit algebraic expression of its arguments. Next this equation has to be solved with respect to $v^{(0)}$ so as to express $v^{(0)}$ as a function of v, w and v_t, i.e. $v^{(0)} = \Phi(v, w, v_t)$. Differentiating both sides of this equation and replacing in the third of equations (5.21) $v^{(0)}, v_t^{(0)}, w^{(0)}$ by their expressions in terms of v, w, v_t, one finally obtains the boundary condition which characterizes the integrable Dirichlet–type second order IBVP, which is indeed a relation between v, w, v_t, w_t and v_{tt}. Here the difficult step is of course giving the function Φ an explicit expression.

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