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On Dispersionless BKP Hierarchy and its Reductions

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Abstract

Integrable dispersionless Kadomtsev-Petviashvili (KP) hierarchy of B type is considered. Addition formula for the τ-function and conformally invariant equations for the dispersionless BKP (dBKP) hierarchy are derived. Symmetry constraints for the dBKP hierarchy are studied.

1 Introduction

Dispersionless integrable hierarchies play an important role in the study of different nonlinear phenomena in various fields of physics and mathematics (see e.g. [1]-[12]). They have attracted recently a considerable interest (see e.g. [13]-[17]).

Quasi-classical $\bar{\partial}$-dressing method, proposed in [18, 19, 20, 21], gives a new approach to study the properties of dispersionless integrable hierarchies, including various addition formulae, symmetry constraints etc. [22, 23, 24]. Most of the results obtained using this approach are connected with dispersionless Kadomtsev-Petviashvili (dKP), modified dispersionless Kadomtsev-Petviashvili (dmKP) and dispersionless 2-dimensional Toda lattice (d2DTL) hierarchies.

In the full 'dispersive' case not only the standard KP hierarchy (the hierarchy of A type), but also BKP, CKP and DKP hierarchies play an important role [25, 26, 27]. In the dispersionless case the study of hierarchies of B type is just in the very beginning [28, 20].

In the present paper we analyze the dBKP hierarchy in detail. We formulate the $\bar{\partial}$-dressing approach to this hierarchy, derive an integral formula for the τ-function, obtain the fundamental equation for the basic homeomorphism which represents a generating equation for the whole hierarchy. We derive also the dispersionless addition formula for the τ-function and obtain conformally invariant equation for its symmetries. We consider symmetry constraints for the dBKP hierarchy and find generating equations and Sato functions for the constrained hierarchies.

2 dKP and dBKP hierarchies

The quasiclassical $\bar{\partial}$-dressing scheme for dispersionless KP hierarchy [18, 19, 20, 21] is based on nonlinear Beltrami equation

$$S_{\bar{z}} = W(z, \bar{z}, S_z), \tag{2.1}$$

where $\bar{\partial}$-data W are localized in the unit disc, $S(z, \bar{z}, \mathbf{t}) = S_0 + \widetilde{S}$, $S_0(z, \mathbf{t}) = \sum_{n=1}^{\infty} t_n z^n$, $\widetilde{S}$ is analytic outside the unit disc, and at infinity it has an expansion $\widetilde{S} = \sum_{i=1}^{\infty} \widetilde{S}_i(\mathbf{t}) z^{-i}$. The quantity $p = \frac{\partial S}{\partial t_1}$ is a basic homeomorphism [19]. Important role in the theory of dKP hierarchy is played by the equation

$$p(z) - p(z_1) + z_1 \exp(-D(z_1)S(z)) = 0, \quad z \in \mathbb{C}, \quad z_1 \in \mathbb{C} \setminus D, \tag{2.2}$$

(where $D(z)$ is the quasiclassical vertex operator, $D(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n} \frac{\partial}{\partial t_n}$, $|z| > 1$) which generates Hamilton-Jacobi equations of the hierarchy by expansion into the powers of z^{-1} at infinity (see, e.g., [22]). This equation also implies existence of the τ-function, characterized by the relation

$$\widetilde{S}(z, \mathbf{t}) = -D(z)F(\mathbf{t}), \tag{2.3}$$

and provides the dispersionless addition formula.

Dispersionless BKP hierarchy is characterized by the symmetry condition

$$S(-z, \mathbf{t}) = -S(z, \mathbf{t}), \tag{2.4}$$

which is preserved only by odd flows of the hierarchy (t_n with odd n), $S_0(z, \mathbf{t}) = \sum_{n=0}^{\infty} t_{2n+1} z^{2n+1}$. Thus dBKP hierarchy is dKP hierarchy with even times frozen at zero plus symmetry (2.4) [28]. In terms of $\bar{\partial}$-data this symmetry is provided by the condition

$$W(-z, S_z) = W(z, S_z).$$

To obtain the analogue of relation (2.2) for dBKP hierarchy, we introduce B-type quasiclassical vertex operator $\mathcal{D}(z) = 2\sum_{n=0}^{\infty} \frac{z^{-(2n+1)}}{2n+1} \frac{\partial}{\partial t_{2n+1}}$, $|z| > 1$, characterized by the property $\mathcal{D}(z_1)S_0(z) = \ln \frac{z_1+z}{z_1-z}$. Then, similar to dKP case [22], we get

$$\frac{p(z_1) - p(z)}{p(z_1) + p(z)} = \exp(-\mathcal{D}(z_1)S(z)), \tag{2.5}$$

This equation generates Hamilton-Jacobi equations of the dBKP hierarchy by expansion into the powers of z^{-1} at infinity. The first two Hamilton-Jacobi equations are

$$S_y = p^3 + 3up \tag{2.6}$$

$$S_t = p^5 + 5up^3 + vp, \tag{2.7}$$

where $x = t_1$, $y = t_3$, $t = t_5$, $u = -\partial_x \widetilde{S}_1$, $v_x = \frac{5}{3}u_y + 5(u^2)_x$. Compatibility condition for these equations gives dispersionless BKP equation [20]

$$\frac{1}{5}u_t + u^2 u_x - \frac{1}{3}uu_y - \frac{1}{3}u_x \partial_x^{-1} u_y - \frac{1}{9}\partial_x^{-1} u_{yy} = 0. \tag{2.8}$$

Relation (2.5) for dBKP hierarchy can be also obtained starting with (2.2). Indeed, using (2.2), in the framework of dKP hierarchy we get a relation

$$\frac{p(z)-p(z_1)}{p(z)-p(-z_1)} = -\exp(-(D(z_1)-D(-z_1))S(z)), \tag{2.9}$$

Then, freezing even times at zero and using symmetry (2.4), we get (2.5) ($\mathcal{D}(z) = D(z) - D(-z)$).

Equation (2.5) also implies existence of the τ-function, characterized by the relation

$$\widetilde{S}(z,\mathbf{t}) = -\mathcal{D}(z)F_{\text{dBKP}}(\mathbf{t}).$$

Comparing this relation with (2.3), we come to the conclusion that

$$2F_{\text{dBKP}} = F_{\text{dKP}},$$

if F_{dKP} is taken at zero even times and symmetry condition (2.4) is satisfied [28]. This symmetry condition is equivalent to a simple condition for the function F_{dKP} itself, namely that its derivatives $\partial F_{\text{dKP}}/\partial t_{2(n+1)}$ taken at zero even times are equal to zero.

In complete analogy with dKP hierarchy, it is possible to find explicit representaion of dBKP τ-function as an action for nonlinear Beltrami equation (2.1) evaluated on its solution.

Proposition 1. *The function*

$$F(\mathbf{t}) = -\frac{1}{2\pi\mathrm{i}}\iint_D \left(\frac{1}{2}\widetilde{S}_{\bar z}(\mathbf{t})\widetilde{S}_z(\mathbf{t}) - W_{-1}(z,\bar z, S_z(\mathbf{t}))\right) dz\wedge d\bar z, \tag{2.10}$$

i.e., the action for the problem (2.1) evaluated on its solution, where $\partial_\eta W_{-1}(z,\bar z,\eta) = W(z,\bar z,\eta)$*, and* W *satisfies a symmetry condition*

$$W(-z,-\bar z, S_z) = W(z,\bar z,S_z),$$

is a τ*-function of dBKP hierarchy.*

Variations of the $\bar\partial$-data define infinitesimal symmetries of the τ-function [23, 24]. One should take into account that these variations should satisfy the symmetry condition. Considering variations localized in the pair of points z_0, $-z_0$, we get a symmetry

$$\delta F = f(S_z(z_0)), \tag{2.11}$$

where f is an arbitrary analytic function. Variations localized on the set of curves lead to infinitesimal symmetry

$$\delta F = \sum_{i=1}^{N} c_i(S_i - \tilde S_i), \tag{2.12}$$

where $S_i = S(z_i)$, $\tilde S_i = S(\tilde z_i)$, z_i, $\tilde z_i$ are some sets of points, and c_i are arbitrary constants. Due to the symmetry $S(-z) = -S(z)$ in dBKP case, it is possible to take $\tilde z_i = -z_i$ and consider symmetries of the form

$$\delta F = 2\sum_{i=1}^{N} c_i S_i. \tag{2.13}$$

3 Addition formula for dBKP hierarchy

Expressing S in terms of F, from equation (2.5) we get

$$\frac{p(z_1)-p(z_2)}{p(z_1)+p(z_2)} = \frac{z_1-z_2}{z_1+z_2} e^{\mathcal{D}(z_1)\mathcal{D}(z_2)F}. \tag{3.1}$$

Using this equation, we obtain a system of linear equations for $p(z_i)$,

$$p(z_i)(f_{ij}-1) - p(z_j)(f_{ij}+1) = 0 \tag{3.2}$$

where

$$f_{ij} = \frac{z_i-z_j}{z_i+z_j} e^{\mathcal{D}(z_i)\mathcal{D}(z_j)F}, \quad 1 \le i,j \le 3. \tag{3.3}$$

To possess nontrivial solutions, this system should have zero determinant,

$$\det \begin{pmatrix} f_{12}-1 & f_{12}+1 & 0 \\ 0 & f_{23}-1 & f_{23}+1 \\ f_{13}+1 & 0 & f_{13}-1 \end{pmatrix} = 0.$$

Thus

$$(f_{23}+1)(f_{31}+1)(f_{12}+1) = (f_{23}-1)(f_{31}-1)(f_{12}-1), \tag{3.4}$$

or, equivalently,

$$f_{23}f_{31}f_{12} + f_{23} + f_{31} + f_{12} = 0. \tag{3.5}$$

This condition gives addition formula for dispersionless BKP hierarchy,

$$1 + c_2c_3 e^{-(\mathcal{D}_3\mathcal{D}_1+\mathcal{D}_1\mathcal{D}_2)F} + c_1c_3 e^{-(\mathcal{D}_2\mathcal{D}_3+\mathcal{D}_1\mathcal{D}_2)F} + c_2c_1 e^{-(\mathcal{D}_3\mathcal{D}_1+\mathcal{D}_2\mathcal{D}_3)F} = 0, \tag{3.6}$$

where $\mathcal{D}_i = \mathcal{D}(z_i)$, $c_i = \frac{z_j+z_k}{z_j-z_k}$ ((i,j,k) is a cyclic permutation of $(1,2,3)$).

4 Conformally invariant equations of dBKP hierarchy

An important object of dispersive integrable hierarchies are discrete Schwarzian equations, which possess Möbius symmetry and have a deep connection with geometry [29, 30]. It was demonstrated in [23] for KP and 2DTL hierarchies that dispersionless analogues of these equations are given by conformally invariant equations of dispersionless hierarchies, which arise also an a naive continuum limit of discrete Schwarzian equations [29].

In the case of dBKP hierarchy we start with the evident analogy of (3.2), (3.4) and the formulae in [30] connected with continuum limit of discrete SBKP equation. Using this analogy, we introduce the function Φ, $p(z_i) = \mathcal{D}_i\Phi$. To demonstrate existence of Φ, let us rewrite relations (3.2) in the form

$$\mathcal{D}_2\Phi = \frac{f_{12}-1}{f_{12}+1}\mathcal{D}_1\Phi, \quad \mathcal{D}_3\Phi = \frac{f_{13}-1}{f_{13}+1}\mathcal{D}_1\Phi$$

Compatibility of this linear system is implied by addition formula (3.5). Thus the function Φ, $p(z_i) = \mathcal{D}_i\Phi$ exists.

Relations (3.1) imply equation for Φ,

$$\mathcal{D}_1 \ln \frac{\mathcal{D}_2\Phi + \mathcal{D}_3\Phi}{\mathcal{D}_2\Phi - \mathcal{D}_3\Phi} = \mathcal{D}_2 \ln \frac{\mathcal{D}_1\Phi + \mathcal{D}_3\Phi}{\mathcal{D}_1\Phi - \mathcal{D}_3\Phi}, \tag{4.1}$$

and this equation coincides with continuum limit of discrete SBKP equation introduced in [30].

Equation (4.1) can be written in symmetric form,

$$\Phi_{23}\Phi_1(\Phi_2^2 - \Phi_3^2) + \Phi_{31}\Phi_2(\Phi_3^2 - \Phi_1^2) + \Phi_{12}\Phi_3(\Phi_1^2 - \Phi_2^2) = 0, \tag{4.2}$$

where subscripts denote vertex derivatives.

It is also easy to find a determinant representation for this equation,

$$\det \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \Phi_2\Phi_3 & \Phi_1\Phi_3 & \Phi_1\Phi_2 \\ \Phi_{23} & \Phi_{13} & \Phi_{12} \end{pmatrix} = 0. \tag{4.3}$$

We have defined the function Φ implicitly through the relation $p_i = \mathcal{D}_i\Phi$. However, it is possible to construct this function explicitly, using the potential u of dispersionless hierarchy. Indeed, $u = -F_x$, $p(z) = -\partial_x\mathcal{D}(z)F + z$, then $p(z) = z + \mathcal{D}(z)u$. Thus Φ is 'almost' u. For u, instead of (4.1), we get a generating equation

$$\mathcal{D}_1 \ln \frac{u_2 + u_3 + z_2 - z_3}{u_2 - u_3 + z_2 - z_3} = \mathcal{D}_2 \ln \frac{u_1 + u_3 + z_1 + z_3}{u_1 - u_3 + z_1 - z_3}. \tag{4.4}$$

This equation gives dBKP hierarchy equations for potential u by expansion into parameters z_1, z_2, z_3.

To transform relation $p(z) = z + \mathcal{D}(z)u$ to the relation $p_i = \mathcal{D}_i\Phi$, we define Φ as $u - \sum_{k=0}^{N} c_{2k+1}t_{2k+1}$. It is easy to see that it is possible to satisfy relations $p_i = \mathcal{D}_i\Phi$ at some set of points z_i by the choice of constants c_{2k+1} (taking sufficiently large N). Indeed,

$$p(z) - D(z)\Phi = z + \sum_{k=0}^{N} \frac{c_{2k+1}}{2k+1} z^{-(2k+1)},$$

and the relation $p_i = \mathcal{D}_i\Phi$ is satisfied if z_i is a zero of polynomial

$$P(z) = (z^2)^{N+1} + \sum_{k=0}^{N} \frac{c_{2k+1}}{2k+1} (z^2)^{N-k}.$$

Taking, e.g., $P(z) = \prod_{i=1}^{I}(z^2 - z_i^2)$, it is possible to express the constants c_{2k+1} through z_i explicitly. Thus solution to equation (4.1) can be expressed in terms of potential u.

A general conformally invariant equation of dBKP hierarchy

It is also possible, using the approach developed in [23], to derive a general equation for the symmetry ϕ of the function F which is invariant under conformal transformation (we mean that $f(\phi)$ is also a symmetry for arbitrary analytic f). We start with addition formula (3.6). Considering a symmetry $\delta F = e^{\Theta\phi}$, where Θ is an arbitrary parameter, we get a system of equations

$$\begin{cases} x + y + z = -1, \\ (\phi_{12} + \phi_{13})x + (\phi_{23} + \phi_{12})y + (\phi_{31} + \phi_{23})z = 0, \\ (\phi_1\phi_2 + \phi_1\phi_3)x + (\phi_2\phi_3 + \phi_1\phi_2)y + (\phi_3\phi_1 + \phi_2\phi_3)z = 0, \end{cases} \tag{4.5}$$

where $x = c_2c_3e^{-(F_{12}+F_{13})}$, $y = c_1c_3e^{-(F_{23}+F_{12})}$, $z = c_1c_2e^{-(F_{31}+F_{23})}$. The first line of system (4.5) (zero order in Θ) is addition formula (3.6), the second (first order in Θ) defines its symmetry, and the third (second order in Θ) follows from conformal invariance.

Using this system, we express x, y, z through ϕ, then find $e^{F_{23}}$, $e^{F_{13}}$ in terms of ϕ and get a compatibility condition

$$\mathcal{D}_1 \ln \frac{(f^1_{23} + f^2_{31} + f^3_{12})(f^1_{32} + f^2_{31} + f^3_{12})}{(f^1_{23} + f^2_{13} + f^3_{12})(f^1_{23} + f^2_{31} + f^3_{21})} =$$

$$\mathcal{D}_2 \ln \frac{(f^1_{23} + f^2_{31} + f^3_{12})(f^1_{23} + f^2_{13} + f^3_{12})}{(f^1_{23} + f^2_{31} + f^3_{21})(f^1_{32} + f^2_{31} + f^3_{12})}, \tag{4.6}$$

where $f^i_{jk} = \mathcal{D}_i \ln \frac{\phi_j}{\phi_k}$. Equation (4.6) is a general equation for conformally-invariant symmetry of the τ-function of dBKP hierarchy.

5 Symmetry constraints for dBKP hierarchy

Using the symmetry (2.13), we define a symmetry constraint

$$F_x = \sum_{i=1}^{N} c_i S_i,$$

or, in terms of potential u,

$$u = 2\sum_{i=1}^{N} c_i p_i, \quad p_i = p(z_i).$$

Evaluating first dBKP Hamilton-Jacobi equation

$$S_y = p^3 + 3up$$

where $y = t_3$, at z equal to z_i, we get a system of hydrodynamic type

$$\partial_y p_k = \partial_x \big(p_k^3 + 6p_k \sum_i c_i p_i\big). \tag{5.1}$$

Higher Hamilton-Jacobi equations will give higher systems of constrained dBKP hierarchy. The Sato function $z(p)$ for this hierarchy is constructed similar to constrained dKP case [24],

$$z = p - \sum_{i=1}^{N} c_i \ln \frac{p - p_i}{p + p_i}. \tag{5.2}$$

Its expansion at infinity is

$$z \to p + \sum_{n=0}^{\infty} v_{2n+1} p^{-(2n+1)}, \quad v_{2n+1} = \frac{2}{2n+1} \sum_{i=1}^{N} c_i p_i^{2n+1}. \tag{5.3}$$

From (2.5) we obtain a generating system for the constrained hierarchy,

$$\mathcal{D}(z) p_k = -\partial_x \ln \frac{p - p_k}{p + p_k}, \tag{5.4}$$

where p is a function of z, $(p_1, \ldots, p_N)$, defined by the relation (5.2). Expanding both sides of this system into the powers of z^{-1}, one gets the systems (5.1) and its higher counterparts. Expansion of $p(z)$ at infinity is given by the formula

$$p(z) = z + \sum_{n=0}^{\infty} \frac{1}{2n+1} \mathrm{res}_{p=\infty} \big(z(p)^{2n+1}\big) z^{-(2n+1)}.$$

In the same manner, it is possible to define constrained hierarchy using symmetry (2.12) (which can be considered as a special case of (2.13)) and (2.11). We will give the basic formulae for constrained hierarchy connected with (2.12) and obtain constrained hierarchy for the symmetry of the type (2.11) as a limit.

Using the symmetry (2.12), we define a symmetry constraint

$$F_x = \frac{1}{2} \sum_{i=1}^{N} c_i (S_i - \tilde{S}_i),$$

or, in terms of u,

$$u = \sum_{i=1}^{N} c_i (p_i - \tilde{p}_i).$$

The first hydrodynamic type system of constrained hierarchy is

$$\begin{cases} \partial_y p_k = \partial_x \big((p_k^3) + 3 p_k \sum_i c_i (p_i - \tilde{p}_i)\big) \\ \partial_y \tilde{p}_k = \partial_x \big((\tilde{p}_k^3) + 3 \tilde{p}_k \sum_i c_i (p_i - \tilde{p}_i)\big) \end{cases} \tag{5.5}$$

The Sato function for constrained hierarchy is given by

$$z = p - \tfrac{1}{2} \sum_{i=1}^{N} c_i \ln \frac{p - p_i}{p + p_i} \frac{p + \tilde{p}_i}{p - \tilde{p}_i},$$

The generating equation for the constrained hierarchy is

$$\mathcal{D}(z)p_k = -\tfrac{1}{2}\partial_x \ln \frac{p-p_k}{p+p_k}\frac{p+\tilde{p}_i}{p-\tilde{p}_i}.$$

Finally, we will consider symmetry constraint connected with the symmetry (2.13),

$$F_x = \sum_{i=1}^{N} c_i S_z(z_i), \quad u = \sum_{i=1}^{N} c_i \phi_i, \quad \phi_i = \partial_x S_z(z_i).$$

Though it is possible to consider this constrained hierarchy directly, we will obtain it as a limit of the previous case, when $p_i \to \tilde{p}_i$. Then from (5.5) we obtain a first system of constrained hierarchy,

$$\begin{cases} \partial_y p_k = \partial_x\big((p_k^3) + 3p_k \sum_i c_i\phi_i\big) \\ \partial_y \phi_k = \partial_x\big(3p_k^2\phi_k + 3\phi_k \sum_i c_i\phi_i\big) \end{cases} \tag{5.6}$$

The Sato function for the constrained hierarchy is

$$z = p + \sum_{i=1}^{N} c_i \frac{p\phi_i}{p^2 - p_i^2}.$$

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