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# Reductions of Integrable Lattices

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**Abstract**

We present a novel method for the reduction of integrable two-dimensional discrete systems to one-dimensional mappings. The procedure allows for the derivation of nonautonomous systems, which are typically discrete (difference or $q$) Painlevé equations, or of autonomous ones. In the latter case we produce the discrete analogue of an integrable subcase of the Hénon-Heiles system.

## 1 Introduction

The systematic construction of reductions of integrable evolution equations has contributed significantly to the understanding of these integrable systems. The solution of the reduced equation is, in principle, easier to obtain than that of the full partial differential equation. Thus the reductions lead to solutions of the initial system which can furnish useful insights into the behaviour of the general solution. Since one expects all the reductions of an integrable system to be also integrable, Ablowitz, Ramani and Segur [1] have formulated the conjecture linking the integrability of a partial differential equation to the Painlevé property [2] of its reductions. From a different perspective, the fact that many similarity reductions of partial differential equations are just Painlevé equations furnished a method for the integration of the latter [3] by adapting the spectral methods used for the solution of the former.

The similarity reductions of two-dimensional evolution equations are easy to understand. One seeks self-similar solutions by introducing a single independent variable which combines the two variables of the initial equation in an adequate way (the general method for the construction of such solutions relies on the study of the symmetries of the evolution equation). The standard example of a similarity reduction leading to a Painlevé equation is that of the modified-KdV equation $u_t - 6u^2u_x + u_{xxx} = 0$. Introducing the appropriate similarity variable $z = x(3t)^{-1/3}$ and the ansatz $u = w(z)(3t)^{-1/3}$ one finds, after one integration, the equation $w'' = 2w^3 + zw + c$ which is precisely $\mathrm{P_{II}}$.

With the discovery of integrable discrete equations, and in particular integrable multi-dimensional systems, the question naturally arose as to the discrete equivalent of similarity reductions. In particular one important question was whether discrete Painlevé equations can be obtained through some "similarity reduction" of integrable difference equations in two dimensions, i.e. integrable lattices. Clearly an approach directly analogous to that of the continuous systems was out of question since the latter was based on manipulations of the independent variable that have no discrete analogue. The question has already been answered in a certain way. In [4] Nijhoff and Papageorgiou obtained a "similarity constraint" to a given integrable lattice in the form of a nonautonomous nonlinear discrete equation. The reduction to a discrete Painlevé equation was realised by a semicontinuous limit applied to both equations, the lattice and the constraint, with elimination of derivatives with respect to the now-continuous variable. In [5] three of the present authors have implemented a bilinear formulation of the problem, obtaining a system of an autonomous and a nonautonomous bilinear lattice equation and proceeding through semicontinuous limits. A purely discrete approach was introduced in the works of Nijhoff and collaborators [6], [7], [8], where the reduction was obtained from the equation and its constraint by introducing an appropriate variable and choosing the proper path in the two-dimensional lattice.

In this article we present a novel approach to the question of reduction of integrable lattices. This approach allows us to treat the question of stationary and "similarity" reductions of integrable lattices in the same way. At this point it must be made clear that the term "similarity" is used here purely as an analogy. What we really mean is that we present a way to construct discrete Painlevé equations starting from some integrable lattice. In what follows we illustrate our method on some well-known integrable discrete equations in two dimensions. In the case of similarity reductions we identify the resulting mappings as discrete difference or $q$-Painlevé equations. In the case of stationary reductions we study the stationary flow of the (discrete) Sawada-Kotera equation and obtain the discrete equivalent of one of the integrable cases of the Hénon-Heiles system.

## 2 Reductions of nonautonomous lattices: the method and some examples

As we explained in the introduction, the manipulations of the independent variable, leading to the similarity reductions in the continuous case, are proscribed for discrete equations. Our method for the reduction of integrable lattices starts from a given equation by seeking its nonautonomous integrable extensions. Usually the autonomous integrable equation has some parameter which is related to the lattice spacing. Making it nonautonomous means that we are writing the equation on a lattice where the spacing varies from point to point. The important requirement is that the nonautonomous equation be integrable. Integrability is ensured by the application of a discrete integrability criterion such as singularity confinement [9] or low-growth (algebraic entropy) [10]. To illustrate this point we can start from the discrete mKdV equation:

$$x_{n+1}^{m+1} = x_n^m \frac{x_{n+1}^m - q x_n^{m+1}}{q x_{n+1}^m - x_n^{m+1}} \tag{2.1}$$

and deautonomise it by letting $q$ depend on $n$ and $m$. The integrable deautonomisation was presented in [11], where we have shown that $q$ must satisfy the equation $q_{n+1}^{m+1} q_n^m = q_{n+1}^m q_n^{m+1}$, with solution $q_n^m = f(n)g(m)$ in which $f$ and $g$ are free functions. Next we restrict the evolution to one dimension by considering a periodic reduction of the (nonautonomous) lattice, i.e. by demanding that $x_{n+k}^m = x_n^{m+l}$, where $k$ and $l$ are integers (and the same equation for $q_n^m$). The first example we are going to present is precisely based on the mKdV. Since the reduction $x_n^{m+1} = x_{n+1}^m$ is trivial (linear), we proceed to $x_n^{m+1} = x_{n+2}^m$. Neglecting the (common) index $m$ we find the mapping

$$x_{n+3} = x_n \frac{x_{n+1} - q_n x_{n+2}}{q_n x_{n+1} - x_{n+2}}. \tag{2.2}$$

Introducing the variable $y_n = x_{n+2}/x_{n+1}$ we can rewrite (2.2) as

$$y_{n+1} y_{n-1} = \frac{1 - q_n y_n}{y_n(q_n - y_n)}, \tag{2.3}$$

where $q_n$ satisfies the equation $q_{n+3} q_n = q_{n+1} q_{n+2}$ the solution of which is $\log q_n = an + b + c(-1)^n$. Equation (2.3) with $c = 0$ is precisely a form of the $q$-$\mathrm{P_{II}}$ equation discovered in [12]. The full freedom of (2.3), with $c \neq 0$, was first identified in [13], where it was shown that the full equation is a $q$-discrete form of $\mathrm{P_{III}}$. The above reduction is not the only one to a $q$-Painlevé equation we can obtain from mKdV. Introducing $x_n^{m+1} = x_{n+3}^m$ we find

$$x_{n+4} = x_n \frac{x_{n+1} - q_n x_{n+3}}{q_n x_{n+1} - x_{n+3}} \tag{2.4}$$

and again we rewrite using $y_n = x_{n+2}/x_{n+1}$ to

$$y_{n+2} y_{n+1} y_n y_{n-1} = \frac{1 - q_n y_n y_{n+1}}{q_n - y_n y_{n+1}}. \tag{2.5}$$

The obvious substitution now is $w_n = y_n y_{n+1}$ leading to

$$w_{n+1} w_{n-1} = \frac{1 - q_n w_n}{q_n - w_n}, \tag{2.6}$$

where now $\log q_n = an + b + cj^n + dj^{2n}$ with $j^3 = 1$. Equation (2.6) in the symmetric case $c = d = 0$ is again a $q$-$\mathrm{P_{II}}$ identified in [12] and was shown in [13] to be a $q$-$\mathrm{P_V}$ in the generic case. What is really interesting in the case of mKdV is the fact that the reductions lead to $q$-discrete equations. Our second example is based on the KdV equation. Its nonautonomous extension can be recovered from the results of [14] and has the form

$$x_{n+1}^{m+1} - x_n^m = \frac{z_n^{m+1}}{x_n^{m+1}} - \frac{z_{n+1}^m}{x_{n+1}^m}. \tag{2.7}$$

Here $z$ satisfies the equation $z_{n+1}^{m+1} + z_n^m = z_{n+1}^m + z_n^{m+1}$ the solution of which is $z_n^m = f(n) + g(m)$ with $f$ and $g$ two arbitrary functions. Again the reduction $x_n^{m+1} = x_{n+1}^m$ is trivial (linear) and we proceed to $x_n^{m+1} = x_{n+2}^m$. We find the mapping

$$x_{n+3} - x_n = \frac{z_{n+2}}{x_{n+2}} - \frac{z_{n+1}}{x_{n+1}} \tag{2.8}$$

which can be easily shown to be the discrete derivative of the mapping

$$x_{n+1} + x_n + x_{n-1} = \frac{z_n}{x_n} + c. \tag{2.9}$$

The equation for $z$ is $z_{n+2} - z_{n+1} - z_n + z_{n-1} = 0$ with solution $z_n = \alpha n + \beta + \gamma(-1)^n$. In the symmetric, $\gamma = 0$, case the mapping (2.9) is just the discrete Painlevé I equation. The full asymmetry was shown in [15] to lead to a discrete form of $\mathrm{P_{II}}$. The reduction $x_n^{m+1} = x_{n+3}^m$ can also be shown to lead to a discrete Painlevé equation. We start from

$$x_{n+4} - x_n = \frac{z_{n+3}}{x_{n+3}} - \frac{z_{n+1}}{x_{n+1}}. \tag{2.10}$$

Again one integration is possible leading to

$$x_{n+3} + x_{n+2} + x_{n+1} + x_n = \frac{z_{n+2}}{x_{n+2}} + \frac{z_{n+1}}{x_{n+1}} + 2c. \tag{2.11}$$

Next we put $y = x_{n+2} + x_n - z_{n+1}/x_{n+1} - c$ and obtain for $y$ the equation $y_{n+1} + y_n = 0$ with solution $y = \theta(-1)^n$. The equation for $z$, $z_{n+3} - z_{n+2} - z_n + z_{n-1} = 0$, has the solution $z_n = \alpha n + \beta + \gamma j^n + \delta j^{2n}$ where $j$ is a cubic root of unity. The even-odd dependence introduced by $\theta$ can always be gauged away and thus we find finally the mapping

$$x_{n+1} + x_{n-1} = \frac{z_n}{x_n} + c \tag{2.11}$$

which, when $\gamma = \delta = 0$, is just another form of d-$\mathrm{P_I}$ [12]. In the case $\gamma\delta \neq 0$ this mapping was shown to be a discrete Painlevé IV equation [16]. We remind at this point that in [5] using the semicontinuous limit approach we were able to obtain another similarity reduction of the discrete KdV, namely a discrete $\mathrm{P_{34}}$ equation.

Another interesting example is that of the discrete sine-Gordon equation [17]. We have given its nonautonomous extension in [14]:

$$x_{n+1}^{m+1} x_n^m = \frac{1 + q_n x_{n+1}^m x_n^{m+1}}{q_n + x_{n+1}^m x_n^{m+1}} \tag{2.12}$$

with the same $q$ as in the mKdV case, i.e. $q_n^m = f(n)g(m)$. The first reduction of d-sG is obtained for $x_n^{m+1} = x_{n+1}^m$. We find readily the mapping

$$x_{n+1} x_{n-1} = \frac{1 + q_n x_n^2}{q_n + x_n^2} \tag{2.13}$$

with $q_{n+1} q_{n-1} = q_n^2$ the solution of which is $\log q_n = an + b$. This mapping is a special (nongeneric) case of the $q$-$\mathrm{P_{III}}$ equation. An equivalent form can be obtained [18] if we put $y_n = x_n^2$ which results to

$$y_{n+1} y_{n-1} = \left(\frac{1 + q_n y_n}{q_n + y_n}\right)^2. \tag{2.14}$$

The next reduction, $x_n^{m+1} = x_{n+2}^m$, also leads to a $q$ Painlevé equation. Starting from

$$x_{n+3}x_n = \frac{1 + q_n x_{n+2}x_{n+1}}{q_n + x_{n+1}x_{n+2}} \tag{2.15}$$

and putting $y_n = x_{n+1}x_{n+2}$ we find

$$y_{n+1}y_{n-1} = \frac{y_n(1 + q_n y_n)}{q_n + y_n}, \tag{2.16}$$

where $\log q_n = an + b + c(-1)^n$. The mapping (2.16) is equivalent to (2.3). It suffices to invert the $y$s of even (or odd) index. The final example we present here is of a different kind. While all the previous equations were S-integrable, in the Calogero terminology [19], i.e. integrable through spectral methods, here we examine the reductions of a C-integrable, i.e. linearisable, case, namely Burgers equation. The nonautonomous form introduced in [14] can be easily shown to be equivalent to

$$x_n^{m+1} = \sigma_n^m x_n^m \frac{1 + x_{n+1}^m}{1 + x_n^m}, \tag{2.17}$$

where $\sigma$ is a *free* funtion of $n$ and $m$. The reduction $x_n^{m+1} = x_{n+1}^m$ leads to a homographic mapping and thus we proceed to $x_n^{m+1} = x_{n+2}^m$. Putting $x = -y - 1$ we obtain readily the mapping

$$y_{n+1}y_{n-1} + y_{n-1} - \sigma_n y_n(y_{n-1} + 1) = 0, \tag{2.18}$$

where $\sigma_n$ is to be understood now as a free function of $n$ only. The mapping (2.18) was identified in [20] as a linearisable system, the linearisation being obtained by a Cole-Hopf transformation. Next we examine the reduction $x_n^{m+1} = x_{n+3}^m$ which leads to

$$x_{n+3} = \sigma_n x_n \frac{1 + x_{n+1}}{1 + x_n}. \tag{2.19}$$

Putting $\sigma_n = \rho_{n+2}/\rho_{n+1}$ we can integrate (2.19) to

$$x_{n+1}x_{n-1} = c\rho_n \frac{1 + x_{n-1}}{x_n}. \tag{2.20}$$

This mapping is a subcase of the generic projective second-order mapping identified in [20] and can thus be reduced to a linear system. It goes without saying that the reductions we presented above are just the first few of an infinite hierarchy of reductions all of which are integrable. They have been chosen because they are the ones which can be reduced to second-order mappings.

## 3 A reduction of the discrete Sawada-Kotera equation

In the previous section we have presented the general method for the construction of reductions of an integrable lattice, focusing on second-order nonautonomous systems, in particular Painlevé equations. In this section we use our approach in a different direction. Our starting point is the well-known result of Fordy [21] on the Hénon-Heiles system:

$$x'' = -\lambda x^2 - y^2 \tag{3.1a}$$

$$y'' = -2xy. \tag{3.1b}$$

Three subcase of this system are integrable [22], corresponding to $\lambda = 1, 6, 16$. Fordy has shown that these subcases of this system, when (3.1) is recast in the form of a fourth-order equation

$$x'''' - (8 + 2\lambda)xx'' - 2(\lambda - 1)x'^2 + \frac{20}{3}\lambda x^3 - 4E = 0 \tag{3.2}$$

(where $E$ is the total energy), are precisely the stationary flows of (respectively) the Sawada-Kotera, Lax5-KdV and Kaup-Kuperschmidt equations, once integrated. (Of course, the integration of these stationary flows introduces a constant which is not *a prori* related to the energy of the Hénon-Heiles system. This means simply that the general solution of the equations of the stationary flows is richer than that of the Hénon-Heiles system which is recovered only when the integration constant is precisely $4E$).

It is thus interesting to perform the analogue of the stationary reduction for the discrete equivalent of these PDEs and thus recover the discrete counterparts of the integrable Hénon-Heiles systems. The only difficulty in this enterprise is that the discrete analogues to the Lax5-KdV and Kaup-Kuperschmidt equations are not yet known (at least to the present authors). Thus in what follows we restrict ourselves to the analysis of the discrete Sawada-Kotera equation.

In [23] Hirota and Tsujimoto have presented the discrete Sawada-Kotera equation. Its nonlinear form is

$$\frac{u_n^{m+1}(1 + \alpha u_{n+1}^{m+1})}{1 + \beta u_n^{m+1} u_{n+1}^{m+1} u_{n+2}^{m+1}} = \frac{u_n^m(1 + \alpha u_{n-1}^m)}{1 + \beta u_n^m u_{n-1}^m u_{n-2}^m}. \tag{3.3}$$

The simplest reduction one can implement on (3.3) is $u_n^{m+1} = u_n^m$, which amounts to ignoring the $m$-dependence. Omitting the common index $m$ we can rewrite (3.3) as

$$\frac{u_n(1 + \alpha u_{n+1})}{1 + \beta u_n u_{n+1} u_{n+2}} = \frac{u_n(1 + \alpha u_{n-1})}{1 + \beta u_n u_{n-1} u_{n-2}}. \tag{3.4}$$

This is a fourth-order mapping which can be readily integrated. Putting

$$w_n = \frac{1 + \alpha u_n}{1 + \beta u_{n+1} u_n u_{n-1}} \tag{3.5}$$

we have from (3.4) $w_{n+1} - w_{n-1} = 0$ with solution $w_n = \gamma + \delta(-1)^n$. Neglecting the $(-1)^n$ dependence we obtain a mapping of the form $u_{n+1}u_{n-1} = a + b/u_n$ which is just a special case of the QRT mapping [24]. If we keep the $(-1)^n$ term, we must distinguish the evolution of the even and odd index terms, which leads to a mapping of an "asymmetric" QRT form. Still it turns out that one can obtain a single, "symmetric", QRT mapping even in this case, the form of which is $(u_{n+1}u_n - 1)(u_n u_{n-1} - 1) = au_n/(u_n + b)$.

The reduction $u_n^{m+1} = u_{n+1}^m$ is more interesting. We obtain

$$\frac{u_{n+1}(1 + \alpha u_{n+2})}{1 + \beta u_{n+1} u_{n+2} u_{n+3}} = \frac{u_n(1 + \alpha u_{n-1})}{1 + \beta u_n u_{n-1} u_{n-2}} \tag{3.6}$$

which is a fifth-order mapping. Its integration is again based on the use of the variable $w_n$ defined by (3.5). Using it in (3.6) we find $u_{n+1}w_{n+2} = u_n w_{n-1}$. It suffices now to introduce

$v_n = u_n w_{n+1} w_n w_{n-1}$ and we can show that $v$ satisfies the equation $v_{n+1} - v_n = 0$. Thus $v$ is constant and we obtain finally the fourth-order mapping:

$$u_n \frac{1+\alpha u_{n+1}}{1+\beta u_{n+2}u_{n+1}u_n} \frac{1+\alpha u_n}{1+\beta u_{n+1}u_n u_{n-1}} \frac{1+\alpha u_{n-1}}{1+\beta u_n u_{n-1}u_{n-2}} = c \tag{3.7}$$

We claim that this mapping is the discrete analogue of (3.2) for $\lambda$=1, and is thus the discrete analogue of the case where the Hénon-Heiles system is integrable through separation of coordinates. Before proceeding to the continuous limit we can exhibit one more reduction which can be written as a fourth-order system, namely $u_n^{m+1} = u_{n+2}^m$. We obtain now the sixth-order equation

$$\frac{u_{n+2}(1+\alpha u_{n+3})}{1+\beta u_{n+2}u_{n+3}u_{n+4}} = \frac{u_n(1+\alpha u_{n-1})}{1+\beta u_n u_{n-1}u_{n-2}}. \tag{3.8}$$

Using the auxiliary quantity $w$ we find, proceeding exactly as in the case of the previous reduction, that $u_n u_{n+1} w_{n-1} w_n w_{n+1} w_{n+2} = c^2$ where $c$ is a constant. Moreover in this case one more integration is possible. Putting $z_n = u_n w_{n+1} w_{n-1}$ we rewrite the previous equality as $z_n z_{n+1} = c^2$ and, neglecting a possible even-odd degree of freedom, we have $z_n = c$. The final result is a fourth-order mapping

$$u_n \frac{1+\alpha u_{n+1}}{1+\beta u_{n+2}u_{n+1}u_n} \frac{1+\alpha u_{n-1}}{1+\beta u_n u_{n-1}u_{n-2}} = c \tag{3.9}$$

which, we claim, is also a discrete form of the stationary flow (3.2) for $\lambda$=1.

In order to prove the relation of (3.7) and (3.9) to (3.2) it suffices to compute the continuous limit of these mappings. We start with (3.7) and, in order to simplify the calculations, we introduce $c = \gamma^3$. Moreover we put $u = 1-\epsilon^2 x$, $\alpha = -2/5 - 2\epsilon^2\phi/5 - \epsilon^4(\rho - \phi^2)/15$, $\beta = -1/10 - 3\epsilon^2\phi/10 - \epsilon^4(\sigma+\rho)/20$ and $\gamma = 2/3 - 2\epsilon^2\phi/9 + \epsilon^4(\sigma-\rho)/27$. We find thus at the limit $\epsilon \to 0$ the equation $x'''' - 10xx'' + \frac{20}{3}\lambda x^3 - 20\phi x'^2 + 10\phi x'' + 5(\sigma - \phi^2)x + \phi\rho = 0$. This is precisely equation (3.2) for $\lambda = 1$. The extra terms present are related to the fact that the Hénon-Heiles potential can be augmented by the addition of quadratic and linear terms and still retain its integrability (provided these terms are chosen in an adequate way) [22]. The canonical form of the Hénon-Heiles Hamiltonian is obtained after a translation of the dependent variable which makes the linear terms vanish, but it is equally possible to introduce a different translation and put the quadratic term to zero. The same transformation can be performed on the Sawada-Kotera equation above, eliminating the $x''$ and $x'^2$ terms whereupon a choice of $\sigma$ leads to the canonical form (3.2). Still it is interesting to remark that the discrete equation (3.7) corresponds to the most general, $\lambda = 1$, Hénon-Heiles case. We turn now to the mapping (3.9). We introduce $c = \gamma^2$ and put $u = 1 - 4\epsilon^2 x/3$, $\alpha = -5/9 - 10\epsilon^2\phi/27 - \epsilon^4(5\rho + \sigma - 2\phi^2)/27$, $\beta = -1/5 - 2\epsilon^2\phi/5 - \epsilon^4(\rho+\sigma)/5$ and $\gamma = 5/9 - 5\epsilon^2\phi/27 + 5\epsilon^4(\sigma - \rho)/54$. We find thus at the limit $\epsilon \to 0$ the equation $x'''' - 10xx'' + \frac{20}{3}\lambda x^3 - 10\phi x'^2 + 5\phi x'' + 2(3\sigma - \phi^2)x + \phi\rho = 0$. Again we find, as expected, the stationary (integrated) Sawada-Kotera equation.

# 4 Conclusions

In this paper we have presented a novel method to obtain reductions of integrable lattices. The method is based on the existing integrable deautonomizations of partial difference equations. We have shown that following our reduction procedure it is possible to obtain mappings which are discrete Painlevé equations. A remarkable result is that some of the latter are of multiplicative, $q$, type while all the previously known lattice similarity reductions led to difference equations (at this point we must mention the approach of Kajiwara, Noumi and Yamada [25] on the reductions of the $q$-KP hierarchy who also obtained $q$-discrete analogues of Painlevé equations). The examples presented here were limited to second-order systems. However, since all reductions of an integrable lattice are integrable in their own right, our approach furnishes a method for the derivation of higher order integrable systems, in particular discrete analogues of higher-order Painlevé equations (a domain which is far from having been thoroughly explored even in the continuous case). In the case of linearisable lattices the method introduced in this paper leads to linearisable mappings, which can in principle be obtained at every order.

One point that has to be investigated concerning these higher-order systems is whether they are independent of the ones obtained at lower order. We believe, in the light of the low-order results where we have shown for instance that two consecutive reductions lead to different Painlevé equations, that the higher order systems are not mere differential consequences of the lower ones, but stand in their own right. (This is not a rigourous statement of course and a more thorough analysis is needed for the study of the higher-order systems.)

The stationary reductions of the Sawada-Kotera equation and its relation to the Hénon-Heiles system offer another interesting perspective: that of the construction of the discrete analogue of integrable Hamiltonian systems. It goes without saying that this study can be undertaken, following our method, only after one has solved a highly nontrivial problem, that of the construction of discrete analogues of integrable evolution equations of higher order.

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