Journal of Nonlinear Mathematical Physics Volume 12, Supplement 1 (2005), 372–379 **Birthday Issue**

Link Invariants and Lie Superalgebras

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This article is part of the special issue published in honor of Francesco Calogero on the occasion of his 70th birthday

Abstract

Berger and Stassen reviewed skein relations for link invariants coming from the simple Lie algebras $\mathfrak{g}$. They related the invariants with decomposition of the tensor square of the $\mathfrak{g}$-module V of minimal dimension into irreducible components. (If $V \not\simeq V^*$, one should also consider the decompositions of $V \otimes V^*$ and $V^* \otimes V^*$.) Here we consider decompositions into irreducible components for $\mathfrak{g}$-modules V of minimal dimension over some simple and close to simple Lie superalgebras $\mathfrak{g}$. For the classical series ($\mathfrak{gl}$, $\mathfrak{sl}$, $\mathfrak{osp}$), as well as for the Poisson and Hamiltonian algebras — "quasi-classical" analogs of $\mathfrak{gl}$ and $\mathfrak{sl}$ — the answer is rather complicated due to the lack of complete reduciblity. Contrariwise, the case of exceptional Lie superalgebras $\mathfrak{g} = \mathfrak{ag}_2$ and $\mathfrak{ab}_3$ turned out to be similar to that of Lie algebras: The $\mathfrak{g}$-module $\mathfrak{g} \otimes \mathfrak{g}$ (here the representation of minimal dimension is the adjoint one) is completely reducible and, remarkably, the spectra of highest weights for $\mathfrak{ag}_2$ are almost identical (in certain coordinates) to that for $\mathfrak{ab}_3$! We also consider $\mathfrak{g} = \mathfrak{osp}_\alpha(4|2)$ for $\alpha \neq 0, 1$.

Introduction

We began this work from curiosity: To see if we can advance further than in [FKV]. And then we read Vassiliev in [V]: "Recently P. Vogel proved that our [Vassiliev's] invariants arising from Lie superalgebras are strictly stronger than the invariants arising from quantum groups." So we hope the experts in knots/links will be interested in our results.

In what follows the ground field is $\mathbb{C}$, algebras and modules are of finite dimension; a Lie superalgebra is said to be "classical" if it is either simple or the result of the following iterated procedures applied to a simple Lie superalgebra $\mathfrak{g}$: Taking either a deformation, or a central extension, or a (containing $\mathfrak{g}$) subalgebra of the algebra of derivations of $\mathfrak{g}$.

For preliminaries on skein relations, see [BS] and [FKV] as well as [CP]; [Se] is a latest review of irreducible finite dimensional representations of simple (and close to them "classical") Lie superalgebras $\mathfrak{g}(A)$ with Cartan matrix A; for presentations of these $\mathfrak{g}(A)$, see [GL3]; for a review of representation of the Lie superalgebras without Cartan matrix, see [GLS]; for a review of indecomposable representations of simple (and close to them "classical") Lie superalgebras, see [L2] and also [G1], [G2].

§1 Setting of the problem

Not being experts in the field, we know only of two (perhaps, there are more) ways to construct link invariants from (quantum) Lie algebras.

One, described in sec. 15.2 of [CP], requires an *enhanced Yang–Baxter operator*; such operators are not completely described even for Lie algebras and at the time we wrote this were known (as far as we could see) for the Lie algebras of classical series and their identity representations only. Observe that these operators may be considered as quantizations of solutions to the classical YB equation, and therefore a solution to the quantum YBE. There are, however, other solutions of QYBE, even in the cases where CYBE has no solutions, cf. [O], [LS].

To describe such invariants for the nontrivial solutions of QYBE (e.g., the ones that have no classical analogs, see [O], [LS]), and for Lie algebras of matrices of complex size and their analogs [GL1] are tempting problems; we hope to consider these problems elsewhere. As far as we know, Nazarov [N] was the only one who advanced a bit along this scenic but unpaved road in search of link invariants.

Another approach is given in [BS]. We follow it; actually, we only consider the related linear algebra. To be able to follow the lines of [BS], we confine ourselves to

$$\text{Lie superalgebras possessing a nondegenerate even invariant bilinear form } B. \qquad (*)$$

We further assume our Lie superalgebras to be "classical", although no restriction excludes semisimple or even more general Lie superalgebras.

For any ("classical") Lie superalgebra $\mathfrak{g}$ satisfying $(*)$, we need to select a set of its representations $\mathcal{S} = \mathcal{S}(\mathfrak{g})$ containing the trivial $\mathfrak{g}$-module $\mathbf{1}$ and such that the family of $\mathfrak{g}$-invariant linear maps

$$\{p_Z^{X\otimes Y} : X \otimes Y \longrightarrow Z \mid X \otimes Y \in \mathcal{S} \setminus \{\mathbf{1}\};\ X, Y, Z \in \mathcal{S}\}$$

have the following properties:

(i) $p_Z^{X\otimes Y} \neq 0$ whenever $\dim \operatorname{Hom}_{\mathfrak{g}}(X \otimes Y, Z) \neq 0$ and $p_{\mathbf{1}}^{\mathrm{ad}\otimes\mathrm{ad}} = aB$ for a complex number a and the form B from $(*)$;

(ii) $p_{\mathbf{1}}^{X\otimes X^*}(x \otimes y) = \pm(-1)^{p(y)p(x)} p_{\mathbf{1}}^{X^*\otimes X}(y \otimes x)$ for any $x \in X$ and $y \in X^*$;

(iii) for any $X, Y, Z \in \mathcal{S} \setminus \{\mathbf{1}\}$, we have

$$p_{X^*}^{Y\otimes Z^*} = \left(\mathrm{id}_{X^*} \otimes p_{\mathbf{1}}^{Z\otimes Z^*}\right) \circ \left(\mathrm{id}_{X^*} \otimes p_Z^{X\otimes Y} \otimes \mathrm{id}_{Z^*}\right) \circ \left(i^{\mathbf{1}}_{X^*\otimes X} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*}\right),$$

where $i^X_{Y\otimes Z} = 0$ if $p_X^{Y\otimes Z} = 0$ and if $p_X^{Y\otimes Z} \neq 0$, then the nontrivial $\mathfrak{g}$-invariant linear map $i^X_{Y\otimes Z}$ is uniquely recovered from the relation

$$p_X^{Y\otimes Z} \circ i^X_{Y\otimes Z} = \mathrm{id}_X.$$

For example, if

$$\dim \operatorname{Hom}_{\mathfrak{g}}(X \otimes Y, Z) \text{ is either 1 or 0 for any } X, Y, Z \in \mathcal{S},$$

then $\mathcal{S}$ qualifies to be selected.

Berger and Stassen showed that, for all simple finite dimensional Lie algebras, one can take for $\mathcal{S}$ a subset of the set of irreducible submodules of the tensor products $V \otimes V$ (and also $V \otimes V^*$ and $V^* \otimes V^*$ if V does not possess a nondegenerate invariant bilinear form), where V is the $\mathfrak{g}$-module of the least dimension.

The passage to superalgebras was performed, so far, along the usual road: One considers first the simplest (most trivial) analogs of $\mathfrak{sl}(2)$ and then goes to higher ranks. However, the seemingly simplest super analogs of $\mathfrak{sl}(2)$, namely, $\mathfrak{sl}(1|1)$ or $\mathfrak{gl}(1|1)$ are not so simple in any sense. First, they are solvable. Second, usually, there is no complete reducibility even for finite dimensional representations of simple Lie superalgebras. Figueroa-O'Farrill, Kimura and Vaintrob [FKV] showed that, to consider $\mathfrak{g} = \mathfrak{gl}(1|1)$, one *has* to compose $\mathcal{S}$ of indecomposable, rather than of irreducible, representations. The indecomposable representations of simple Lie superalgebras, even finite dimensional ones, can be quite complicated or even wild ([L2], [G1], [G2]).

To follow Berger and Stassen, we only have to consider the decomposition of the tensor square $V \otimes V$ of a $\mathfrak{g}$-module V (usually, an irreducible of least dimension); although there is no reason to expect complete reducibility, this is always a tame problem. For the majority of series of simple Lie superalgebras, even to decompose this square $V \otimes V$ into indecomposables is rather complicated and to analyze all the cases à la FKV is impossible without computer's aid.

We show therefore, that undertaking the straightforward superization one will inevitably get stuck with troubles much more serious than already considerable ones encountered by FKV. So our initial intention was to consider Lie (super)algebras which are in many aspects similar to simple finite dimensional Lie algebras, more precisely, to $\mathfrak{gl}(n)$, but which, so far, escaped the limelight. Namely, it is highly tempting to consider $\mathfrak{gl}(\lambda)$, the Lie algebra of complex ($\lambda \in \mathbb{C}$) size matrices and its generalizations ([GL1]). But $\mathfrak{gl}(\lambda)$ is of infinite dimension, so we leave it for a while and consider various "relatives" of the Poisson Lie superalgebra $\mathfrak{po}(0|2n)$, whose deformation is the conventional $\mathfrak{gl}(2^{n-1}|2^{n-1})$.

The algebraic part of the problem solved by Berger and Stassen is to consider a representation V of $\mathfrak{g}$, decompose $V \otimes V$, $V^* \otimes V^*$, and $V \otimes V^*$ into irreducible (in our setting: indecomposable) pieces and select among these pieces a subset $\mathcal{S}$ satisfying the above conditions (i)–(iii), if such a subset exists.

Here we perform the first step in solving this task and consider unconventional super analogs of $\mathfrak{gl}$: finite dimensional Poisson Lie superalgebras $\mathfrak{po}(0|2n)$ and their simple subquotients $\mathfrak{h}'(0|2n)$, the Lie superalgebras of Hamiltonian vector fields. The picture becomes truly bizarre as n grows!

Contrariwise, when, inspired by Berger and Stassen who considered the exceptional Lie algebras, we turned to the exceptional Lie superalgebras we got a nice and tidy answer resembling that for Lie algebras.

§2 Main result

For simple finite dimensional Lie algebras, Berger and Stassen constructed a selected sets of representations from the tensor square of the *minimal representation* (i.e., the irreducible representations of least dimension), see [BS].

As shown in [FKV], for $\mathfrak{gl}(1|1)$, in order to construct a selected set, one has to add to the minimal representation certain indecomposable representations.

We conjecture that similar but more involved will be the case of $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$ and even more complicated for $\mathfrak{psl}(n|n)$. We did not test which of the indecomposable representations should we take for V; we only considered the minimal irreducible representations.

Hereafter the irreducible representations of $\mathfrak{g}$ with a specified (say, by selecting a Cartan matrix, cf. [GL3]) decomposition of its roots into positive and negative ones is given in terms of the coordinates of the highest weight with respect to the basis of Cartan subalgebra corresponding to the Cartan matrix of $\mathfrak{g}$, or its 0th part with respect to the specified $\mathbb{Z}$-grading in case of the absence of Cartan matrix. The highest weight vector of the representation $R(\varphi)$ with highest weight φ is supposed to be even, otherwise the representation is denoted $\Pi(R(\varphi))$.

For simple Lie superalgebras (and their relatives), the minimal irreducible representations are as follows (as one can deduce from [K] and [Se]):

—for matrix Lie superalgebras $\mathfrak{gl}(m|n)$ and its subalgebras $\mathfrak{sl}$ and $\mathfrak{osp}$, this is the identity $(m|n)$-dimensional module;

—for the exceptional Lie superalgebras $\mathfrak{ag}_2$, $\mathfrak{ab}_3$, and $\mathfrak{osp}_\alpha(4|2)$ at the generic value of parameter α, as well as for $\mathfrak{psl}(n|n)$ and $\mathfrak{sh}(0|2n)$, the minimal module is the adjoint module;

—for $\mathfrak{po}(0|2n)$, this is the subquotient representation in $\mathfrak{h}'(0|2n)$;

—for $\mathfrak{g} = \mathfrak{osp}_2(4|2)$ and $\mathfrak{osp}_3(4|2)$ considered in realization with Cartan matrices

$$\begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix},$$

respectively, these are representations $V = R(0,0,2)$ and $W = R(2,3,0)$, respectively, where the coordinates of the highest weight are given with respect to the Chevalley generators $H_i = [X_i^+, X_i^-]$ corresponding to the above Cartan matrices.

It was natural to expect that decompositions into the direct sum of indecomposable $\mathfrak{g}$-modules will be of simpler structure for $\mathfrak{gl}$ and $\mathfrak{osp}$ series. The reality is quite the opposite:

$\underline{\mathfrak{gl}(1|1)}$. Even this case is pretty complicated, see [FKV].

$\underline{\mathfrak{psl}(2|2) \simeq \mathfrak{h}'(0|4)}$. It turns out that the symmetric square is completely reducible, the highest (H) and the lowest (L) weights and the dimensions (D) are as follows (the weight is given with respect to the natural Cartan generators of the $\mathfrak{sl}(2)$-summands of $\mathfrak{psl}(2|2)_{\bar{0}}$ and the grading operator):

(H)	$(2,1,-1)$	$(2,1,1)$	$(0,0,0)$
(L)	$(-2,-1,-1)$	$(-2,-1,-1)$	$(0,0,0)$
(D)	$24\vert 24$	$24\vert 24$	$1\vert 0$

The exterior square is indecomposable, the highest and the lowest weights and the dimensions of the irreducible components of the Jordan-Hölder series are in the following table, they are composed from irreducible pieces over the Lie algebra $\mathfrak{psl}(2|2)_{\bar{0}}$. Here the first module, $(6|8)$, is a submodule; the next five modules are glued to it, each separately;

the last module (6|8) is glued to the sum of all the modules listed above it:

dim	lowest	highest
(6|8)	$(-1,-1,0)$	$(1,1,0)$
(1|0)	$(-2,0,0)$	$(-2,0,0)$
(1|0)	$(0,0,0)$	$(0,0,0)$
(1|0)	$(2,0,0)$	$(2,0,0)$
(18|16)	$(0,-2,0)$	$(2,2,0)$
(18|16)	$(-2,-2,0)$	$(0,2,0)$
(6|8)	$(-1,-1,0)$	$(1,1,0)$

After such horrors, here are small miracles we observed: As is easy to verify,

$$\mathfrak{ag}_2 = \mathfrak{osp}_3(4|2) \oplus \Pi(W); \qquad \mathfrak{ab}_3 = \mathfrak{osp}_2(4|2) \oplus \mathfrak{sl}(2) \oplus V \oplus V^*,$$

where $V \cong V^*$ and $W \cong W^*$ for the minimal $\mathfrak{osp}_2(4|2)$-module $V = R(0,0,2)$ and minimal $\mathfrak{osp}_3(4|2)$-module $W = R(2,3,0)$; Π is the shift of parity functor. As always, we assume that the highest weight vector of the module M is even, unless specified, as above.

$\underline{\mathfrak{osp}_2(4|2)}$ The decompositions of the tensor square of the minimal representation of $\mathfrak{osp}_2(4|2)$ (the symmetric and exterior squares are united in parentheses with a subscript S or Λ, respectively) are **completely reducible** and as follows:

$$V \otimes V \cong (R(0,0,-4) \oplus R(-2,-1,3))_S \oplus (R(-1,-2,-4) \oplus R(0,0,0))_\Lambda$$

The dimensions are:

$$\dim R(0,0,-4) = 18|6; \quad \dim R(-2,-1,3) = 9|8;$$
$$\dim R(-1,-2,-4) = 24|24; \quad \dim R(0,0,0) = 1|0.$$

$\underline{\mathfrak{osp}_3(4|2)}$ The decompositions of the tensor square of the minimal representation of $\mathfrak{osp}_3(4|2)$ (the symmetric and exterior squares are united in parentheses with a subscript S or Λ, respectively) are **completely reducible** and as follows:

$$\Pi(W) \otimes \Pi(W) \cong (R(2,5,0) \oplus R(5,6,1) \oplus R(0,0,0))_S \oplus (R(4,6,0) \oplus R(0,4,0))_\Lambda$$

The dimensions are:

$$\dim R(2,5,0) = 24|24; \quad \dim R(5,6,1) = 26|24; \quad \dim R(0,0,0) = 1|0;$$
$$\dim R(4,6,0) = 40|40; \quad \dim R(0,4,0) = 9|8.$$

$\underline{\mathfrak{ag}_2 \text{ and } \mathfrak{ab}_3}$ We consider $\mathfrak{ag}_2$ and $\mathfrak{ab}_3$ in realization with Cartan matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

respectively; the even parts of these algebras are $\mathfrak{sl}(2)\oplus\mathfrak{g}_2$ and $\mathfrak{sl}(2)\oplus\mathfrak{o}(7)$, respectively. The decompositions of the tensor square of the minimal representation of $\mathfrak{ag}_2$ and $\mathfrak{ab}_3$ (in both cases the minimal representation is the adjoint one; the symmetric and the exterior squares are united in parentheses with a subscript S or Λ, respectively) are as follows.

Let $X_i^{\pm}$ and H_i be the Chevalley generators corresponding to the above Cartan matrix; for the basis of Cartan sublagebra of $\mathfrak{sl}(2)$ we take

$$H_0=\begin{cases}\frac{1}{2}H_1-H_2-\frac{3}{2}H_3 & \text{for } \mathfrak{ag}_2\\ 2H_1+H_2+3H_3+4H_4 & \text{for } \mathfrak{ab}_3.\end{cases}$$

Then the tensor squares of the minimal modules over these algebras are **completely reducible** and as follows:

$$\mathfrak{ag}_2:\quad \mathrm{ad}\otimes\mathrm{ad}\cong(R(8,0,0)\oplus R(7,0,1)\oplus R(0,0,0))_S\oplus(R(8,1,0)\oplus R(4,0,0))_\Lambda$$

The dimensions are (observe that $R(4,0,0)=\mathrm{ad}$):

$$\dim R(8,0,0)=96|96;\quad \dim R(7,0,1)=147|142;\quad \dim R(0,0,0)=1|0;$$
$$\dim R(8,1,0)=224|224;\quad \dim R(4,0,0)=17|14.$$

$$\mathfrak{ab}_3:\quad \mathrm{ad}\otimes\mathrm{ad}\cong(R(6,0,0,0)\oplus R(5,0,1,0)\oplus R(0,0,0,0))_S\oplus(R(6,1,0,0)\oplus R(3,0,0,0))_\Lambda$$

The dimensions are (observe that $R(3,0,0,0)=\mathrm{ad}$):

$$\dim R(6,0,0,0)=152|144;\quad \dim R(5,0,1,0)=147|142;\quad \dim R(0,0,0,0)=1|0;$$
$$\dim R(6,1,0,0)=224|224;\quad \dim R(3,0,0,0)=26|14.$$

Here we encounter a remarkable coincidence obscured by our choice of coordinates for the weights. If, instead of the Chevalley generators H_i for $\mathfrak{g}$, we take the Chevalley generators h_i of the even part $\mathfrak{g}_{\bar 0}$, the weights become

$$(R(4,0,0)\oplus R(2,0,1)\oplus R(0,0,0))_S\oplus(R(3,1,0)\oplus R(2,0,0))_\Lambda\ \text{ for } \mathfrak{ag}_2$$

and

$$(R(4,0,0,0)\oplus R(2,0,1,0)\oplus R(0,0,0,0))_S\oplus(R(3,1,0,0)\oplus R(2,0,0,0))_\Lambda\ \text{ for } \mathfrak{ab}_3.$$

In other words, the coordinates of the weights for $\mathfrak{ab}_3$ are **the same** as for $\mathfrak{ag}_2$, bar the last coordinate 0. (This is a miracle.)

$\mathfrak{osp}_\alpha(4|2)$ for the generic value of parameter α. We consider $\mathfrak{osp}_\alpha(4|2)$ in the incarnation with Cartan matrix

$$\begin{pmatrix}2&-1&0\\1&0&-\alpha\\0&-1&2\end{pmatrix}$$

To make the coordinates of the highest weight look more symmetric, we consider them with respect to the first and third copies of $\mathfrak{sl}(2)$, corresponding to the 2's on the diagonal of the Cartan matrix, as expected, i.e., with respect to H_1 and H_3, whereas instead of H_2 we take

$$-H_2 = -2[X_2^+, X_2^-] + [X_1^+, X_1^-] + [X_3^+, X_3^-].$$

The symmetric square of ad is completely reducible, namely, apart from the Casimir element, $R(0,0,0)$, it contains 3 identical, up to permutation of the weights, modules of dimension 24|24 each. Their weights (in the above coordinates) are as follows:

$$\begin{array}{lllllll} (A) & (4,0,0) & (3,1,1) & (2,2,0) & (2,0,2) & (1,1,1) & (0,0,0) \\ (B) & (0,4,0) & (1,3,1) & (2,2,0) & (0,2,2) & (1,1,1) & (0,0,0) \\ (C) & (0,0,4) & (1,1,3,) & (2,0,2) & (0,2,2) & (1,1,1) & (0,0,0) \end{array}$$

The exterior square is indecomposable of dimension 72|72. Its content as $\mathfrak{g}_0$ module:

$$\begin{array}{l} (3,1,1),\ (1,3,1),\ (1,1,3,),\ (2,2,2),\ (2,2,0),\ (2,0,2),\ (0,2,2),\ (1,1,1) \\ (2,0,0),\ (0,2,0),\ (0,0,2),\ (1,1,1) \end{array}$$

The second line is an irreducible submodule (isomorphic to ad); the quotient module is indecomposable; its dimension is equal to 54|56. The quotient of the exterior square modulo the maximal submodule is an irreducible module (isomorphic to ad) of dimension $(72|72)/(63|64) = 9|8$.

The above results are examples of what can be performed with the aide of *Mathematica*-based package **SuperLie**. For the documentation, see [SLie]; for other results (to be) obtained with its aide, see [GL4].

Acknowledgements

Financial support of NFR and MPIMiS and prompts by O. Viro and A. Vaintrob are gratefully acknowledged.

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