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Solitons of Wave Equation

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Abstract

Modulated progressive wave solutions (solitons) to $(3+1)$–dimensional wave equation are discussed within a general geometrical framework. The role of geodesic coordinates defined by hypersurfaces of Riemannian spaces is pointed out in this context. In particular in E^3 orthogonal geodesic coordinates defined by Dupin cyclides are used to simplify derivation of the most nontrivial results of Friedlander on solitons of $(3+1)$–dimensional wave equations and to correct some of them. The essence of this novel approach is use of the technique of separation of variables in the Kalnins–Miller formulation.

1 Problem

Consider $(n+1)$–dimensional wave equation

$$(\Delta_n - \partial_t^2)\mathcal{F} = 0\,, \tag{1.1}$$

where $\Delta_n = \sum_{i=1}^n \frac{\partial^2}{\partial {x^i}^2}$ is the n–dimensional Euclidean Laplacian and x^i are standard Cartesian coordinates in E^n.

Hyperspherical coordinates $(r, \vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1})$ in Euclidean n–space E^n are defined by

$$x^1 = r\cos\vartheta_1, \quad x^2 = r\cos\vartheta_2\sin\vartheta_1, \quad x^3 = r\cos\vartheta_3\sin\vartheta_1\sin\vartheta_2, \quad \ldots,$$
$$x^n = r\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{n-1}\,.$$

In hyperspherical coordinates eq. (1.1) is

$$\Big\{\partial_r^2 + (n-1)r^{-1}\partial_r + \ldots + r^{-2}\frac{1}{\prod_{j=1}^{i-1}\sin^2\vartheta_j}\Big[\frac{\partial^2}{\partial\vartheta_i^2} + (n-1-i)\cot\vartheta_i\frac{\partial}{\partial\vartheta_i}\Big] + \ldots$$
$$+r^{-2}\frac{1}{\prod_{j=1}^{n-2}\sin^2\vartheta_j}\frac{\partial^2}{\partial\vartheta_{n-1}^2}\Big\}\mathcal{F} - \frac{\partial^2\mathcal{F}}{\partial t^2} = 0\,. \tag{1.2}$$

When $\mathcal{F} = \mathcal{F}(r,t)$ ("spherically" symmetric wave), eq. (1.2) is reduced to

$$\Big[\partial_r^2 + (n-1)r^{-1}\partial_r - \partial_t^2\Big]\mathcal{F} = 0\,. \tag{1.3}$$

In order to eliminate the linear term in (1.3) we apply the standard procedure $\mathcal{F} \mapsto \Phi$, where

$$\mathcal{F} = \Phi \exp\big(-\frac{1}{2}\int \frac{n-1}{r}dr\big) = \Phi r^{\frac{1-n}{2}}$$

and Φ solves the equation

$$\Big[\partial_r^2 - \frac{1}{4}r^{-2}(n-1)(n-3) - \partial_t^2\Big]\Phi = 0\,. \tag{1.4}$$

Certainly eq. (1.4) selects $n = 1$ and $n = 3$. In the three–dimensional case we have the standard spherical wave

$$\mathcal{F}(r,t) = \frac{1}{r}G(r \mp t)\,, \tag{1.5}$$

where G is an arbitrary function of $C^2(R)$ class.

Inspired by (1.4) and (1.5) one can formulate the following two problems.

a) What dimensions $n > 1$ admit solutions to eq. (1.1) of the form

$$\mathcal{F}(x^i,t) = F(x^i)G\Big[w(x^i) \mp t\Big]\,, \tag{1.6}$$

where G denotes an *arbitrary* $C^2(R)$–function while F and w are *fixed* C^2–functions?

b) For n being a positive answer to a) find a general solution (1.6).

In a) we discard plane waves. For obvious reasons in general a solution (1.6) is called *soliton of the wave equation.*

Presumably H. Schmidt was the first to pose the problem b) for $n = 3$ which is a positive answer to a) [1]. He was unable, however, to solve this problem completely. The same problem was undertaken independently and solved almost completely by F.G. Friedlander [2]. His 10–page proof contains rather tedious calculations and some mistaken final formulae.

The aim of this paper is to simplify derivation of the most nontrivial results of [2] in correct form. The approach is based on the technique of separation of variables [3, 4, 5, 6]. This paper is an extended version of [7]. The final considerations are preceded by some geometrical preliminaries.

The answer to a) seems to be unknown. Our hypothesis is: only $n = 3$ admits (1.6). E.g. one can show *nonexistence* of spherically symmetric solitons for $n \neq 1, 3$. Indeed we assume

$$\mathcal{F} = \mathcal{F}(r,t) = F(r)G(r \mp t) \tag{1.7}$$

and insert (1.7) into (1.3). Since G is by assumption arbitrary, one obtains

$$F''(r) + \frac{n-1}{r}F'(r) = 0\,, \tag{1.8}$$

and

$$F'(r) + \frac{n-1}{2r}F(r) = 0\,. \tag{1.9}$$

If one excludes the case $n = 1$ ($F = const$), (1.8) and (1.9) are compatible iff $n = 3$ [1].

2 "Nonlinearization" of the problem

Again the functional independence of G, G' and G'' implies (1.6) is a solution to (1.1) iff

$$\big(\nabla_n w\big)^2 = \Big(\frac{\partial w}{\partial x^1}\Big)^2 + \Big(\frac{\partial w}{\partial x^2}\Big)^2 + \ldots + \Big(\frac{\partial w}{\partial x^n}\Big)^2 = 1\,, \tag{2.1}$$

$$2\nabla_n F \cdot \nabla_n w + F\Delta_n w = 2\Big(\frac{\partial F}{\partial x^1}\frac{\partial w}{\partial x^1} + \ldots + \frac{\partial F}{\partial x^n}\frac{\partial w}{\partial x^n}\Big) + F\Delta_n w = 0\,, \tag{2.2}$$

$$\Delta_n F = 0\,. \tag{2.3}$$

(2.1)–(2.3) is an overdetermined *nonlinear* system for the unknowns F and w.

Certainly (2.1)–(2.3) can be easily rewritten in *any curvilinear coordinates* $(x^1, x^2, \ldots, x^n)$ in E^n as follows

$$g^{ik}\frac{\partial w}{\partial x^i}\frac{\partial w}{\partial x^k} = 1\,, \tag{2.4}$$

$$2g^{ik}\frac{\partial F}{\partial x^i}\frac{\partial w}{\partial x^k} + F\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\Big(\sqrt{g}g^{ik}\frac{\partial w}{\partial x^k}\Big) = 0\,, \tag{2.5}$$

$$\frac{\partial}{\partial x^i}\Big(\sqrt{g}g^{ik}\frac{\partial F}{\partial x^k}\Big) = 0\,. \tag{2.6}$$

Obviously $g_{ik}(x^j)$ ($g^{ik}(x^j)$) are components of covariant (contravariant) metric tensor of E^n and $g = \det[g_{ik}]$.

Eq. (2.1) for $n = 3$ is nothing else but the basic equation of geometrical optics, i.e. the branch of optics which is defined by asymptotically vanishing wavelengths, called the *eikonal equation* (from Greek $\epsilon\iota\kappa\omega\nu$ — image, icon). To be more precise eq. (2.1) ($n = 3$) corresponds to the case of a homogeneous isotropic medium. For this reason (2.1) or (2.4) (also in the case when E^n is replaced by *arbitrary* Riemannian space V^n) is called an *eikonal equation* as well.

One possible strategy to solve the system (2.4–2.6) is as follows. To treat the eikonal equation (even in Riemannian case) we have at our disposal an efficient geometrical method based upon the classical Beltrami theorem (1869) on geodesic and normal congruences in Riemannian spaces [8]. For details see section 3.

Next on inserting a selected solution w to (2.4) into (2.5) we convert it into a linear equation for F. Finally, we attempt to find F to be harmonic (satisfying (2.6)).

It is worthwhile to mention that the n–dimensional Laplace eq. (2.3) is also tractable (see e.g. the "Anhang" in the famous treatise [9] by Maxime Bôcher).

3 The method to treat the eikonal equation

The strategy described above does not seem to be very promising. Soon we see that sometimes it can work.

Given an n–dimensional Riemannian space V^n and some $(n-1)$–dimensional hypersurface $\Sigma \subset V^n$ equipped with local coordinates $(u^2, u^3, \ldots, u^n)$. At each point $P(u^2, u^3, \ldots, u^n)$ of Σ in the normal direction to Σ starts a uniquely defined geodesic γ of the ambient space. Let $Q \in \gamma$. The signed distance from P to Q measured along γ is denoted by u^1. In this way for sufficiently small values of u^1 we construct the so called *geodesic coordinates* $Q \mapsto (u^1, u^2, \ldots, u^n)$ in V^n. One can prove the following

Theorem 1.

a) The metric of V^n in geodesic coordinates assumes the form

$$ds^2 = (du^1)^2 + \sum_{i,k\geq 2} \overline{g}_{ik}(u^j)du^i du^k = \overline{g}_{ik}(u^j)du^i du^k \,. \tag{3.1}$$

b) Hypersurfaces $u^1 = const$ are orthogonal sections of an $(n-1)$ – parameter family of geodesics γ emanating orthogonally from points of Σ.

c) $w(u^j) = u^1$ trivially satisfies the eikonal equation (2.4).

All the content of this theorem can be found in [10]. Hypersurfaces $u^1 = const$ are called *geodesically parallel*. We change coordinates $u^i = u^i(x^j)$. Now the metric (3.1) is

$$ds^2 = g_{ik}(x^j)dx^i dx^k \,. \tag{3.2}$$

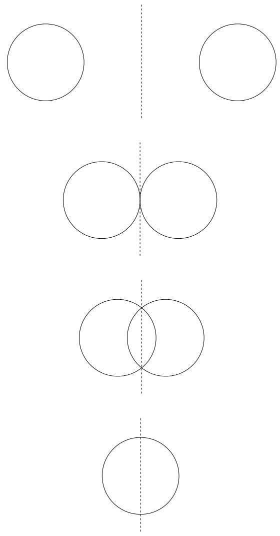
From (c) of the Theorem 1 we infer that $w(x^j) := u^1(x^j)$ satisfies the eikonal equation (2.4). In particular, when $V^n = E^n$ and x^i are Cartesian coordinates, $w(x^j)$ solves (2.1).

Example 1. $V^n = E^n$ equipped with the hyperspherical coordinates of section 1. Σ is a hypersphere $r = r_0 = const > 0$. The corresponding family of geodesics γ consists of all straight lines emanating from the center of the coordinate system. Now $u^1 = r - r_0, u^2 = \vartheta_1, \ldots, u^n = \vartheta_{n-1}$. On returning to Cartesian coordinates $(x^1, x^2, \ldots, x^n)$ we arrive at

$$w(x^j) = \Big[(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2\Big]^{\frac{1}{2}} - r_0 \tag{3.3}$$

as a solution to the eikonal equation (2.1).

Example 2. $V^3 = E^3$. The ordinary sphere (Σ of the previous example) can be treated as a degenerate case of a torus. Fig. 1 shows normal sections of the torus evolving to become eventually a sphere.



(3.4)

(3.5)

(3.6)

(3.7)

Fig. 1

Σ is a standard regular torus parametrically given by $x = (a + b_0 \cos\varphi)\cos\alpha$, $y = (a + b_0 \cos\varphi)\sin\alpha$, $z = b_0 \sin\varphi$, $(0 < b_0 < a)$, where $\varphi, \alpha \in [0, 2\pi)$, b_0 is a radius of circular generator while a is a distance from center of generator to z–axis. One can easily check that in this case

$$w(x, y, z; a) = \left(x^2 + y^2 + z^2 + a^2 - 2a\sqrt{x^2 + y^2}\right)^{\frac{1}{2}} - b_0 \tag{3.8}$$

indeed solves the eikonal equation

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = 1\,. \tag{3.9}$$

If $a = 0$ (Fig. 1 (3.7)), then (3.8) is identical with (3.3) for $n = 3$. Moreover (3.3) for $n = 3$ is the w–function of spherical solitons of the $(3 + 1)$ – dimensional wave equation. This suggests the following question: can one find a 1–parameter family of *harmonic* functions $F = F(x, y, z; a)$ such that for any $a \geq 0$ the functions (3.8) and $F(x, y, z; a)$ solve eq. (2.2)? Certainly this idea is in accordance with the strategy to solve (2.1)–(2.3) and described in the end of section 2.

4 The role of geodesic coordinates

Since $w = u^1$ is in fact a geodesic coordinate, it is natural to use geodesic coordinates in our study. Moreover in the simple cases of V^2 and E^3 geodesic coordinates can always be chosen as *orthogonal*.

V^2–case. We put $u^1 = w$ and $u^2 = u$. The metric (3.1) is rewritten as

$$ds^2 = H_1^2(u, w)du^2 + dw^2\,. \tag{4.1}$$

E^3–case. Σ is any surface in E^3. All straight lines normal to Σ (geodesics γ in this case) form the so–called *normal congruence*. In general a *congruence* in E^3 is a two–dimensional submanifold in four–dimensional manifold of all straight lines in E^3. Geodesically parallel surfaces are called *parallel surfaces* in this case.

As coordinates (u^2, u^3) on Σ we select the so called *principal* (*curvature*) *coordinates* [11] which are uniquely defined (modulo "scaling"). We put $u^1 = w, u^2 = u, u^3 = v$. We have the following classical

Theorem 2.

a) Geodesic coordinates (u, v, w) are orthogonal, i.e., the metric (3.1) of E^3 is of the form

$$ds^2 = H_1^2(u, v, w)du^2 + H_2^2(u, v, w)dv^2 + dw^2 \tag{4.2}$$

b) Parametric surfaces $u = const$ and $v = const$ are developable surfaces, i.e. surfaces isometric to the plane.

5 Example 2 revisited

We come back to the Example 2 of section 3. Now we admit $0 < a \leq b_0$ i.e., we admit singularities on our torus (see Fig. 1 (3.5) and (3.6)). Our aim is to use the coordinates of Theorem 2 (Σ is our torus) to solve the system (2.4)–(2.6) ($n = 3$).

We identify: $\varphi = u$ and $\alpha = v$ as principal coordinates. The corresponding metric (4.2) is given by

$$ds^2 = (b_0 + w)^2 du^2 + \big[a + (b_0 + w)\cos u\big]^2 dv^2 + dw^2 \,. \tag{5.1}$$

Certainly the eikonal equation (2.4) is trivially satisfied by the coordinate w. Eq. (2.5) is reduced to

$$2\frac{\partial F}{\partial w} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial w}\sqrt{g} = 0\,, \tag{5.2}$$

where $\sqrt{g} = (b_0 + w)\big[a + (b_0 + w)\cos u\big]$. Eq. (5.2) implies

$$F(u, v, w) = (b_0 + w)^{-\frac{1}{2}}\big[a + (b_0 + w)\cos u\big]^{-\frac{1}{2}} g(u, v)\,, \tag{5.3}$$

where $g(u, v)$ is an arbitrary C^2–class function. When $a = 0$ and $g(u, v) = \sqrt{\cos u}$, we return to the spherical soliton case. For arbitrary $a > 0$ we assume $g(u, v) = U(u)V(v)$ and finally the question is: under what conditions is

$$F(u, v, w) = (b_0 + w)^{-\frac{1}{2}}\big[a + (b_0 + w)\cos u\big]^{-\frac{1}{2}} U(u)V(v) \tag{5.4}$$

an harmonic function? Upon substituting (5.4) in (2.6) we arrive at

$$H_1^{\frac{1}{2}} H_2^{-\frac{3}{2}} U\big(V'' + \frac{1}{4}V\big) + H_1^{-\frac{3}{2}} H_2^{\frac{1}{2}} V\big(U'' + \frac{1}{4}U\big) = 0 \tag{5.5}$$

and obviously (5.5) is satisfied iff $U'' = -\frac{1}{4}U$ and $V'' = -\frac{1}{4}V$.

Theorem 3. *Let (u,v,w) be geodesic coordinates defined by the torus (surface Σ) $x = (a + b_0 \cos u)\cos v, y = (a + b_0 \cos u)\sin v, z = b_0 \sin u$. Then*

$$F(u,v,w;a) = (b_0 + w)^{-\frac{1}{2}}\big[a + (b_0 + w)\cos u\big]^{-\frac{1}{2}} U(u)V(v)G[w \mp t] \tag{5.6}$$

is a family ($a > 0$) of solitons of the (3+1) – dimensional wave equation provided $U(u) = c_1 \cos\frac{u}{2} + c_2 \sin\frac{u}{2}$ and $V(v) = c_3 \cos\frac{v}{2} + c_4 \sin\frac{v}{2}$ ($c_1, \ldots, c_4 = const$).

Solutions (5.6) can be expressed in Cartesian coordinates as $F(x,y,z;a)$ making use of the following identities

$$b_0 + w = \big(x^2 + y^2 + z^2 + a^2 - 2a\sqrt{x^2 + y^2}\big)^{\frac{1}{2}}\,, \tag{5.7}$$

$$a + (b_0 + w)\cos u = \sqrt{x^2 + y^2}\,, \tag{5.8}$$

$$\cos\frac{u}{2} = \pm 2^{-\frac{1}{2}}\Big[1 + \big(\sqrt{x^2+y^2} - a\big)\big(x^2 + y^2 + z^2 + a^2 - 2a\sqrt{x^2+y^2}\big)^{-\frac{1}{2}}\Big]^{\frac{1}{2}}, \tag{5.9}$$

$$\sin\frac{u}{2} = \pm 2^{-\frac{1}{2}}\Big[1 - \big(\sqrt{x^2+y^2} - a\big)\big(x^2 + y^2 + z^2 + a^2 - 2a\sqrt{x^2+y^2}\big)^{-\frac{1}{2}}\Big]^{\frac{1}{2}}, \tag{5.10}$$

$$\cos\frac{v}{2} = \pm 2^{-\frac{1}{2}}\big[1 + x(x^2 + y^2)^{-\frac{1}{2}}\big]^{\frac{1}{2}}\,, \tag{5.11}$$

$$\sin\frac{v}{2} = \pm 2^{-\frac{1}{2}}\big[1 - x(x^2 + y^2)^{-\frac{1}{2}}\big]^{\frac{1}{2}}\,. \tag{5.12}$$

We conclude this section with two inspiring remarks.

1. Tori are algebraic surfaces of the fourth–order. For instance the torus of sec. 3 is defined by

   $$(x^2 + y^2 + z^2)^2 - 2(a^2 + b_0^2)(x^2 + y^2) + 2(a^2 - b_0^2)z^2 + (a^2 - b_0^2)^2 = 0\,. \tag{5.13}$$

   The torus is the simplest example of the so–called *Darboux cyclide*. Darboux cyclides constitute an important class of surfaces from the standpoint of conformal geometry [12, 13]. They are surfaces of the third– or fourth–order of the form

   $$\begin{aligned}&\epsilon(x^2 + y^2 + z^2)^2 + (x^2 + y^2 + z^2)P_1(x,y,z) + P_2(x,y,z) + \\ &Q_1(x,y,z) + P_0 = 0\,,\end{aligned} \tag{5.14}$$

   where $\epsilon = 0$ or $\epsilon = 1$, P_i are homogeneous polynomials of the degree i and Q_1 is an homogeneous polynomial of the first degree.

   The totality of all Darboux cyclides is invariant with respect of conformal transformations in E^3. The proper subset of Darboux cyclides are the *Dupin cyclides* [11]. Again the torus is the simplest example of a Dupin cyclide. They can be defined in many ways. Probably the simplest one is: these are surfaces all the *principal* lines [11] of which are circles. The interesting property of Dupin cyclides (evident for tori) is that all parallels to Dupin cyclide are Dupin cyclides as well. Dupin cyclides are conformally invariant too.

2. Geodesic coordinates built on a torus as a Σ–surface prove to be useful in our study. It is a natural idea to construct and then to apply to our problem geodesic coordinates built on *Dupin cyclides* as Σ–surfaces.

Indeed with some modifications this idea is realized in the following sections. Modifications are: a) we do not apply geodesic coordinates to the system (2.4)-(2.6) ($n = 3$) directly, instead we apply them to the Helmholtz equation,

$$\Delta_3 f + k^2 f = 0 \quad (k = const)\,, \tag{5.15}$$

and b) to solve (5.15) we make use of the technique of separation of variables in the Kalnins–Miller formulation. Exactly thanks to a) and b) we are able to achieve the main goal of the paper as it was formulated is section 1.

6 On the Kalnins–Miller approach to separation of variables

The most general setting to treat separation of variables is due to E.G. Kalnins and W. Miller, Jr. [3, 4] as a far–reaching extension of T. Levi–Civita's idea to handle the additive separation for the n–dimensional Hamilton–Jacobi equation [14]. In this context see also important contributions [5, 6]. According to Kalnins and Miller a generic additive separation can be either *regular* or *nonregular.* Regular separation is encoded in the requirement of complete integrability of the first–order differential system canonically associated with a given PDE: see equations (1.1), (1.6) and (1.7) of [4]. Any other separation is nonregular. These notions can be easily translated to the case of multiplicative R-separation, i.e. ordinary multiplicative separation preceded by some variable dependent *fixed* factor R. In particular Kalnins and Miller were able to show that, if a stationary n-dimensional Schrödinger equation defined on a pseudo–Riemannian manifold (V^n, g) equipped with orthogonal coordinates x^i admits regular multiplicative R–separation, then the corresponding metric components $g_{ij}(x^k)$ are necessarily in the so called Stäckel form [4, 15, 16]. Surely the last result applies to Helmholtz equation (5.15) which describes in the quantum mechanics a spectrum of the free particle in E^3.

Hence any *regular* orthogonal multiplicative R-separation for (5.15) leads to a flat Stäckel metric. On the other hand according to the Weinacht [17] - Eisenhart [18] theorem all metrics in E^3 which can be put in the Stäckel form are those of standard confocal quadrics or their appropriate degenerations.

The problem of the existence of *nonregular* orthogonal multiplicative R-separations, which moreover could be used in physical problems, seems to be both fresh and interesting. Indeed according to Willard Miller, Jr. [19]: It is straightforward to find (fairly trivial) examples of nonregular R-separation from Lie theory, but nontrivial examples. . .are not uncovered very often.

Below we show that geodesic coordinates (u, v, w) defined by any Dupin cyclide as a Σ–surface are multiplicatively R–separable for (5.15). Since the metric (4.2) in this case is *not* in the Stäckel form, the resulting R–separation is *not* regular.

7 Maxwellian construction of Dupin cyclides

For our purposes the most useful approach to Dupin cyclides is one given in 1867 by James Clerk Maxwell [11, 20]. Actually J.C. Maxwell solved the following problem: to find two curves in E^3 such that the congruence of straight lines meeting both curves is normal, i.e. one which admits an one–parameter family of orthogonal sections. The solution is:

the curves are a couple of *focal conics* [21]. Simultaneously the corresponding orthogonal sections (parallel surfaces) are just Dupin cyclides. The generic case of focal conics is a couple of ellipse and hyperbola (e-h) case. When one focus of the ellipse tends to infinity, focal conics become two parabolas (p-p) case. Finally confluence of two foci of the ellipse results in a pair of a circle and a straight line cutting the circle plane at the center of the circle and this is exactly the case of the tori discussed earlier. From now on we discuss (e-h) and (p-p) cases only.

We recall that Dupin cyclides are algebraic surfaces of the fourth– order ((e-h) case) and the third–order ((p-p) case). In all cases parametric surfaces $u = const$ and $v = const$ are circular cones and hence developable surfaces (see Theorem 2). In view of the Weinacht–Eisenhart theorem all our "cyclidic" coordinates are not in the Stäckel form.

1. (e-h) case

   The Cartesian coordinates are given in terms of (u, v, w) coordinates by

$$x = \left(b^2 \cos u \cosh v + (c \cosh v - a \cos u)w\right) / (a \cosh v - c \cos u) \ , \tag{7.1}$$

$$y = (b \sin u(a \cosh v - w)) / (a \cosh v - c \cos u) \ , \tag{7.2}$$

$$z = (b \sinh v(w - c \cos u)) / (a \cosh v - c \cos u) \ , \tag{7.3}$$

   where a, b and c are positive parameters ($b < a$, $c^2 = a^2 - b^2$). We select $u \in [0, 2\pi)$, $v \in R$ and $c \cos u < w < a \cosh v$.

   The Euclidean metric (4.2) is

$$ds^2 = \frac{b^2(a \cosh v - w)^2}{(a \cosh v - c \cos u)^2}\, du^2 + \frac{b^2(w - c \cos u)^2}{(a \cosh v - c \cos u)^2}\, dv^2 + dw^2 \, . \tag{7.4}$$

2. (p-p) case

   The Cartesian coordinates are given in terms of (u, v, w) coordinates by

$$x = \left(u(8a^2 + v^2 + 8aw)\right) / \left(u^2 + v^2 + 16a^2\right) \ , \tag{7.5}$$

$$y = -\left(v(8a^2 + u^2 - 8aw)\right) / \left(u^2 + v^2 + 16a^2\right) \ , \tag{7.6}$$

$$z = \left(16a^2 w + v^2(2a - w) - u^2(2a + w)\right) / \left(u^2 + v^2 + 16a^2\right) \ , \tag{7.7}$$

   where a is a positive parameter. We select $u, v \in R$ and $-(a + v^2/(8a)) < w < (a + u^2/(8a))$.

   The Euclidean metric (4.2) is

$$ds^2 = \frac{(8a^2 + v^2 + 8aw)^2}{(u^2 + v^2 + 16a^2)^2}\, du^2 + \frac{(8a^2 + u^2 - 8aw)^2}{(u^2 + v^2 + 16a^2)^2}\, dv^2 + dw^2 \, . \tag{7.8}$$

8 Theorem on R–separability of the Helmholtz equation

Theorem 4. *The Helmholtz equation (5.15) admits the following nonregular R-separations:*

a) In the (e-h) case:

$$f(u,v,w) = (w - c\cos u)^{-1/2}(a\cosh v - w)^{-1/2}U(u)V(v)W(w)\,, \tag{8.1}$$

where $U'' + U/4 = 0$, $V'' - V/4 = 0$ *and* $W'' + k^2W = 0$.

b) In the (p-p) case:

$$f(u,v,w) = (8a^2 + u^2 - 8aw)^{-1/2}(8a^2 + v^2 + 8aw)^{-1/2}U(u)V(v)W(w) \tag{8.2}$$

where $U'' = 0$, $V'' = 0$ *and* $W'' + k^2W = 0$.

To prove this theorem we modify the procedure formulated by Kalnins and Miller in [22] as “a precise operational definition of orthogonal R-separation” encoded in the identity (2.5) of their paper. Below we confine ourselves to the (e-h) case only. In our (non-Stäckel!) case we rewrite identity (2.5) of [22] as follows

$$R^{-1} \circ \Delta_3 \circ R + k^2 = \sum_{i=1}^{3} g_i(\partial_i^2 + l_i\partial_i + m_i)\,, \tag{8.3}$$

where $i = 1,2,3$ corresponds to u,v,w respectively ($g_i = g_i(u,v,w)$, $l_1 = l_1(u)$ etc. and $m_1 = m_1(u)$ etc.). The simplifying assumption $l_i \equiv 0$ ($i = 1,2,3$) defines R uniquely (modulo a constant factor) as $R = (w - c\cos u)^{-1/2}(a\cosh v - w)^{-1/2}$. Moreover, with the choices $m_1 = 1/4$, $m_2 = -1/4$ and $m_3 = k^2$, the identity (8.3) is satisfied due to another identity:

$$R^{-1}\Delta R = (H_1^{-2} - H_2^{-2})/4\,. \tag{8.4}$$

The separations (8.1) and (8.2) are by no means regular since Dupin cyclides are surfaces of the fourth– order or the third–order (see section 6).

9 Friedlander’s results revisited

Now we are in a position to formulate and prove the following

Theorem 5. *To every geodesic coordinate system (u,v,w) defined by a Dupin cyclide as a Σ–surface there corresponds the soliton $\mathcal{F}(u,v,w,t)$ of the wave equation (1.1) ($n = 3$).*
In particular in the (e-h) case

$$\mathcal{F}(u,v,w,t) = (w - c\cos u)^{-1/2}(a\cosh v - w)^{-1/2}U(u)V(v)G(w \mp t)\,, \tag{9.1}$$

where $U'' + U/4 = 0$ *and* $V'' - V/4 = 0$, *and in the (p-p) case*

$$\mathcal{F}(u,v,w,t) = (8a^2 + u^2 - 8aw)^{-1/2}(8a^2 + v^2 + 8aw)^{-1/2}U(u)V(v)G(w \mp t)\,, \tag{9.2}$$

where $U'' = 0$ *and* $V'' = 0$.

This result is a direct consequence of Theorem 4. In the proof we confine ourselves to (e-h) case only. Let $\xi \in R$ and consider the 1–parameter family of solutions to the wave equation (1.1) ($n = 3$)

$$\mathcal{F}_\xi(u,v,w,t) = (w - c\cos u)^{-1/2}\,(a\cosh v - w)^{-1/2}\; U(u)\; V(v)\; \exp\left(-i\xi(w-t)\right), \quad (9.3)$$

where $U'' + U/4 = 0$ and $V'' - V/4 = 0$. Select any function $\varphi(\xi)$ which is absolutely integrable together with $\xi\varphi(\xi)$ and $\xi^2\varphi(\xi)$. Certainly

$$\int_{-\infty}^{\infty} \varphi(\xi)\mathcal{F}_\xi(u,v,w,t)d\xi = (w - c\cos u)^{-1/2}\,(a\cosh v - w)^{-1/2}\; U(u)\; V(v)\; G(w-t) \quad (9.4)$$

is a solution (“cyclidic” soliton) to the wave equation where $G(\sigma) = \int_{-\infty}^{\infty} \varphi(\xi)\; \exp\left(-i\xi\sigma\right)\; d\xi$ is the Fourier transform of a pretty general function $\varphi(\xi)$.

The “cyclidic” soliton (9.1) is, modulo notation, the result (10.9) of [2]. The “cyclidic” soliton (9.2) corresponds to the result (11.6) of [2] and this one is wrong. Note that the “solution” (11.6) of [2] is in fact not in R–separation form! Indeed by a change of variables α and β it is transformable to the ordinary separation form. With the original notation of Friedlander the result (11.6) should be corrected as:

$$u(\alpha,\beta,\gamma,t) = (p\alpha^2 + \gamma)^{-1/2}(p\beta^2 + p - \gamma)^{-1/2}A(\alpha)B(\beta)F(t-\gamma)\,, \quad (9.5)$$

where $A'' = 0$ and $B'' = 0$.

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