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Multiscale Analysis of Discrete Nonlinear Evolution Equations: The Reduction of the dNLS

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Abstract

In this paper we consider multiple lattices and functions defined on them. We introduce some slow varying conditions and define a multiscale analysis on the lattice, i.e. a way to express the variation of a function in one lattice in terms of an asymptotic expansion with respect to the other. We apply these results to the case of the multiscale expansion of the differential-difference Nonlinear Schrödinger equation.

1 Introduction

The reductive perturbation method [10] allows to deduce a set of simplified equations starting from a basic model without losing its main characteristic features. The method consists essentially in the construction of an asymptotic series, based on the existence of different scales. The scales are directly related to the (small) amplitude of the field, which satisfies at the lowest order a linear equation. The deviation from the linear equation is induced by the nonlinearity.

The success of the method relies mainly on the nice property of the resulting reduced models, which are simpler than the starting equations and still providing useful information. Moreover, as emphasized in [2], there exists a general property of the reductive perturbation approach which implies, in a qualitative way, that the reduced systems are often integrable, i.e. that they have an infinite set of conserved quantities, a bi-Hamiltonian formulation and are solvable (in some sense).

The situation is quite different in the case of nonlinear lattice equations (continuous time and discrete space) for which a reliable reductive perturbative method which would produce reduced discrete systems up to now does not exist. Leon and Manna [6] proposed a set of tools which allow to perform multiscale analysis on a discrete evolution equation. These tools rely on the definition of a large grid scale via the comparison of the magnitude of the related difference operators, and on the expansion of the wavenumber in powers of frequency variations due to nonlinearity. The results, however, are not very promising as the reduced models are neither simpler nor more integrable than the original one. Starting from an integrable model, like the Toda lattice, the Leon and Manna reduction technique produce a non-integrable differential difference equation of the discrete Nonlinear Schrödinger type [8, 9].

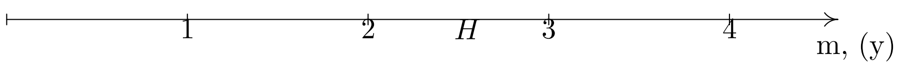
We here introduce two different lattices. The lattice spacing of one is related to the other by an integer parameter, the inverse of the infinitesimal scaling factor ϵ. We then introduce a slow variation condition for a function defined on the lattice and consequently consider the asymptotic expansion of the variations of a function on a small grid scale in terms of variations on the large grid scale. All passages are defined in such a way that are always consistent with the continuous limit, when the lattice spacings on the two grids go to zero.

In Section 2 we introduce the multiple lattice, the slow varying conditions and the asymptotic expansions of the variations and in Section 3 we apply the resulting formulas to the case of the multiscale expansion of the discrete Nonlinear Schrödinger equation (dNLS). In Section 4 we present a discussion of the results obtained and some concluding remarks.

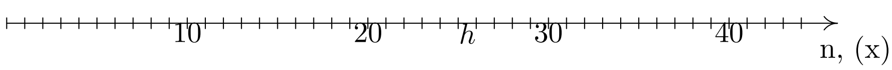
2 Multiple lattices and the variation of a function on them

2.1 Rescaling on the lattice

Let us consider two different lattices, depicted in (2.1) and (2.2), characterized by two different apriori arbitrary real lattice spacings, respectively H and h. For convenience we will define by m the running index of the points separated by H and n those separated by h. Moreover we can introduce two real variable $x = hn$ and $y = Hm$.



(2.1)



(2.2)

We assume that there exists an integer number N such that $H = Nh$. If N is a large number then $\epsilon = \frac{1}{N}$ will be a small number. The variables x and y will go over to continuous variables when respectively $h \to 0$, $n \to \infty$ and $H \to 0$, $m \to \infty$ in such a way that their product nh and mH are finite.

Let us assume that in the asymptotic region the two continuous variables x and y are such that $y = \epsilon x$. This implies that for $x \sim \frac{1}{\epsilon}$ we get $y \sim 1$. This assumption will reflect onto the relation between the lattice variables n and m as

$$y = Hm = hNm = \epsilon x = \frac{1}{N}hn \quad \Rightarrow \quad m = \frac{n}{N^2}. \tag{2.3}$$

Consequently, in the asymptotic region of the lattice we need to move N^2 points in n to shift m of one point.

2.2 Slowly varying functions and their asymptotic expansion

Let us consider a function f defined on the lattice points of the two lattices (2.1, 2.2). We will denote $f(n) = f(nh)$ and $f(m) = f(mH)$ for the values of f in the x and y lattices, respectively. Abusing notation we will write $f(n) = f(m)$ in a common reference point in both lattices (for example, $n = m = 0$) and, in the asymptotic region, $f(m \pm k) = f(n \pm kN^2)$.

We will analyze the action of a finite difference operator in the two lattices in the asymptotic region. Let us consider the M^{th} order variation in n, given, for example, by

$$\Delta_M^h f(n) = \sum_{j=0}^{M} \frac{(-1)^{M-j}\binom{M}{j} f(n+j)}{h^M}. \tag{2.4}$$

For the lattice m the corresponding M^{th} order variation will be:

$$\Delta_M^H f(m) = \sum_{j=0}^{M} \frac{(-1)^{M-j}\binom{M}{j} f(m+j)}{H^M}. \tag{2.5}$$

A function $f(n)$ will be a **slow varying function of order M** if, for all values n, the variations of $f(n)$ of order M are all equal and those of order $M+1$ are of the order of zero.

For example, if we consider a slow varying function of order 1 the second variation must be zero. Let us see which consequences we can derive from this hypothesis on the variation of the function $f(m)$ on the large grid. As the second variation is zero, we have:

$$\begin{aligned} f(m+1) - f(m) &= f(n+N^2) - f(n) \\ &= f(n+N^2) - f(n+N^2-1) \\ &\quad + f(n+N^2-1) - f(n+N^2-2) + \ldots + f(n+1) - f(n) \\ &= N^2[f(n+1) - f(n)]. \end{aligned} \tag{2.6}$$

So eq.(2.6) implies:

$$\begin{aligned} \Delta_m^H f(m) &\equiv \frac{f(m+1) - f(m)}{H} = \frac{N^2[f(n+1) - f(n)]}{Nh} \\ &= N\frac{f(n+1) - f(n)}{h} = N\Delta_n^h f(n), \end{aligned} \tag{2.7}$$

which, in the continuum limit, reads

$$\frac{df}{dy} = \frac{1}{\epsilon}\frac{df}{dx}.$$

Before going to the general case, let us consider a second example, the case of a slow varying function of order 2. Defining the shift operator T such that $Tf(n) = f(n+1)$, we can calculate the second difference

$$T^{2N^2} - 2T^{N^2} + 1 = (T^{N^2} - 1)^2 = (T^{N^2-1} + T^{N^2-2} + \cdots + 1)^2 (T-1)^2.$$

The slow varying condition of order 2 implies, for any integer α, $T^\alpha (T-1)^2 = (T-1)^2$ and thus

$$\begin{aligned} (T^{N^2-1} + T^{N^2-2} + \cdots + 1)^2 (T-1)^2 &= (T^{N^2-1} + T^{N^2-2} + \cdots + 1) N^2 (T-1)^2 \\ &= N^4 (T-1)^2. \end{aligned}$$

From this simple calculation we deduce that

$$f(m+2) - 2f(m+1) + f(m) = N^4 (f(n+2) - 2f(n+1) + f(n))$$

and by trivial shifting (using the slow varying condition) we can derive that also

$$f(m+1) - 2f(m) + f(m-1)f(n-2) = N^4 (f(n+1) - 2f(n) + f(n-1)). \qquad (2.8)$$

Let us calculate now the first difference:

$$\begin{aligned} T^{N^2} - 1 &= (T^{N^2-1} + T^{N^2-2} + \cdots + 1)(T-1) \\ &= N^2 (T-1) + \\ &\quad + [T^{N^2-2} + 2T^{N^2-3} + \cdots + iT^{N^2-i-1} + \cdots + (N^2-1)](T-1)^2 \\ &= N^2 (T-1) + \frac{N^2(N^2-1)}{2} (T-1)^2 \end{aligned}$$

Thus

$$f(m+1) - f(m) = N^2 [f(n+1) - f(n)] + \frac{N^2(N^2-1)}{2} [f(n+1) - 2f(n) + f(n-1)]. \qquad (2.9)$$

From eqs. (2.8,2.9) we get:

$$f(n+1) - 2f(n) + f(n-1) = \frac{1}{N^4} [f(m+1) - 2f(m) + f(m-1)] \qquad (2.10a)$$

$$\begin{aligned} f(n+1) - f(n) &= \frac{1}{2N^2} [f(m+1) - f(m-1)] \\ &\quad + \frac{1}{2N^4} [f(m+1) - 2f(m) + f(m-1)]. \end{aligned} \qquad (2.10b)$$

Let us pass now to the general case. Eq. (4) in Section §76 of [5] tell us that

$$\Delta_j^H f(m) = \sum_{i=0}^{\infty} \frac{h^i}{H^j} \frac{j!}{i!} P(i,j) \Delta_i^h f(n) \qquad (2.11)$$

where $P(i,j)$ is defined by

$$P(i,j) = \sum_{\alpha=j}^{i} \left(\frac{H}{h}\right)^\alpha S_i^\alpha \mathfrak{S}_\alpha^j$$

with S_i^j, $\mathfrak{S}_i^j$ Stirling numbers of the first and second kind respectively. The slow varying condition of order p means that

$$\Delta_{p+1}^h f(n) = 0.$$

In the case $p = 2$, we easily recover the equations (2.8,2.9). As one can interchange in eq. (2.11) the role of h and H, because in eq. (2.11) they are arbitrary numbers, we get:

$$\begin{aligned}
\Delta_2^h f(n) &= \frac{H^2}{h^2 N^4} \Delta_2^H f(m) \\
\Delta^h f(n) &= \frac{H}{hN^2} \Delta^H f(m) + \frac{(1-N^2)H^2}{2hN^4} \Delta_2^H f(m)
\end{aligned}$$

that are equivalent to eqs. (2.10). The formulas for $p = 3$ are

$$\begin{aligned}
\Delta_3^h f(n) &= \frac{H^3}{h^3 N^6} \Delta_3^H f(m) \\
\Delta_2^h f(n) &= \frac{H^2}{h^2 N^4} \Delta_2^H f(m) + \frac{(1-N^2)H^3}{h^2 N^6} \Delta_3^H f(m) \\
\Delta^h f(n) &= \frac{H}{hN^2} \Delta^H f(m) + \frac{(1-N^2)H^2}{2hN^4} \Delta_2^H f(m) + \frac{(1-3N^2+2N^4)H^3}{6hN^6} \Delta_3^H f(m)
\end{aligned}$$

From the results presented above we can see that, differently from the continuous case, where the Taylor expansion of a perturbed function is uniquely defined, in the discrete case it depends from the order of slow variation we are considering. For example, taking into account the results presented above, $f(m+1)$, when the function $f(m)$ is a slow varying function of order 1, is given by

$$f(n+1) = f(m) + \frac{1}{N^2}[f(m+1) - f(m)] + O\left(\frac{1}{N^4}\right); \tag{2.12}$$

when the function $f(m)$ is a slow varying function of order 2 is given by

$$\begin{aligned}
f(n+1) = f(m) &+ \frac{1}{2N^2}[-f(m+2) + 4f(m+1) - 3f(m)]+ \\
&+ \frac{1}{2N^4}[f(m+2) - 2f(m+1) + f(m)] + O\left(\frac{1}{N^6}\right);
\end{aligned} \tag{2.13}$$

when the function $f(m)$ is a slow varying function of order 3 is given by

$$\begin{aligned}
f(n+1) = f(m) &+ \frac{1}{6N^2}[2f(m+3) - 9f(m+2) + 18f(m+1) - 11f(m)]+ \\
&+ \frac{1}{2N^4}[-f(m+3) + 4f(m+2) - 5f(m+1) + 2f(m)]+ \\
&+ \frac{1}{6N^6}[f(m+3) - 3f(m+2) + 3f(m+1) - f(m)] + O\left(\frac{1}{N^8}\right).
\end{aligned} \tag{2.14}$$

In the following we will consider integrable discrete equations which must depend symmetrically on the discrete variable [7] , i.e. the discrete equation must be invariant with respect to the inversion of n. So the starting integrable equation will contain both $f(n \pm 1)$

and to preserve integrability, we need to consider slow varying conditions which also do so. When considering $f(n \pm 1)$, in place of eqs. (2.12, 2.13, 2.14) we have:

$$f(n+1) = f(m) + \frac{1}{N^2}[f(m+1) - f(m)] + O\left(\frac{1}{N^4}\right), \tag{2.15a}$$

$$f(n-1) = f(m) + \frac{1}{N^2}[f(m-1) - f(m)] + O\left(\frac{1}{N^4}\right), \tag{2.15b}$$

when the function $f(n)$ is a slow varying function of order 1,

$$\begin{aligned} f(n+1) = f(m) &+ \frac{1}{2N^2}[f(m+1) - f(m-1)]+ \\ &+ \frac{1}{2N^4}[f(m+1) - 2f(m) + f(m-1)] + O\left(\frac{1}{N^6}\right), \end{aligned} \tag{2.16a}$$

$$\begin{aligned} f(n-1) = f(m) &- \frac{1}{2N^2}[f(m+1) - f(m-1)]+ \\ &+ \frac{1}{2N^4}[f(m+1) - 2f(m) + f(m-1)] + O\left(\frac{1}{N^6}\right), \end{aligned} \tag{2.16b}$$

when the function $f(n)$ is a slow varying function of order 2. In the case when the function $f(n)$ is a slow varying function of order 3, we are not able to construct completely symmetric third order derivatives and thus $f(n+1)$ and $f(n-1)$ cannot be expressed in a symmetric form. In the case when the function $f(n)$ is a slow varying function of order 4, we have:

$$\begin{aligned} f(n+1) = f(m) &- \frac{1}{12N^2}[f(m+2) - 8f(m+1) + 8f(m-1) - f(m-2)] \\ &- \frac{1}{24N^4}[f(m+2) - 16f(m+1) + 30f(m) - 16f(m-1) + f(m-2)] \\ &+ \frac{1}{12N^6}[f(m+2) - 2f(m+1) + 2f(m-1) - f(m-2)]+ \\ &+ \frac{1}{24N^8}[f(m+2) - 4f(m+1) + 6f(m) - 4f(m-1) + f(m-2)] + O\left(\frac{1}{N^{10}}\right), \end{aligned} \tag{2.17a}$$

$$\begin{aligned} f(n-1) = f(m) &+ \frac{1}{12N^2}[f(m+2) - 8f(m+1) + 8f(m-1) - f(m-2)] \\ &- \frac{1}{24N^4}[f(m+2) - 16f(m+1) + 30f(m) - 16f(m-1) + f(m-2)] \\ &- \frac{1}{12N^6}[f(m+2) - 2f(m+1) + 2f(m-1) - f(m-2)]+ \\ &+ \frac{1}{24N^8}[f(m+2) - 4f(m+1) + 6f(m) - 4f(m-1) + f(m-2)] + O\left(\frac{1}{N^{10}}\right). \end{aligned} \tag{2.17b}$$

3 Reduction of the dNLS

In this Section we will apply the results presented above to the case of the dNLS [3]:

$$\begin{aligned} i\partial_t f(n,t) + \frac{f(n+1,t) - 2f(n,t) + f(n-1,t)}{2h^2} &= \\ = \varepsilon |f(n,t)|^2 \frac{f(n+1,t) + f(n-1,t)}{2}&, \end{aligned} \tag{3.1}$$

where $\varepsilon = \pm 1$ and $f(n,t)$ is a complex function of two independent variables, one continuous t and one discrete n. By defining $x = nh$ and expanding in Taylor series of h as $h \to 0$ we get the standard Nonlinear Schrödinger equation (NLS):

$$i\partial_t f(x,t) + \frac{1}{2}\partial_{xx} f(x,t) = \varepsilon |f(x,t)|^2 f(x,t). \tag{3.2}$$

It is well known [11] that by a multiscale expansion we can reduce the NLS (3.2) to the Korteweg - de Vries (KdV) equation

$$\partial_\tau u(\xi,\tau) - \frac{1}{8}\partial_{\xi\xi\xi} u(\xi,\tau) + \frac{3}{2} u(\xi,\tau)\partial_\xi u(\xi,\tau) = 0. \tag{3.3}$$

The KdV (3.3) is obtained from the NLS (3.2) in the case when $\epsilon = 1$ by defining $f(x,t) = \sqrt{\nu(x,t)}e^{i\phi(x,t)}$, rewriting the two fields $\nu(x,t)$ and $\phi(x,t)$ in powers of a small parameter λ around its trivial solution

$$\nu(x,t) = 1 + \lambda^2 u(x_1,t_1,\tau) + \ldots, \tag{3.4}$$

$$\phi(x,t) = -t + \lambda w(x_1,t_1,\tau) + \ldots, \tag{3.5}$$

where the new independent slow variables are given by

$$x_1 = \lambda x; \qquad t_1 = \lambda t; \qquad \tau = \lambda^3 t, \tag{3.6}$$

and expanding the real and imaginary part of the NLS in Taylor series in λ. At lowest order in λ we get:

$$u(x_1,t_1,\tau) = \partial_{x_1} w(x_1,t_1,\tau) \tag{3.7}$$

$$\partial_{x_1} u(x_1,t_1,\tau) + \partial_{t_1} u(x_1,t_1,\tau) = 0 \tag{3.8}$$

and at a higher order the KdV equation in the slow varying time τ

$$\partial_\tau u(x_1,t_1,\tau) - \frac{1}{8}\partial_{x_1x_1x_1} u(x_1,t_1,\tau) + \frac{3}{2} u(x_1,t_1,\tau)\partial_{x_1} u(x_1,t_1,\tau) = 0. \tag{3.9}$$

Let us now pass to the discrete case and consider the reduction of the dNLS (3.1). The dNLS is an integrable equation of the same category as the NLS. So, as in the case of the NLS we got an integrable model, the same we expect here [2]. Hence we expect to have a resulting discrete equation which is symmetric in the inversion of m, i.e. if it depends on $m+j$ it shall depend in the same way on $m-j$. So in the transformation of the discrete independent variables we just use formulas (2.15, 2.16, 2.17). Following the continuous case let us define $f(n,t) = \sqrt{\nu_d(n,t)}e^{i\phi_d(n,t)}$. In such a way the dNLS reduces to the following system of equations:

$$\begin{aligned}\partial_t \nu_d(n,t) + [\frac{1}{h^2} - \epsilon\nu_d(n,t)]\{\sqrt{\nu_d(n,t)\nu_d(n+1,t)}\sin[\phi_d(n+1,t) - \phi_d(n,t)] + \\ - \sqrt{\nu_d(n-1,t)\nu_d(n,t)}\sin[\phi_d(n,t) - \phi_d(n-1,t)]\} = 0,\end{aligned} \tag{3.10a}$$

$$\begin{aligned}\partial_t \phi_d(n,t) + \frac{1}{h^2} - \frac{1}{2}[\frac{1}{h^2} - \epsilon\nu_d(n,t)]\{\sqrt{\frac{\nu_d(n+1,t)}{\nu_d(n,t)}}\cos[\phi_d(n+1,t) - \phi_d(n,t)] + \\ + \sqrt{\frac{\nu_d(n-1,t)}{\nu_d(n,t)}}\cos[\phi_d(n,t) - \phi_d(n-1,t)]\} = 0.\end{aligned} \tag{3.10b}$$

It is easy to check that by taking a straightforward Taylor series expansion of formulas (3.10) for $h \to 0$ we get the corresponding equations obtained from the NLS. Rewriting the two fields ν_d and ϕ_d in terms of a small parameter $\lambda = \frac{1}{N}$ (different from h, the parameter which governs the continuous limit),

$$\nu_d(n,t) = 1 + \lambda^2 u_d(n, t_1, \tau) + \dots, \tag{3.11}$$

$$\phi_d(n,t) = -\epsilon t + \lambda w_d(n, t_1, \tau) + \dots, \tag{3.12}$$

and taking into account eqs. (2.16) we get at the lowest order in λ, the following set of equations

$$\partial_{t_1} u_d(m, t_1, \tau) + \left[\frac{w_d(m+1, t_1, \tau) + w_d(m-1, t_1, \tau) - 2w_d(m, t_1, \tau)}{H^2}\right] = 0 \tag{3.13a}$$

$$\partial_{t_1} w_d(m, t_1, \tau) + \epsilon u_d(m, t_1, \tau) = 0 \tag{3.13b}$$

$$\begin{aligned} \partial_\tau u_d(m, t_1, \tau) + u_d(m, t_1, \tau) \left[\frac{w_d(m+1, t_1, \tau) + w_d(m-1, t_1, \tau) - 2w_d(m, t_1, \tau)}{H^2}\right] + \\ + \left[\frac{u_d(m+1, t_1, \tau) - u_d(m-1, t_1, \tau)}{2H}\right] \left[\frac{w_d(m+1, t_1, \tau) - w_d(m-1, t_1, \tau)}{2H}\right] \\ - \epsilon \left[w_d(m+1, t_1, \tau) + w_d(m-1, t_1, \tau) - 2w_d(m, t_1, \tau)\right] = 0. \end{aligned} \tag{3.13c}$$

Eq. (3.13b) defines u_d in terms of derivatives of w_d with respect to the fast time t_1 (3.6). Taking this into account eq. (3.13a) is just a differential-difference wave equation for the variable w_d in the variables t_1 and m. Eq. (3.13c) provides us with the evolution of u_d (or, taking into account eq. (3.13b) $\partial_{t_1} w_d$) with respect to the slow time τ (3.6). Eq. (3.13c) is a nonlinear equation with quadratic nonlinearities.

4 Discussion of the results and conclusions

Many of the problems encountered in the derivation of system (3.13) are due to the fact that we are considering an equation with one discrete and one continuous variable. In such a situation eq. (3.13a) is a linear differential difference equation and we cannot solve it completely. We can just provide special solutions, obtained by separation of variables. The corresponding continuous equation (3.8) is the wave equation which is identically solved by introducing a new variable $\xi = x_1 - t_1$. A natural way out to this problem is to consider a completely discrete NLS and carry out on it the multiscale expansion. The integrable completely discrete NLS is a nonlocal difference - difference equation [1]. The reductive perturbation technique is more complicated, but work on this is in progress.

The KdV equation is the continuous limit of any equation of the Toda Lattice Hierarchy by an appropriate choise of the limiting transformation. In [4] we show this for the Toda lattice itself, the Volterra equation and a higher Volterra equation. The difference of the three cases is mainly in the choice of the independent variables. The only case which, like in the case presented above for the dNLS, goes naturally into KdV is the higher Volterra equation which contains the function $u_d(m)$, $u_d(m \pm 1)$ and $u_d(m \pm 2)$ and has a cubic nonlinearity. By choosing a higher order slow varying function we could get $u_d(m \pm 2)$ but it is not clear to us now how we can get cubic a nonlinearity. Moreover the problem of solving the linear differential difference equation still persists.

One can consider eq. (3.13c), which is an evolution equation in the slow time τ and depends parametrically on the quick time t_1 as part of the system (3.13). As a system, this equation may still be integrable. The integrability properties of a system like eq. (3.13) has never been studied and work on this is in progress.

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