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Some Water Wave Equations and Integrability

Yishen LI

Department of Mathematics and Center of Nonlinear Science
University of Science and Technology of China
Hefei 230026, China

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Abstract

A theory of bidirectional solitons on water is developed by using the classical Boussinesq equation. Moreover, analytical multi-solitons of Camassa-Holm equation are presented.

1 Unidrectional soliton

In 1965 Zabusky and Kruskal introduced the concept of a soliton for the Korteweg - de Vries(KdV) equation [1]. Two years latter, by using the inverse scattering method on the Schrodinger equation, Gardner et al.(GGKM) solved the KdV equation for exact N-solitons [2], which can be used to model the interation of unidirectional solitary waves on water. Their discovery establishes the mathematical foundation of the unidirectional water wave interaction. The KdV equation is the leading-order approximation of the Euler equation from a perturbation scheme under the assumption that the wave height is relatively small and the wavelength is relatively long compared with the water depth. It also assumes that the wave propagation is in one direction, which is not a good assumption to model the reflection of water wave on a vertical wall. For reflection of water wave, we need a model that allows the bidirectional wave interactions, including head-on and overtaking collections. In soliton theory it is known that the Boussinesq one equation is often used in the literature to model a bidirectional soliton [3], but the result of the head on collision is not physically meaningful for water waves. The fact that the Boussinesq one equation cannot be used to physically describe the head-on collision of solitons has been pointed out in the fluid mechanics literature. A more detail discussion is given in ref[4].

2 Bidirectional soliton

The 2 + 1 dimensional nonlinear dispersive wave equation

$$\begin{cases} u_t + uu_x + vu_y + \zeta_x = 0 \\ v_t + uv_x + vv_y + \zeta_y = 0 \\ \zeta_t + [(1+\zeta)u]_x + [(1+\zeta)v]_y + \frac{u_{xxx}+u_{xyy}+v_{xxy}+v_{yyy}}{3} = 0 \end{cases} \tag{1}$$

where (u, v) is the horizonal projection of the surface velocity of a water, ς is wave elevation which is regarded as Wu - Zhang(WZ) equation by Ref.[5]. The WZ equation is derived in Ref.[6] from the Euler equation with perturbation schems under the assumption that the amplitude of wave elevation is small and the wave is long compared with the water depth(scaled to be 1). The WZ equation can be used to model the three dimensional behavior of solitary waves on a uniform layer of water, such as oblique interaction, oblique reflection from a vertical wall and turning in a curved channel. In Ref.[5], it point out that the WZ equation can not pass the Painleve test, but in Ref.[7] it points out that the reduction of this equation to 1 + 1 dimensional case in any direction, it is reduced to the classical Boussinesq equation

$$\left\{ \begin{array}{l} \zeta_t + [(1+\zeta)u]_x = -\frac{1}{3}u_{xxx}, \\ u_t + uu_x + \zeta_x = 0, \end{array} \right. \tag{2}$$

In Ref.[4,8-10], we consider the bidirectional soliton of this equation. With scaling transformation

$$\frac{\sqrt{3}}{2}x \to x, \quad \frac{\sqrt{3}}{2}t \to t, \tag{3}$$

equation (2) becomes

$$\left\{ \begin{array}{l} \zeta_t + [(1+\zeta)u]_x + \frac{1}{4}u_{xxx} = 0, \\ u_t + uu_x + \zeta_x = 0. \end{array} \right. \tag{4}$$

The Lax pair of the system (4) is

$$\left\{ \begin{array}{l} \phi_{xx} = (\lambda^2 + \lambda u + \frac{1}{4}u^2 - \zeta - 1)\phi, \\ \phi_t = \frac{1}{4}u_x\phi + (\lambda - \frac{1}{2}u)\phi_x. \end{array} \right. \tag{5}$$

By using the transformation

$$u = -v, \quad \zeta = -1 + w - \frac{1}{2}v_x, \tag{6}$$

we can convert the system (4) to Broer-Kaup(BK) system

$$\left\{ \begin{array}{l} v_t = \frac{1}{2}(v^2 + 2w - v_x)_x, \\ w_t = (vw + \frac{1}{2}w_x)_x. \end{array} \right. \tag{7}$$

Introducing the following transformation

$$q = e^{\int u dx}, \quad r = -\left(1 + \zeta - \frac{1}{2}u_x\right)e^{-\int u dx} \tag{8}$$

or

$$u = \frac{q_x}{q}, \quad \zeta = -1 - qr + \frac{1}{2}u_x, \tag{9}$$

we have an equivalent system for q and r,

$$\left\{ \begin{array}{l} q_t + \frac{1}{2}q_{xx} - q^2 r - q = 0, \\ r_t - \frac{1}{2}r_{xx} + qr^2 + r = 0, \end{array} \right. \tag{10}$$

which is a member of the AKNS system.

The Lax pair of the system (10) reads

$$\begin{aligned} &\Psi_x = M\Psi, \qquad \Psi = (\psi_1, \psi_2)^T, \quad M = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \\ &\Psi_t = N\Psi, \qquad N = \begin{pmatrix} -\lambda^2 + \frac{1}{2}qr + \frac{1}{2} & \lambda q - \frac{1}{2}q_x \\ \lambda r + \frac{1}{2}r_x & \lambda^2 - \frac{1}{2}qr - \frac{1}{2} \end{pmatrix}. \end{aligned} \tag{11}$$

The Darboux transformation on the AKNS system, available in a textbook [11], is given as follows: Let

$$\phi' = T\phi, \qquad T = \lambda^n I + \sum_{j=1}^{n} T_j \lambda^{n-j}, \qquad T_j = \begin{pmatrix} a_{2j-1} & a_{2j} \\ b_{2j-1} & b_{2j} \end{pmatrix}, \tag{12}$$

where I is a 2×2 identity matrix, ϕ is a solution of equations (11), then ϕ' is a solution of equation

$$\phi'_x = M'\phi', \qquad \phi'_t = N'\phi', \tag{13}$$

where M' and N' are the same as M and N in equations (11), but with q, r, q_x and r_x replaced by q', r', q'_x and r'_x. We assume $\lambda_i \neq \lambda_j$ for $i \neq j$, $i = 1, 2, \cdots, 2n$, and denote

$$\phi_{1,j} = \phi_1(x, \lambda_j), \qquad \phi_{2,j} = \phi_2(x, \lambda_j). \tag{14}$$

We define a $2n \times 2n$ matrix H to be the following

$$H = \begin{pmatrix} \lambda_1^{n-1}\phi_{1,1} & \lambda_1^{n-1}\phi_{2,1} & \lambda_1^{n-2}\phi_{1,1} & \lambda_1^{n-2}\phi_{2,1} & \cdots & \phi_{1,1} & \phi_{2,1} \\ \lambda_2^{n-1}\phi_{1,2} & \lambda_2^{n-1}\phi_{2,2} & \lambda_2^{n-2}\phi_{1,2} & \lambda_2^{n-2}\phi_{2,2} & \cdots & \phi_{1,2} & \phi_{2,2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{2n}^{n-1}\phi_{1,2n} & \lambda_{2n}^{n-1}\phi_{2,2n} & \lambda_{2n}^{n-2}\phi_{1,2n} & \lambda_{2n}^{n-2}\phi_{2,2n} & \cdots & \phi_{1,2n} & \phi_{2,2n} \end{pmatrix}. \tag{15}$$

Solving the equations

$$H\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n} \end{pmatrix} = \begin{pmatrix} -\lambda_1^n\phi_{1,1} \\ -\lambda_2^n\phi_{1,2} \\ \vdots \\ -\lambda_{2n}^n\phi_{1,2n} \end{pmatrix} \equiv A, \qquad H\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2n} \end{pmatrix} = \begin{pmatrix} -\lambda_1^n\phi_{2,1} \\ -\lambda_2^n\phi_{2,2} \\ \vdots \\ -\lambda_{2n}^n\phi_{2,2n} \end{pmatrix} \equiv B \tag{16}$$

gives us a_i and b_i, $i = 1, 2, 3, \cdots, 2n$. Then

$$q' = q + 2a_2, \qquad r' = r - 2b_1, \tag{17}$$

where

$$a_2 = \frac{\det H_2}{\det H}, \qquad b_1 = \frac{\det H_1}{\det H}, \tag{18}$$

and H_2 is a $2n \times 2n$ matrix of H with the second column replaced by A, H_1 is a $2n \times 2n$ matrix of H with the first column replaced by B.

Now we discuss the mechanics of soliton interaction by using the Darboux transformation. For a layer of quiescent water without any waves, wave elevation is $\zeta = 0$ and velocity is $u = 0$, and corresponding $q = 1$ and $r = -1$. Therefore we take $(q, r) = (1, -1)$

as our initial seed to implement Darboux transformation. With this initial seed, we have the following two sets of basic solutions for the spectral problem (11):

$$\phi_{1,j} = \cosh \xi_j, \qquad \phi_{2,j} = c_j \sinh \xi_j + \lambda_j \cosh \xi_j, \qquad j \text{ is an odd number}, \tag{19}$$

$$\phi_{1,j} = \sinh \xi_j, \qquad \phi_{2,j} = c_j \cosh \xi_j + \lambda_j \sinh \xi_j, \qquad j \text{ is an even number}, \tag{20}$$

where $\xi_j = c_j(x + \lambda_j t)$ and $c_j = \sqrt{\lambda_j^2 - 1}$. The eigenvalue λ_j is the wave speed of a soliton. The soliton is right-going if $\lambda_j < -1$, left-going if $\lambda_j > 1$.

For a single right-going soliton solution, we can take the following eigenvalues and eigenfunctions:

$$\begin{aligned}
&\lambda_1^* = -1, \quad \phi_{1,1} = 1, \quad \phi_{2,1} = -1, \quad \xi = \frac{\sqrt{3}}{2} c(x - \lambda t), \quad c = \sqrt{\lambda^2 - 1}, \\
&\lambda_2^* = -\lambda < -1, \quad \phi_{1,2} = \sinh \xi, \quad \phi_{2,2} = c \cosh \xi - \lambda \sinh \xi,
\end{aligned}$$

where x and t have been converted to the original coordinates before the scaling transformation (3). We obtain a single-soliton solution of equation (2)

$$u_B(x - \lambda t; \lambda) = \frac{2(\lambda^2 - 1)}{\lambda + \cosh \sqrt{3(\lambda^2 - 1)}(x - \lambda t)}, \tag{21}$$

$$\zeta_B(x - \lambda t; \lambda) = \frac{2(\lambda^2 - 1)\left(1 + \lambda \cosh \sqrt{3(\lambda^2 - 1)}(x - \lambda t)\right)}{\left(\lambda + \cosh \sqrt{3(\lambda^2 - 1)}(x - \lambda t)\right)^2}. \tag{22}$$

The wave speed λ and the wave amplitude a satisfy

$$\lambda = 1 + \frac{1}{2} a. \tag{23}$$

Integrating the wave elevation (22) over the whole space domain gives us the mass under the soliton

$$m_B(\lambda) = \int_{-\infty}^{\infty} \zeta_B(s; \lambda) ds = \frac{4}{\sqrt{3}} \sqrt{\lambda^2 - 1} = \frac{4}{\sqrt{3}} \sqrt{(1 + a/4)a}. \tag{24}$$

Differentiating (22) twice and evaluating at the origin gives us

$$\zeta_B''(0; \lambda) = -6(2 - \lambda)(\lambda - 1)^2. \tag{25}$$

Therefore the soliton has a single peak when $\lambda < 2$ and double peaks when $\lambda > 2$. The soliton appears to have some remarkable features. It is single-peaked when the wave amplitude is not larger than 2, and *double-peaked* when the wave amplitude is larger than 2. As is well-known that the Boussinesq model is only valid for the water waves with small amplitude, i.e., the wave amplitude smaller than water depth (scaled to be 1 here). Therefore the new feature of double-peaked soliton is not physically meaningful for the water wave.

We now construct a multisoliton solution with $2m$ left-going and $2l$ right-going solitons, the power of the eigenvalue in the Darboux transformation is taken to be $n = m + l$. First we rank the solitons by their amplitudes (or speeds). For the $2m$ left-going solitons, we assume $\lambda_{2m} > \lambda_{2m-1} > \cdots > \lambda_1 > 1$. For the $2l$ right-going solitons, we assume $\lambda^*_{2l} < \lambda^*_{2l-1} < \cdots < \lambda^*_1 < -1$. With the eigenfunctions defined in (19, 20) for both λ_j and λ^*_j, we can obtain the soliton solution as follows:

$$u = \frac{q'_x}{q'}, \qquad \zeta = -1 - q'r' + \frac{1}{2}u_x, \tag{26}$$

where

$$q' = 1 + 2a_2, \qquad r' = -1 - 2b_1, \tag{27}$$

a_2 and b_1 are defined by (18). This is the solution for the interaction of an even number of solitons in both directions. To obtain an odd number of solitons, we can simply set the first eigenvalue to be 1 for a left-going soliton and -1 for a right-going soliton. In other words, an odd number of soliton solution can be treated as an even number of solitons in which one of the solitons has zero amplitude.

For a solution with two-soliton overtaking collision, we take $m = 0$ and $l = 1$ with the following eigenvalues and eigenfunctions:

$$\begin{aligned} &\lambda^*_1 = -\lambda_1 < -1, \qquad \phi_{1,1} = \cosh\xi_1, \qquad \phi_{2,1} = c_1 \sinh\xi_1 - \lambda_1\cosh\xi_1, \\ &\lambda^*_2 = -\lambda_2 < -\lambda_1, \qquad \phi_{1,2} = \sinh\xi_2, \qquad \phi_{2,2} = c_2\cosh\xi_2 - \lambda_2\sinh\xi_2, \end{aligned}$$

where λ_1 and λ_2 are two positive numbers. The solution to system (2), given by (26), can be written in a closed form as

$$u = \frac{2(\lambda_2-\lambda_1)[c_2^2 - c_1^2\tanh^2\xi_1\tanh^2\xi_2 - (\lambda_2^2-\lambda_1^2)\tanh^2\xi_2]}{(c_2 - c_1\tanh\xi_1\tanh\xi_2)^2 - (\lambda_2-\lambda_1)^2\tanh^2\xi_2}, \tag{28}$$

$$\begin{aligned} \zeta = -1 + &\frac{c_2 - c_1\tanh\xi_1\tanh\xi_2 + (\lambda_2-\lambda_1)\tanh\xi_2}{c_2 - c_1\tanh\xi_1\tanh\xi_2 - (\lambda_2-\lambda_1)\tanh\xi_2} \times \\ &\left[1 - 2\frac{(\lambda_2-\lambda_1)(c_1\tanh\xi_1-\lambda_1)(c_2-\lambda_2\tanh\xi_2)}{c_2 - c_1\tanh\xi_1\tanh\xi_2 - (\lambda_2-\lambda_1)\tanh\xi_2}\right] + \frac{1}{\sqrt{3}}u_x, \end{aligned} \tag{29}$$

$$\xi_i = \frac{\sqrt{3}}{2}c_i(x-\lambda_i t), \qquad c_i = \sqrt{\lambda_i^2 - 1}, \qquad i = 1, 2, \qquad \lambda_2 > \lambda_1 > 1,$$

where x and t have been converted to the original coordinates before the scaling transformation (3), λ_1 and λ_2 are the speeds of the two solitons, with λ_2 larger than λ_1. The soliton with the speed λ_2 is taking over the soliton with the speed λ_1. The process of overtaking interaction can be easily seen with the asymptotic limit of the solution (28, 29): as $t \to -\infty$,

$$\begin{aligned} \zeta(x,t) &\to \zeta_B(x-\lambda_1 t - \Delta_1;\lambda_1) + \zeta_B(x - \lambda_2 t + \Delta_2;\lambda_2), \\ u(x,t) &\to u_B(x-\lambda_1 t - \Delta_1;\lambda_1) + u_B(x - \lambda_2 t + \Delta_2;\lambda_2), \end{aligned}$$

and as $t \to +\infty$,

$$\begin{aligned} \zeta(x,t) &\to \zeta_B(x-\lambda_1 t + \Delta_1;\lambda_1) + \zeta_B(x - \lambda_2 t - \Delta_2;\lambda_2), \\ u(x,t) &\to u_B(x-\lambda_1 t + \Delta_1;\lambda_1) + u_B(x - \lambda_2 t - \Delta_2;\lambda_2), \end{aligned}$$

where $\zeta_B(s;\lambda)$ and $u_B(s;\lambda)$ are the wave elevation and surface velocity of the single-soliton solution given by (22, 21), and the total phase shift of the two solitons are given by the following

$$2\Delta_1 = \frac{2}{\sqrt{3(\lambda_1^2-1)}}\ \text{arccosh}\frac{\lambda_1\lambda_2-1}{\lambda_2-\lambda_1}, \qquad 2\Delta_2 = \frac{2}{\sqrt{3(\lambda_2^2-1)}}\ \text{arccosh}\frac{\lambda_1\lambda_2-1}{\lambda_2-\lambda_1}. \quad (30)$$

Since the mass has been obtained previously in (24) as

$$m_i = \frac{4}{\sqrt{3}}\sqrt{\lambda_i^2-1}, \qquad i=1,2, \quad (31)$$

the conservation of momentum can be easily verified by

$$2m_1\Delta_1 = 2m_2\Delta_2 = \frac{8}{3}\ \text{arccosh}\frac{\lambda_1\lambda_2-1}{\lambda_2-\lambda_1}. \quad (32)$$

For a solution with two-soliton head-on collision, we take $m=0$ and $l=1$ with the following eigenvalues and eigenfunctions:

$$\begin{aligned} &\lambda_1^* = -\lambda_1 < -1, \qquad \phi_{1,1} = \cosh\xi_1, \qquad \phi_{2,1} = c_1\sinh\xi_1 - \lambda_1\cosh\xi_1, \\ &\lambda_2^* = \lambda_2 > 1, \qquad \phi_{1,2} = \cosh\xi_2, \qquad \phi_{2,2} = c_2\sinh\xi_2 + \lambda_2\cosh\xi_2. \end{aligned}$$

The solution of system (2) given by (26) can be written in a closed form as follows

$$u = \frac{2(\lambda_1+\lambda_2)(\lambda_2^2-\lambda_1^2-c_2^2\tanh^2\xi_2+c_1^2\tanh^2\xi_1)}{(c_2\tanh\xi_2-c_1\tanh\xi_1)^2-(\lambda_1+\lambda_2)^2}, \quad (33)$$

$$\begin{aligned} \zeta = -1 + &\frac{c_2\tanh\xi_2-c_1\tanh\xi_1-\lambda_1-\lambda_2}{c_2\tanh\xi_2-c_1\tanh\xi_1+\lambda_1+\lambda_2}\times \\ &\left[1+2\frac{(\lambda_1+\lambda_2)(c_1\tanh\xi_1-\lambda_1)(c_2\tanh\xi_2+\lambda_2)}{c_2\tanh\xi_2-c_1\tanh\xi_1+\lambda_1+\lambda_2}\right] + \frac{1}{\sqrt{3}}u_x, \end{aligned} \quad (34)$$

$$\xi_1 = \frac{\sqrt{3}}{2}c_1(x-\lambda_1 t), \qquad \xi_2 = \frac{\sqrt{3}}{2}c_2(x+\lambda_2 t), \qquad c_i = \sqrt{\lambda_i^2-1}, \ \ \lambda_i > 1, \ \ i=1,2,$$

where x and t are the original coordinates before the scaling transformation. The soliton with speed λ_1 is moving from the left to the right. The soliton with speed λ_2 is moving from the right to the left. At $t=0$, the two solitons merge into a single peak. One may verify that $\zeta_x(0,0)=0$, i.e., the solution is symmetric about the origin. Therefore the maximum amplitude appears at the origin, i.e.,

$$\zeta_{\max} = \zeta(0,0) = a_1 + a_2 + \frac{1}{2}a_1a_2. \quad (35)$$

For the head-on collision of two solitons with the same amplitude, $a_1 = a_2 = a$, the wave elevation at $t=0$ can be simplified and given by

$$\zeta(x,0) = \left(2a + \frac{1}{2}a^2\right)\text{sech}^2\left[\frac{1}{4}\sqrt{3a(4+a)}x\right], \quad (36)$$

and the velocity at $t=0$ is zero for all x.

After the head-on collision, each soliton experiences a backward phase shift. The asymptotic analysis of the solution (33, 34) leads to the following limits: as $t \to -\infty$,

$$\zeta(x,t) \to \zeta_B(x - \lambda_1 t - \Delta_1; \lambda_1) + \zeta_B(x + \lambda_2 t + \Delta_2; \lambda_2), \tag{37}$$

$$u(x,t) \to u_B(x - \lambda_1 t - \Delta_1; \lambda_1) - u_B(x + \lambda_2 t + \Delta_2; \lambda_2), \tag{38}$$

and as $t \to +\infty$,

$$\zeta(x,t) \to \zeta_B(x - \lambda_1 t + \Delta_1; \lambda_1) + \zeta_B(x + \lambda_2 t - \Delta_2; \lambda_2), \tag{39}$$

$$u(x,t) \to u_B(x - \lambda_1 t + \Delta_1; \lambda_1) - u_B(x + \lambda_2 t - \Delta_2; \lambda_2), \tag{40}$$

where $\zeta_B(s;\lambda)$ and $u_B(s;\lambda)$ are the wave elevation and surface velocity of the single-soliton solution given by (22, 21), and the total phase shift of the two solitons are given by

$$2\Delta_1 = \frac{2}{\sqrt{3(\lambda_1^2 - 1)}} \operatorname{arccosh} \frac{\lambda_1\lambda_2 + 1}{\lambda_1 + \lambda_2}, \qquad 2\Delta_2 = \frac{2}{\sqrt{3(\lambda_2^2 - 1)}} \operatorname{arccosh} \frac{\lambda_1\lambda_2 + 1}{\lambda_1 + \lambda_2}.$$

The conservation of momentum can be easily verified by

$$2m_1\Delta_1 = 2m_2\Delta_2 = \frac{8}{3} \operatorname{arccosh} \frac{\lambda_1\lambda_2 + 1}{\lambda_1 + \lambda_2}. \tag{41}$$

For the asymptotic behavior of the N-soliton solution for large t, the phase shift of each soliton after the interaction can be derived from our solution. We have following proposition:

Proposition. *For the N right-going overtaking soliton solution given by (26), the asymptotic behavior of the solution is*

$$\lim_{t \to -\infty} \zeta(x,t) = \sum_{j=1}^{N} \zeta_B(x - \lambda_j t + \Delta_j), \qquad \lim_{t \to \infty} \zeta(x,t) = \sum_{j=1}^{N} \zeta_B(x - \lambda_j t - \Delta_j), \tag{42}$$

where the phase shift of the jth soliton is given by

$$\Delta_j = \sum_{\substack{i=1,\\ i \neq j}}^{N} \operatorname{sign}(\lambda_j - \lambda_i) \frac{1}{\sqrt{3(\lambda_j^2 - 1)}} \operatorname{arccosh} \left| \frac{\lambda_j\lambda_i - 1}{\lambda_j - \lambda_i} \right|. \tag{43}$$

The phase shift for N head-on colliding soliton solution has a similar result. It has been proved in ref.[12].

3 The Camassa - Holm(CH) equation

The Camassa-Holm (CH) equation

$$u_t + 2\omega u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{44}$$

which can also be written as

$$q_t + uq_x + 2qu_x = 0, \qquad q = m + \omega = u - u_{xx} + \omega, \tag{45}$$

was derived in [13, 20] to model unidirectional nonlinear dispersive waves on a uniform layer of water with depth proportional to $4\omega^2$. The equation was first found in Ref. [21] by using the method of recursion operators. It was also found in Ref. [22] to model nonlinear waves in cylindrical hyperelastic rods, and in Ref. [23] to model the motion of a non-newtonian fluid of second grade in the limit when the viscosity tends to zero. The role of the equation within the classical model for water waves was explored and justified in Ref. [24]; similar result was obtained earlier in Ref. [25]. Recently more general equations is obtained in [26].

For the case $\omega = 0$, the soliton solution of the CH equation is peaked and its two-peakon solution was presented in [13]. It should be pointed out that the n-peakon interaction was obtained in the paper [14], while the explicit peakon-antipekon interaction is given in [15], and the stability of the peakons is proved in [16].

However this case corresponds to the waves in water with zero quiescent depth. The amplitude-depth ratio of the peakon is infinite. This violates the assumption that the ratio has to be small in deriving the model. The case $\omega > 0$, $\omega = \frac{1}{2}$ in particular, is more relevant to the application in water waves. The case $\omega \neq 0$ also represents the equation for the geodesics on the Bott-Virasoro group[27]. It should be pointed out that the case $\omega = 0$ represents the equation for geodesics on the diffeomorphism group of the line or, in the periodic case, the diffeomorphism group of the circle, cf. [17, 18]. In reference [19] it is proved that the solitary waves of the Camassa-Holm equation(for $\omega \neq 0$)are stable solitons. It is known that the CH equation has a Lax pair. It should be mentioned that the scattering problem is discussed in [28] and in the paper [29], while the periodic case is in [30, 31]. On the other hand, Parker [32] used the Bilinear form to get solitary waves of CH equation.

Regarding the multi-soliton solution of the CH equation for the case $\omega > 0$, considerable progress has been made in Ref. [33, 34], but a nice and clean solution is still not available yet. The clean solution is given in the following proposition 1 [35].

The CH equation is known to have a Lax pair[28] as follows

$$\psi_{xx} = \frac{1}{4}\psi + \lambda(m+\omega)\psi, \qquad -\infty < x < \infty \tag{46}$$

$$\psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi, \tag{47}$$

where the assumption

$$\int_{-\infty}^{\infty} (1+|x|)|m(x)|dx < \infty, \qquad m+\omega > 0,$$

has been made so that the system has only a finite number of eigenvalues, λ. The Liouville transformation

$$\phi(y) = (m(x)+\omega)^{1/4}\psi(x) = q^{1/4}\psi, \tag{48}$$

$$\frac{dy}{dx} = \sqrt{q}, \qquad q(y) = m+\omega, \tag{49}$$

converts spectral problem (46) into

$$-\frac{d^2\phi}{dy^2} + Q\phi = \mu\phi, \qquad \mu = -\frac{1}{4\omega} - \lambda, \tag{50}$$

where

$$Q(y) = \frac{1}{4q(y)} + \frac{q_{yy}(y)}{4q(y)} - \frac{3q_y^2(y)}{16q^2(y)} - \frac{1}{4\omega}. \tag{51}$$

Given $Q(y)$, in order to find a solution for u, one needs to

(i) solve equation (51) for q satisfying the condition $\lim_{|y|\to\infty} q = \omega$;

(ii) integrate (49) to build up the relation between x and y with the known q;

(iii) solve $q = u - u_{xx} + \omega$ for u with the known q;

(iv) introduce a variable t in the solution of u and make it satisfy the CH equation (44), i.e., (45).

The solutions to steps (i), (ii) and (iv), first obtained in [34], are cited here for the convenience of reader.

By using the Darboux transformation[11], the $Q(y)$ of the n-soliton for equation (51) can be generated from that for $Q(y) = 0$ in the form given by

$$Q(y) = -2\frac{\partial^2}{\partial y^2} \ln W(\Psi_1, \Psi_2, \cdots, \Psi_n), \tag{52}$$

where the Wronskian determinant W of n functions Ψ_1, Ψ_2, $\cdots$, Ψ_n is defined by

$$W(\Psi_1, \Psi_2, \cdots, \Psi_n) = \det A, \qquad A_{ij} = \frac{d^{i-1}\Psi_j}{dy^{i-1}}, \qquad i, j = 1, 2, \cdots, n. \tag{53}$$

and the functions Ψ_i $(k_1 < k_2 < \cdots < k_n)$ are defined by

$$\Psi_i(y) = \begin{cases} \cosh k_i y, & i \text{ is an odd number} \\ \sinh k_i y, & i \text{ is an even number} \end{cases}. \tag{54}$$

The solution of q satisfying equation (51) with the boundary condition $\lim_{|y|\to\infty} q = \omega$ is given in Ref. [34] as

$$\sqrt{q} = \frac{f_1 f_2}{W(f_1, f_2)} = \frac{\sqrt{\omega} f_1 f_2}{\prod_{i=1}^n \left(\frac{1}{4\omega} - k_i^2\right)}, \tag{55}$$

where f_1 and f_2 are given by

$$f_1(y) = \frac{W(\Psi_1, \cdots, \Psi_n, e^{\frac{1}{2\sqrt{\omega}}y})}{W(\Psi_1, \cdots, \Psi_n)}, \qquad f_2(y) = \frac{W(\Psi_1, \cdots, \Psi_n, e^{-\frac{1}{2\sqrt{\omega}}y})}{W(\Psi_1, \cdots, \Psi_n)}. \tag{56}$$

With the known q, the relation between x and y can be obtained by solving equation (49). The solution is given in Ref. [34] as

$$x = \ln \frac{Ef_1}{f_2} = \ln \left| \frac{f_1}{f_2} \right|, \tag{57}$$

where E, being an integration constant, is taken to be either 1 or -1 to ensure meaningfulness of the logarithmic function.

It has shown in Ref. [34] that the time variable t can be introduced by following transformation

$$y \to y - c_i\sqrt{\omega}t, \qquad \text{with} \qquad c_i = \frac{2\omega}{1-4k_i^2\omega} \tag{58}$$

in equation (54).

Lemma 1. If y and u are treated as a functions of (x,t), then the solution of the CH equation can be written as

$$u = -\frac{\partial y}{\partial t}\Big/\frac{\partial y}{\partial x}. \tag{59}$$

Proof. Equation (49) can be written as

$$\frac{\partial y}{\partial x} = \sqrt{u - u_{xx} + \omega}, \tag{60}$$

therefore

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right) = \frac{u_t - u_{xxt}}{2\sqrt{u-u_{xx}+\omega}} = \frac{-3uu_x - 2\omega u_x + 2u_xu_{xx} + uu_{xxx}}{2\sqrt{u-u_{xx}+\omega}}, \tag{61}$$

where the CH equation was used in the last equality. We also have another equality

$$\begin{aligned}\frac{\partial}{\partial x}\left(u\frac{\partial y}{\partial x}\right) &= \frac{\partial}{\partial x}\left(u\sqrt{u-u_{xx}+\omega}\right)\\ &= u_x\sqrt{u-u_{xx}+\omega} + u\frac{u_x - u_{xxx}}{2\sqrt{u-u_{xx}+\omega}}\\ &= \frac{3uu_x + 2\omega u_x - 2u_xu_{xx} - uu_{xxx}}{2\sqrt{u-u_{xx}+\omega}}.\end{aligned} \tag{62}$$

Comparing equations (61) and (62) yields

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) = -\frac{\partial}{\partial x}\left(u\frac{\partial y}{\partial x}\right). \tag{63}$$

Integrating with respect to x gives

$$\frac{\partial y}{\partial t} = -u\frac{\partial y}{\partial x}, \qquad \text{or} \qquad u = -\frac{\partial y}{\partial t}\Big/\frac{\partial y}{\partial x}, \tag{64}$$

where the integration constant is taken to be zero by applying the result to a particular solution, e.g., single soliton solution. Q.E.D.

Proposition 1. The multiple-soliton solution, $u(x,t)$, of the CH equation is given in a parametric form by

$$u(y,t) = \frac{\partial}{\partial t}\left(\ln\frac{f_1}{f_2}\right), \tag{65}$$

$$x(y,t) = \ln\left|\frac{f_1}{f_2}\right|, \tag{66}$$

where $f_1(y,t)$ and $f_2(y,t)$ are given by (56) with following Ψ_i functions ($k_1 < k_2 < \cdots < k_n$)

$$\Psi_i(y,t) = \left\{ \begin{array}{ll} \cosh k_i(y - c_i\sqrt{\omega}\ t), & i \text{ is an odd number} \\ \sinh k_i(y - c_i\sqrt{\omega}\ t), & i \text{ is an even number} \end{array} \right. , \qquad k_i = \frac{1}{2\sqrt{\omega}}\sqrt{1-\frac{2\omega}{c_i}}. \quad (67)$$

Proof. The relation between x, y and t is written implicitly as an equation

$$F(x,y,t) = x - \ln\left|\frac{f_1(y,t)}{f_2(y,t)}\right| = 0, \quad (68)$$

and

$$dF = F_x dx + F_y dy + F_t dt = 0, \quad (69)$$

therefore

$$\frac{\partial y}{\partial t} = -\frac{F_t}{F_y}, \qquad \frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{1}{F_y}. \quad (70)$$

With Lemma 1, we have

$$u = -\frac{\partial y}{\partial t}\Big/\frac{\partial y}{\partial x} = -F_t = \frac{\partial}{\partial t}\left(\ln\frac{f_1}{f_2}\right). \quad (71)$$

Q.E.D.

Example 1, Single soliton. For a solution with one soliton, $\Psi_1 = \cosh\xi_1$, where $\xi_1 = k_1(y - c_1\sqrt{\omega}t)$, with $k_1 = \frac{1}{2\sqrt{\omega}}\sqrt{1-\frac{2\omega}{c_1}}$. The two functions f_1 and f_2, obtained from equation (56) are

$$\begin{aligned} f_1 &= \exp\left[\frac{1}{2\sqrt{\omega}}y\right]\left(\frac{1}{2\sqrt{\omega}} - k_1\tanh\xi_1\right), \\ f_2 &= \exp\left[-\frac{1}{2\sqrt{\omega}}y\right]\left(-\frac{1}{2\sqrt{\omega}} - k_1\tanh\xi_1\right). \end{aligned}$$

Applying Proposition 1 gives a single-soliton solution of the CH equation in a parametric form

$$\begin{aligned} u(y,t) &= \frac{\partial}{\partial t}\left(\ln\frac{f_1}{f_2}\right) = \frac{\partial}{\partial t}\left(\ln\frac{\frac{1}{2\sqrt{\omega}} - k_1\tanh\xi_1}{-\frac{1}{2\sqrt{\omega}} - k_1\tanh\xi_1}\right) = \frac{c_1 - 2\omega}{1 + (2\omega/c_1)\sinh^2\xi_1}, \\ x(y,t) &= \ln\left|\frac{f_1}{f_2}\right| = \frac{y}{\sqrt{\omega}} + \ln\frac{\frac{1}{2\sqrt{\omega}} - k_1\tanh\xi_1}{\frac{1}{2\sqrt{\omega}} + k_1\tanh\xi_1}, \end{aligned}$$

which has been obtained in [33, 34, 36, 37, 38].

Example 2, Two solitons. For a solution with two solitons, $\Psi_1 = \cosh\xi_1$ and $\Psi_2 = \sinh\xi_2$, where $\xi_i = k_i(y - c_i\sqrt{\omega}t)$, with $k_i = \frac{1}{2\sqrt{\omega}}\sqrt{1-\frac{2\omega}{c_i}}$, $i = 1, 2$, $c_1 < c_2$. The two functions f_1 and f_2, obtained from equation (56) are

$$f_1 = \frac{\exp\left[\frac{1}{2\sqrt{\omega}}y\right]\begin{vmatrix} \cosh\xi_1 & \sinh\xi_2 & 1 \\ k_1\sinh\xi_1 & k_2\cosh\xi_2 & \frac{1}{2\sqrt{\omega}} \\ k_1^2\cosh\xi_1 & k_2^2\sinh\xi_2 & \frac{1}{4\omega} \end{vmatrix}}{\begin{vmatrix} \cosh\xi_1 & \sinh\xi_2 \\ k_1\sinh\xi_1 & k_2\cosh\xi_2 \end{vmatrix}} = \exp\left[\frac{1}{2\sqrt{\omega}}y\right]\frac{\Delta_1}{\Delta},$$

$$f_2 = \frac{\exp\left[-\frac{1}{2\sqrt{\omega}}y\right]\begin{vmatrix} \cosh\xi_1 & \sinh\xi_2 & 1 \\ k_1\sinh\xi_1 & k_2\cosh\xi_2 & -\frac{1}{2\sqrt{\omega}} \\ k_1^2\cosh\xi_1 & k_2^2\sinh\xi_2 & \frac{1}{4\omega} \end{vmatrix}}{\begin{vmatrix} \cosh\xi_1 & \sinh\xi_2 \\ k_1\sinh\xi_1 & k_2\cosh\xi_2 \end{vmatrix}} = \exp\left[-\frac{1}{2\sqrt{\omega}}y\right]\frac{\Delta_2}{\Delta},$$

where

$$\Delta_1 = a - b, \qquad \Delta_2 = a + b, \qquad \Delta = 4\omega(k_2 - k_1\tanh\xi_1\tanh\xi_2),$$
$$a = k_2(1-4k_1^2\omega) - k_1(1-4k_2^2\omega)\tanh\xi_1\tanh\xi_2, \qquad b = 2(k_2^2-k_1^2)\sqrt{\omega}\tanh\xi_2.$$

One may verify that both Δ_1 and Δ_2 are positive under the condition that $\frac{1}{2\sqrt{\omega}} > k_2 > k_1 > 0$. Applying Proposition 1 gives a two-soliton solution of the CH equation in a parametric form

$$u(y,t) = \frac{\partial}{\partial t}\left(\ln\frac{f_1}{f_2}\right) = \frac{\partial}{\partial t}\left(\ln\frac{\Delta_1}{\Delta_2}\right) = \frac{\Delta_{1t}}{\Delta_1} - \frac{\Delta_{2t}}{\Delta_2} = \frac{a_t - b_t}{a-b} - \frac{a_t+b_t}{a+b} = \frac{D}{A},$$
$$x(y,t) = \ln\left|\frac{f_1}{f_2}\right| = \frac{y}{\sqrt{\omega}} + \ln\frac{\Delta_1}{\Delta_2},$$

where

$$D = 2a_t b - 2ab_t = D_1 + D_2\tanh\xi_1\tanh\xi_2 + D_3\tanh^2\xi_1\tanh^2\xi_2 + D_4\tanh^2\xi_2,$$
$$D_1 = \frac{8k_2^2(k_2^2-k_1^2)\omega^2(1-4k_1^2\omega)}{(1-4k_2^2\omega)}, \qquad D_2 = 0,$$
$$D_3 = -\frac{8k_1^2(k_2^2-k_1^2)\omega^2(1-4k_2^2\omega)}{(1-4k_1^2\omega)}, \qquad D_4 = -\frac{8(k_2^2-k_1^2)^2\omega^2(1-16k_1^2k_2^2\omega^2)}{(1-4k_1^2\omega)(1-4k_2^2\omega)};$$
$$A = a^2 - b^2 = A_1 + A_2\tanh\xi_1\tanh\xi_2 + A_3\tanh^2\xi_1\tanh^2\xi_2 + A_4\tanh^2\xi_2,$$
$$A_1 = k_2^2(1-4k_1^2\omega)^2, \qquad A_2 = -2k_1k_2(1-4k_1^2\omega)(1-4k_2^2\omega),$$
$$A_3 = k_1^2(1-4k_2^2\omega)^2, \qquad A_4 = -4(k_2^2-k_1^2)^2\omega.$$

With some algebra, one may easily verify that the solution agrees with the one provided in [34].

We use $u_1(x,t;c)$ and $u_2(x,t;c_1,c_2)$ to denote single-soliton and two-soliton solutions. Asymptotic analysis shows that the two-soliton solution becomes the summation of two single-soliton solutions, i.e.,

$$\lim_{t\to-\infty} u_2(x,t;c_1,c_2) = u_1(x+\Delta x_2,t;c_2) + u_1(x-\Delta x_1,t;c_1),$$
$$\lim_{t\to+\infty} u_2(x,t;c_1,c_2) = u_1(x-\Delta x_2,t;c_2) + u_1(x+\Delta x_1,t;c_1),$$

where the phase-shifts of the two solitons are

$$\begin{aligned} \Delta x_2 &= \frac{1}{2k_2\sqrt{\omega}} \ln \frac{k_2+k_1}{k_2-k_1} - \ln \frac{1+2k_1\sqrt{\omega}}{1-2k_1\sqrt{\omega}}, \\ \Delta x_1 &= \frac{1}{2k_1\sqrt{\omega}} \ln \frac{k_2+k_1}{k_2-k_1} - \ln \frac{1+2k_2\sqrt{\omega}}{1-2k_2\sqrt{\omega}}. \end{aligned}$$

The results agree with those provided in [33].

Example 3, Three solitons. For a solution with three solitons, $\Psi_1 = \cosh \xi_1$, $\Psi_2 = \sinh \xi_2$ and $\Psi_3 = \cosh \xi_3$, where $\xi_i = k_i(y - c_i\sqrt{\omega}t)$, with $k_i = \frac{1}{2\sqrt{\omega}}\sqrt{1-\frac{2\omega}{c_i}}$, $i = 1, 2, 3$, $c_1 < c_2 < c_3$. The two functions f_1 and f_2, obtained from equation (56) are

$$f_1 = \exp\left[\frac{1}{2\sqrt{\omega}}y\right]\frac{\Delta_1}{\Delta}, \qquad f_2 = \exp\left[-\frac{1}{2\sqrt{\omega}}y\right]\frac{\Delta_2}{\Delta},$$

where

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} \cosh\xi_1 & \sinh\xi_2 & \cosh\xi_3 & 1 \\ k_1\sinh\xi_1 & k_2\cosh\xi_2 & k_3\sinh\xi_3 & \frac{1}{2\sqrt{\omega}} \\ k_1^2\cosh\xi_1 & k_2^2\sinh\xi_2 & k_3^2\cosh\xi_3 & \frac{1}{4\omega} \\ k_1^3\sinh\xi_1 & k_2^3\cosh\xi_2 & k_3^3\sinh\xi_3 & \frac{1}{8\sqrt{\omega^3}} \end{vmatrix}, \\ \Delta_2 &= \begin{vmatrix} \cosh\xi_1 & \sinh\xi_2 & \cosh\xi_3 & 1 \\ k_1\sinh\xi_1 & k_2\cosh\xi_2 & k_3\sinh\xi_3 & -\frac{1}{2\sqrt{\omega}} \\ k_1^2\cosh\xi_1 & k_2^2\sinh\xi_2 & k_3^2\cosh\xi_3 & \frac{1}{4\omega} \\ k_1^3\sinh\xi_1 & k_2^3\cosh\xi_2 & k_3^3\sinh\xi_3 & -\frac{1}{8\sqrt{\omega^3}} \end{vmatrix}, \\ \Delta &= \begin{vmatrix} \cosh\xi_1 & \sinh\xi_2 & \cosh\xi_3 \\ k_1\sinh\xi_1 & k_2\cosh\xi_2 & k_3\sinh\xi_3 \\ k_1^2\cosh\xi_1 & k_2^2\sinh\xi_2 & k_3^2\cosh\xi_3 \end{vmatrix}. \end{aligned}$$

Applying Proposition 1 gives a three-soliton solution of the CH equation in a parametric form

$$\begin{aligned} u(y,t) &= \frac{\partial}{\partial t}\left(\ln\frac{f_1}{f_2}\right) = \frac{\partial}{\partial t}\left(\ln\frac{\Delta_1}{\Delta_2}\right) = \frac{\Delta_{1t}}{\Delta_1} - \frac{\Delta_{2t}}{\Delta_2}, \\ x(y,t) &= \ln\left|\frac{f_1}{f_2}\right| = \frac{y}{\sqrt{\omega}} + \ln\left|\frac{\Delta_1}{\Delta_2}\right|. \end{aligned}$$

Their analytical expressions are long, therefore not presented here for brevity. The analytical solution for multiple solitons can be presented in a similar formula with the same procedure.

The function Ψ in the Darboux transformation in the examples presented so far are chosen carefully for the applications in modelling water waves. Other solutions may be generated by choosing other function Ψ. The interesting features of those solutions are under investigation and will be reported in a subsequent study. A comparative study on the soliton solutions of different integrable water wave equations, including unidirectional models of the KdV, the CH and higher order KdV and bidirectional model of Boussinesq equations [34], is another topic for further research.

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