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A unique continuation principle for steady symmetric water waves with vorticity

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Abstract

We consider a symmetric, steady, and periodic water wave. It is shown that a locally vanishing vertical velocity component implies a flat or oscillating surface profile.

1 Introduction

From focusing mainly on irrotational flows, mathematical research has shifted partly towards water waves travelling on currents with vorticity. Several recent papers deal with this issue [1, 2, 4, 5, 6, 7, 11, 13, 14, 15] with much interest devoted to steady symmetric periodic waves with vorticity. Whereas waves travelling into still water can be suitably regarded as irrotational [10], other situations require taking the vorticity into account. As the wind blows over a flat water surface capillary waves of small amplitude arise, due to the restoring action of surface tension. As the process continues, the waves grow slightly and the influence of gravity becomes relevant. We get gravity-capillary waves, governed by surface tension as well as by gravity. Finally, as the waves further increase in amplitude, the role of gravity outplays that of surface tension. This is the dominating regime of the open sea: gravity waves. An important effect of this process is vorticity. For water waves created by the wind vorticity appears as a process starting first at the surface and thereafter penetrating deeper into the fluid. The presence of a nonvanishing vorticity in the fluid is guaranteed when studying waves propagating into a water flow with a current.

We shall consider such waves that are periodic and steady, i.e. they travel with a constant speed and unchanged shape. Under a growth condition on the vorticity, it was recently proved that for such gravity waves, strict monotonicity of the surface between trough and crest implies symmetry around the crest [1, 2]. In [8] we made the observation that such gravity waves are nowhere flat (unless they are so everywhere). Whereas those papers draw conclusions about the entire fluid motion from the surface behaviour, in this paper we investigate the effects of having a small region within the fluid where the water moves solely horizontally. In the case of irrotational flow or of a flow of constant vorticity, this implies that there is no vertical movement in all of the fluid domain. In that setting, it is a consequence of the real analyticity of the vertical velocity. When a general vorticity is present it might be possible that this result does not hold. However, we prove that if a symmetric wave has finitely many peaks and troughs in each period with a strictly monotone profile in between, any small region of purely horizontal flow forces the surface to be flat.

2 Formulation and preliminaries

Let $\eta \in C^3(\mathbb{R}, \mathbb{R})$ be the surface, periodic of period L, with the trough $\eta_{\min}$ at $x = \pm L/2$, and the crest $\eta_{\max}$ at $x = 0$. Assume that the origin is located at the mean water level, i.e. $\int_0^L \eta(x)\,dx = 0$. Define the fluid domain to be

$$\Omega_\eta = \{(x, y) \in R^2\,;\; x \in \mathbb{R}, -d < y < \eta(x)\},$$

where we accept also $d = \infty$, i.e. the setting of infinite depth. Let $u, v \in C^2(\overline{\Omega_\eta}, \mathbb{R})$ be the horizontal and vertical velocity, respectively, with the properties that for a fixed speed constant $c > 0$, we have $u - c < 0$. This last assumption is supported by experimental and field data [12]. The stream function

$$\psi(x, y) = \psi_0 - \int_0^x v(\xi, -d_0)\,d\xi + \int_{-d_0}^y [u(x, \xi) - c]\,d\xi,$$

where $-d \le -d_0 < \eta_{\min}$, then satisfies $-\psi_x = v$, $\psi_y = u - c < 0$, and $\psi(x, \eta(x)) = 0$. Also, let the vorticity function $\gamma \in C^1$ be a real-valued function defined in the range of ψ, and let $\alpha > 0$ represent the surface tension. With these prerequisites, we deal with the problem (see e.g. [5] for further details)

$$\begin{cases} \Delta\psi = -\gamma(\psi), & (x, y) \in \Omega_\eta \\ |\nabla\psi|^2 + 2gy - \alpha\eta''(1 + {\eta'}^2)^{-3/2} = C, & y = \eta(x) \\ \psi_x + \psi_y\eta' = 0, & y = \eta(x) \\ \psi_x \to 0 \text{ as } y \to -d, & \text{uniformly for } x \in \mathbb{R}. \end{cases} \tag{2.1}$$

In order to simplify the proofs to come, we recall two classical results (see [9]). Here $\Omega \subseteq R^2$ denotes a region in the plane with a C^2 boundary[1].

Lemma 2.1 (Hopf's boundary point lemma)**.** *For any function $c \in C(\overline{\Omega}, \mathbb{R})$, put $\mathcal{L} = \Delta + c(x)$, and let $u \in C^2(\overline{\Omega})$ be such that $\mathcal{L}u \ge 0$. If there exists $x_0 \in \partial\Omega$ such that*

$$0 = u(x_0) > u(x)_{x\in\Omega},$$

then the directional derivative $\frac{\partial u}{\partial \mu}(x_0) > 0$ for any normal μ pointing outwards from Ω at x_0.

Lemma 2.2 (The strong maximum principle)**.** *For a non-positive function $c \in C(\overline{\Omega}, \mathbb{R})$, let $\mathcal{L} = \Delta + c(x)$, and let $u \in C^2(\overline{\Omega})$ be such that $\mathcal{L}u \ge 0$. If there exists $x_0 \in \Omega$ such that*

$$0 \le \max_{x\in\overline{\Omega}} u(x) = u(x_0),$$

then $u(x) \equiv u(x_0)$ throughout $\overline{\Omega}$. Moreover, if $u(x_0) = 0$, the sign condition on the function c is not relevant.

For further use, we note that differentiating the first line of (2.1) with respect to x yields

$$\mathcal{L}v = (\Delta + \gamma'(\psi))v = 0. \tag{2.2}$$

[1]Hopf's boundary point lemma originally concerns a boundary point satisfying an interior ball condition. Our regularity assumption guarantees that this is the case.

3 Results

3.1 Surface profile for monotone symmetric water waves

Before dealing with our main problem we would like to comment on [8]. One need not restrict the attention only to gravity waves. Indeed, it has nothing to do with the Bernoulli surface condition, and is therefore equally valid for capillary and capillary-gravity waves. Moreover, the formulation with a locally flat surface may be loosened. Without giving the explicit proof (which is based on Lemma 2.1 and Lemma 2.2), we state here a somewhat improved version of the main result in [8]: symmetric monotone waves are nowhere flat.

Theorem 3.1. *For a steady symmetric water wave with non-increasing vorticity $\gamma'(\psi) \leq 0$, if the surface profile is monotone from trough to crest, i.e. $\eta'(x) \geq 0, -L/2 \leq x \leq 0$, it is in fact strictly monotone. More precisely, $\eta'(x) = 0$ implies that $\eta'''(x) > \eta''(x) = 0$ for any $x \in (-L/2, 0)$.*

Remark 3.2. In [1, 2] it is proved that if the vorticity is non-increasing with greater depth, i.e. $\gamma'(\psi) \leq 0$, and we deal with gravity waves, then strict monotonicity of the surface profile between troughs and crests implies symmetry. It is thus quite natural to assume symmetry of a monotone wave.

Remark 3.3. As follows from Lemma 2.2, steady water waves given by (2.1) consists of open regions where $v > 0$ and $v < 0$, respectively, and they are separated by closed sets where $v = 0$. This is true regardless of the vorticity and the surface conditions.

3.2 Surface profiles for water waves with locally vanishing vertical velocity

Since proving the main theorem of this section is a somewhat lengthy process, we first state it. The assertion is equally valid for capillary waves, gravity-capillary waves and gravity waves, as well as for finite and infinite depth.

Theorem 3.4. *Let (u, v, η) define a steady symmetric periodic water wave with finitely many peaks and troughs in each period, in between which the surface profile is strictly rising or falling, i.e. $\eta' \neq 0$ here. If there is an open ball $\mathcal{B} \subseteq \Omega_\eta$ where the vertical velocity v vanishes, then the surface profile is flat.*

Remark 3.5. As was noted in the Introduction, if the vorticity is constant, the surface profile must be flat. This follows since v is then harmonic.

Throughout this section, we deal with an open *bounded* region Ω with a boundary given by a piecewise continuous curve, and a function $v \in C^2(\overline{\Omega})$, satisfying the maximum principle of Lemma 2.2 for both v and $-v$. We say that a nonempty open region $\Omega_0 \subset \Omega$ is an Ω_0-set if $v = 0$ on Ω_0 and Ω_0 is maximal, i.e.

$$\Omega_0 = \sup\{\Omega_* \colon v(\Omega_*) = 0,\, \Omega_0 \subseteq \Omega_* \subseteq \Omega\},$$

Ω_* being an open region. In a similar fashion we say that a maximal region $\Omega_+ \subseteq \overline{\Omega}$ with $v(\Omega_+) > 0$ is an Ω_+-set, and a maximal region $\Omega_- \subseteq \overline{\Omega}$ with $v(\Omega_-) < 0$ is an Ω_--set. Note that an $\Omega_\pm$-set need not be open since it may contain some part of the free boundary $\partial\Omega$.

Lemma 3.6. *Let Ω_0 be an Ω_0-set and let $x_0 \in \partial\Omega_0 \cap \Omega$. Then there is a sequence $\{x_+^n\}$ and a sequence $\{x_-^n\}$, both converging to x_0 as $n \to \infty$, for which $v(x_+^n) > 0$ and $v(x_-^n) < 0$, $n \in \mathbb{N}$.*

Proof. By definition $\mathcal{B}(x_0, \frac{1}{n})$ contains inner points for which $v = 0$ as well as points for which $v \neq 0$. If v has the same sign in all of $\mathcal{B}(x_0, \frac{1}{n})$, then either v or $-v$ attains its maximum $v = 0$ at an interior point, so that by the strong maximum principle $v \equiv 0$ in $\mathcal{B}(x_0, \frac{1}{n})$. This however contradicts the definition of the boundary of Ω_0, and thus there exist $x_+^n, x_-^n \in \mathcal{B}(x_0, \frac{1}{n})$ with $v(x_+^n) > 0$ and $v(x_-^n) < 0$. ■

Before moving on, we note that if Ω_0 is an Ω_0-set, then $\partial\Omega_0 \cap \Omega$ is either void or it contains infinitely many points.

Lemma 3.7. *Any $x_+ \in \Omega$ with $v(x_+) > 0$ is part of an Ω_+-set intersecting the boundary $\partial\Omega$. The pre-image of this intersection is open.*

Proof. Let Ω_+ be the Ω_+-set containing x_+. Suppose that this set does not intersect $\partial\Omega$. By continuity and the definition of Ω_+, $v(\partial\Omega_+) = 0$ and v attains its maximum at an interior point of $\overline{\Omega}_+$. Lemma 2.2 then forces $v \equiv \alpha > 0$ in $\overline{\Omega}_+$, contradicting $v(\partial\Omega_+) = 0$. Hence $\partial\Omega \cap \Omega_+$ is non-void. It now follows by continuity of v that the pre-image of any connected part of this intersection, regarded as part of the curve $\partial\Omega$, is open. ■

We can now state the following: if we have an open ball with $v = 0$ inside the region Ω, then either v oscillates on $\partial\Omega$, or Ω_0 is connected to the boundary $\partial\Omega$ via Ω_+-sets.

Lemma 3.8. *Let $\Omega_0 \subseteq \Omega$ be an Ω_0-set. Then either*

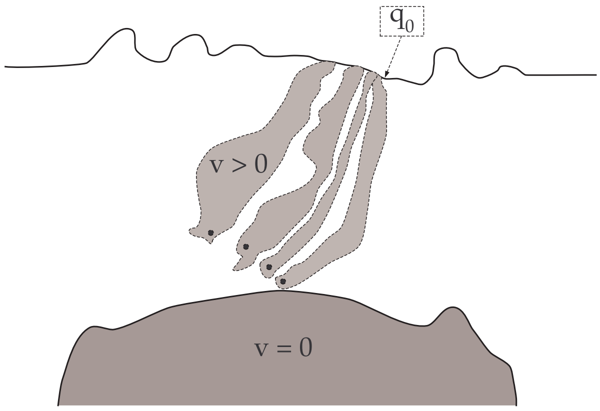
i. the boundary $\partial\Omega$ includes a limit point x_0 for two sequences $\{x_0^n\}, \{x_+^n\} \subset \partial\Omega$, with $v(x_+^n) > v(x_0^n) = 0$, $n \in \mathbb{N}$, or

ii. for any $x_0 \in \partial\Omega_0 \cap \Omega$ there exists an Ω_+-set intersecting $\partial\Omega$ and with distance 0 to x_0. Moreover, the intersection $\Omega_+ \cap \Omega$ is open.

Proof. By Lemma 3.6 there is a sequence of points $\{x_+^n\}$ converging to x_0, all satisfying $v(x_+^n) > 0$. Also, for any such x_+^n, there is a corresponding set Ω_+^n as in Lemma 3.7 in which v is positive and that includes some point p_n at the boundary $\partial\Omega$. Since the support of v on the boundary,

$$\mathcal{S} = \overline{\{x \in \partial\Omega;\, v(x) \neq 0\}},$$

is compact, by Bolzano-Weierstrass $\{p_n\}_{n\in\mathbb{N}}$ has a point of accumulation q_0 in $\overline{\mathcal{S}}$.



If v oscillates on $\partial\Omega$, i.e. if (i) holds, there is nothing to prove. Suppose not. $v(p_n) > 0$ for all n then implies that there is a connected subset of $\partial\Omega$ of positive curve length in which $v > 0$ with distance 0 to q_0. Thus $v > 0$ in a connected part of $\mathcal{S}$ including infinitely many p_n's, implying that infinitely many Ω_+^n's are in fact connected in their union $\bigcup \Omega_+^{n_k}$ which we call $\Omega_+^{q_0}$. Then there is a subsequence

$$\{x_+^{n_k}\} \subseteq \{x_+^n\} \quad \text{with } x_+^{n_k} \in \Omega_+^{q_0}, \ \lim_{k\to\infty} x_+^{n_k} = x_0.$$

Hence $\Omega_+^{q_0}$ has distance 0 to x_0, and $\Omega_+^{q_0}$ is a connected set including some part of the surface and reaching to $x_0 \in \Omega_0$. Furthermore, since $\Omega_+^n \cap \Omega$ is open by Lemma 3.7, so is $\Omega_+^{q_0} \cap \Omega$. ■

Lemma 3.9. *Let $\Omega_0 \subset \Omega$ be an Ω_0-set, and suppose that v does not oscillate on $\partial\Omega$, so that (ii) of Lemma 3.8 holds. Then, for $x_1 \neq x_2$ in $\partial\Omega_0 \cap \Omega$, there are two corresponding Ω_+-sets, Ω_+^1 and Ω_+^2, with*

$$d(\Omega_+^j, x_j) = 0, \quad \Omega_+^j \cap \partial\Omega \neq \emptyset, \qquad j = 0, 1$$

and

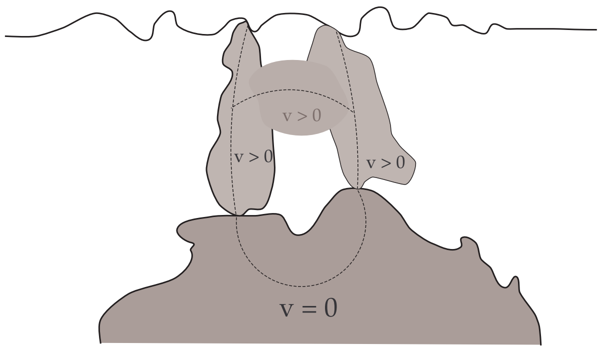
$$d(\Omega_+^1, \Omega_+^2) > 0.$$

Moreover, these two sets are separated at the boundary curve $\partial\Omega$ by a curve piece where $v < 0$.

Proof. By Lemma 3.8, there exist sets $\Omega_+^{q_1}$ and $\Omega_+^{q_2}$, corresponding to x_1 and x_2, respectively. Here, $\Omega_+^{q_j}$ is a maximal connected set for which $v > 0$ with $d(\Omega_+^{q_j}, x_j) = 0$ and including a point $q_j \in \partial\Omega$. Suppose, for a contradiction, that $\Omega_+^{q_1} \cap \Omega_+^{q_2} \neq \emptyset$. The continuity of v then implies

$$\Omega_+^{q_1} \cap \Omega_+^{q_2} \cap \Omega \neq \emptyset,$$

and we denote this intersection by Ω_+.



Ω_+ being open and connected, it is possible to construct a simple continuous curve $\gamma_+ \colon (0,1) \longrightarrow \Omega_+$, with

$$\lim_{t\downarrow 0} \gamma_+(t) = x_1 \neq x_2 = \lim_{t\uparrow 1} \gamma_+(t).$$

Since also Ω_0 is open and connected there exists a simple continuous curve $\gamma_0 \colon [1,2] \longrightarrow \Omega_0$, with

$$\gamma_0(2) = x_1 \neq x_2 = \gamma_0(1).$$

γ_+ and γ_- have no points in common since $v > 0$ in Ω_+ and $v = 0$ in Ω_0. Combining these curves we get a closed continuous simple curve $\gamma \colon [0,2] \longrightarrow \Omega$. By the Jordan Curve Theorem this curve separates the plane into two regions, one of them being the curve's interior, which we call $\mathring{\gamma}$. Then, $v \geq 0$ on the boundary $\partial\mathring{\gamma}$ so that applying Lemma 2.2 shows that $v(\mathring{\gamma}) \geq 0$.

Now, pick $x_0 \in \Omega_0 \cap \mathring{\gamma}$ and $x_+ \in \Omega_+ \cap \mathring{\gamma}$. Since $v(x_0) = 0$ and $v(x_+) > 0$ and these points are connected by arcs within $\mathring{\gamma}$ there is an interior point $\mathring{x} \in \mathring{\gamma}$ which belongs to the boundary $\partial\Omega_0$. But then Lemma 3.6 implies that $\mathring{x}$ is an accumulation point for a sequence $\{x_-^n\}$ of points for which $v < 0$. This contradicts $v(\mathring{\gamma}) \geq 0$ and shows that

$$\Omega_+^{q_1} \cap \Omega_+^{q_2} = \emptyset.$$

Suppose then that $\Omega_+^{q_1}$ and $\Omega_+^{q_2}$ were connected on the boundary $\partial\Omega$ via a set for which $v \geq 0$. Since $\partial\Omega$ is a piecewise continuous curve we could follow the same method as above and once again create an open region $\mathring{\gamma}$ for which $v \geq 0$, contradicting Lemma 3.6. Hence, $\Omega_+^{q_1}$ and $\Omega_+^{q_2}$ are separated at the boundary $\partial\Omega$ by a curve piece where $v < 0$. Take $\Omega_+^{q_j} = \Omega_+^j$, $j = 1, 2$. ■

We are now in a position to state the main lemma. Theorem 3.4 is then just an application of this.

Lemma 3.10. *Let $\Omega_0 \subseteq \Omega$ be an Ω_0-set, and suppose that v does not oscillate on the boundary $\partial\Omega$, i.e. (ii) of Lemma 3.8 holds. Then $\Omega_0 = \Omega$.*

Proof. Suppose that $\Omega_0 \subset \Omega$ in the strict sense. Iterated application of Lemma 3.9 to a sequence of points $\{x_n\} \in \partial\Omega_0$, shows that there is a corresponding sequence of points,

$$\{y_n\} \subset \partial\Omega, \qquad v(y_n) > 0,$$

all pairwise separated on $\partial\Omega$ by curve pieces where $v < 0$. But since $\partial\Omega$ is compact by assumption, the Bolzano-Weierstrass lemma implies that there is an accumulation point $y = \lim_{k\to\infty} y_{n_k}$, and it follows that v oscillates near y. Since this contradicts the assumptions, we must have $\Omega_0 = \Omega$. ■

Proof of Theorem 3.4. It is enough to consider

$$\Omega = \{(x, y) \in \Omega_\eta\,;\ -L/2 < x < 0\}.$$

Here, symmetry forces $v(x) = 0$ for $x = 0$, $x = -L/2$, so v certainly does not oscillate on the vertical boundaries. As in (2.1) we have that

$$v(x, y) \to 0 \text{ as } y \to -d \quad \text{uniformly in } x \in \mathbb{R}.$$

In view of (2.2), applying the maximum principle of Lemma 2.2 to v on a suitable cut-off

$$\Omega_n = \{(x, y) \in \Omega\,;\ -n < y < \eta(x)\},$$

using $n = d$ in the case of finite depth, we find that any region where $v > 0$ or $v < 0$ must reach the surface $\{(x, \eta(x))\,;\ x \in [-L/2, 0]\}$. Thus this is the only part of the boundary that is of interest. From the surface condition $\psi_x + \psi_y\eta' = 0$ and the assumption that $\psi_y < 0$, we deduce that if η' does not oscillate on the surface, nor does $v = -\psi_x$. The proposition is then an immediate consequence of Lemma 3.10. ■

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