

On Nonlinear Differential Equations That Describe Localized Processes

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Abstract

In this paper we want to characterize nonlinear differential equations that describe processes allowing a localization operation in each subdomain of domain in which we consider the process. We formulate this localization condition by means of visual representations and give this operation a mathematical sense. Then we obtain a general form for such equation as well as put in it certain general physical contents, taking into account the fact that nonlinear operators from physically intelligent equations satisfy this condition.

In many problems of mechanics and physics we often meet a situation when a physical process running in a medium allows a localization in each subdomain to the extent that the influence of external to the subdomain processes and forces can be replaced by boundary forces (boundary conditions) so that inside of the subdomain the process runs in the same way as it would run without such intervention. Below we offer a mathematical formulation of possibility of such replace (see below the condition (L)) which we call a condition of physical localization, and explain what equations satisfy this condition.

Let $\mathcal{O} \in \mathbb{R}^n$ be a domain filled out a medium and let function u be a characteristic of the process, which will be called an amplitude; f be external influences, which will be named forces, and let the equation describing the process has the form (t – time, $a_{\alpha\beta\ldots\gamma} \in \mathbb{C}$)

$$u'_t - Pu = f, \quad (1)$$

where

$$Pu = \sum a_{\alpha\beta\ldots\gamma} (D^\alpha u)^{k_\alpha} (D^\beta u)^{k_\beta} \cdot \ldots \cdot (D^\gamma u)^{k_\gamma}$$

is a polynomial of derivatives of the function u , $a_{\alpha\beta\ldots\gamma} \in \mathbb{C}$ are constant coefficients, D^β – partial derivative of order β (multiindex) and let $P0 = 0$. Instead of (1) one can consider the equations of forms $Pu = f$, $u'' - Pu = f$ or systems. It is clearly that if $u_j \rightarrow u_0$ in $C^\infty(\mathcal{O})$ then $Pu_j \rightarrow Pu_0$ in $C^\infty(\mathcal{O})$. Let Ω is any subdomain with smooth boundary in \mathcal{O} , $\theta_\Omega = 1$ in Ω , $\theta_\Omega = 0$ outside Ω , $\theta_{\Omega,j} = \theta_\Omega * \delta_j$, $*$ is the convolution, δ_j is a δ -sequence, i.e. $C_0^\infty(\mathcal{O}) \ni \delta_j \rightarrow \delta$ in $\mathcal{D}'(\mathcal{O})$, $\delta(x)$ is the Dirac measure.

We offer the following condition of physical localization:

for each smooth solution of equation (1) and each subdomain Ω
 there exists a distribution $f_{\partial\Omega}$, $\text{supp } f_{\partial\Omega} \subset \partial\Omega$ with property
 $(\partial/\partial t + P)(u \cdot \theta_{\Omega,j}) \xrightarrow{j \rightarrow \infty} f \cdot \theta_{\Omega} + f_{\partial\Omega}$ in the space $\mathcal{D}'(\mathcal{O})$

or (forgetting the term $\partial u/\partial t$) so:

for each smooth in \mathcal{O} function u and each subdomain Ω
 there exists a distribution $f_{\partial\Omega}$, $\text{supp } f_{\partial\Omega} \subset \partial\Omega$ with property (L)
 $P(u \cdot \theta_{\Omega,j}) \xrightarrow{j \rightarrow \infty} Pu \cdot \theta_{\Omega} + f_{\partial\Omega}$ in the space $\mathcal{D}'(\mathcal{O})$.

Physically it means that for each subdomain Ω and small neighborhood V of its boundary $\partial\Omega$, for any amplitude there exist forces which are nonzero only in V such that corresponding amplitude equals to zero outside of $V \cup \Omega$, but inside of $\Omega \setminus V$ the amplitude of the process is the same as without such intervention of these forces. In other words, there exists a force wall not influencing upon process in Ω and not releasing it from Ω .

One can understand a mathematical sense of this condition as that for each subdomain Ω with smooth boundary the influence of external to Ω part of a solution onto interior part of the solution may be replaced by a setting of boundary values of this solution and its derivatives as in linear case. Here we have in mind the following. If we will continue a solution u of some linear differential equation $Lu = f$ in Ω by means of zero outside of Ω , substitute the function $\tilde{u} = u \theta_{\Omega}$ (let $u \in C^\infty(\mathbb{R}^n)$) in the equation, multiple the equation on an arbitrary smooth ϕ and integrate then we will obtain

$$(L\tilde{u}, \phi)_{L_2(\mathbb{R}^n)} = \int_{\Omega} f \bar{\phi} dx + \int_{\partial\Omega} \sum_{q=0}^{l-1} L_{l-1-q} u \partial_{\nu}^q \bar{\phi} ds, \quad l = \deg L$$

with some linear differential expressions $L_p u$ of the order p and normal derivatives ∂_{ν} in the right part side which can be understood as the boundary part of the Green formula for the operator L . Here the boundary values (which should be coordinated with f and among themselves) of a smooth u give us a generalized function with support on $\partial\Omega$ in the right side of the equation in \mathbb{R}^n . And it is not difficult to see that any linear operator P satisfies condition (L). A mathematical sense of the condition (L) is to allocate nonlinear operators P for which such construction holds.

Note that in the formulation (L) we could use any other solution of the equation (1) instead zero solution outside Ω , i.e. the condition (L) could be written as

for each two smooth solutions u, v of equation (1) in the domain \mathcal{O} and each subdomain Ω there exists a distribution $f_{\partial\Omega}$, $\text{supp } f_{\partial\Omega} \subset \partial\Omega$ such that
 $(\partial/\partial t + P)(u \cdot \theta_{\Omega,j}) - (\partial/\partial t + P)(v \cdot (1 - \theta_{\Omega,j})) \xrightarrow{j \rightarrow \infty} f + f_{\partial\Omega}$ in $\mathcal{D}'(\mathcal{O})$.

or (forgetting term $\partial u/\partial t$) so:

for each two smooth in \mathcal{O} functions u, v and each subdomain Ω there exists a distribution $f_{\partial\Omega}$, $\text{supp } f_{\partial\Omega} \subset \partial\Omega$ with property: in $\mathcal{D}'(\mathcal{O})$
 $P(u \cdot \theta_{\Omega,j}) - P(v \cdot (1 - \theta_{\Omega,j})) \xrightarrow{j \rightarrow \infty} Pu \cdot \theta_{\Omega} - P(v \cdot (1 - \theta_{\Omega})) + f_{\partial\Omega}$.

Now one can understand a mathematical sense of the condition (L) simply:

The operator P is determined on jumps.

Consider for simplicity the case of one variable.

Theorem 1. *Nonlinear ordinary differential operator P with constant coefficients satisfies condition (L) if and only if it has the form*

$$Pu = \sum_{k,m} b_{km} (u^k)^{(m)} \quad (2)$$

where u^k is a power and (m) means a m -derivative as usually.

Proof. Consider an ordinary differential operator with constant coefficients

$$Pu = P[u] = \sum_a a_\alpha u^{\alpha_0} (u')^{\alpha_1} (u'')^{\alpha_2} \cdot \dots \cdot (u^{(l)})^{\alpha_l} =: Q(u, u', \dots, u^{(l)}),$$

where Q is a polynomial, $a_\alpha \in \mathbb{C}$. Here and below $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_l)$ is a multiindex of powers, $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_l$ is its degree, $\|\alpha\| = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots + l\alpha_l$ is a nonlinear order. **Let $\chi \in \mathbb{X}$ mean that $\chi \in C^\infty(\mathbb{R})$ and $\chi(x) = 0$ for $x \leq -1$, $\chi(x) = 1$ for $x \geq 1$.** Then, it is evident, for $\chi \in \mathbb{X}$

$$\frac{d}{dx} \chi \left(\frac{x}{\epsilon} \right) \xrightarrow{\epsilon \rightarrow 0} \delta(x) \text{ in } \mathcal{D}'(\mathbb{R}).$$

Let's read the condition (L) as follows:

$$\forall u \in C^\infty(\mathbb{R}) \text{ and } \forall \chi \in \mathbb{X} \text{ there exists } \lim_{\epsilon \rightarrow 0} P \left[u(x) \chi \left(\frac{x}{\epsilon} \right) \right] \text{ in } \mathcal{D}'(\mathbb{R}). \quad (L')$$

Intuitively, the condition (L) is not satisfied if among terms of P there are terms $(u')^k$, $k > 1$ and similar as far as δ^k doesn't exist there.

Sufficiency is almost evident. Indeed,

$$\left(\theta_\Omega * \chi \left(\frac{x}{\epsilon} \right) \right)^k \xrightarrow{\epsilon \rightarrow 0} \theta_\Omega, \text{ because } \theta_\Omega * \chi \left(\frac{x}{\epsilon} \right) \xrightarrow{\epsilon \rightarrow 0} \theta_\Omega.$$

The action of a linear operator $L_k = \sum_m b_{km} \left(\frac{d}{dx} \right)^m$ upon product $v \cdot \theta_\Omega$ inside Ω is reduced to the action upon the function v :

$$L_k[v \theta_\Omega] = \sum_m b_{km} D^{m-1} (\theta_\Omega \cdot D v + v \cdot D \theta_\Omega) = \dots = \theta_\Omega \sum_m b_{km} D^m v + f_{\partial\Omega},$$

where $f_{\partial\Omega}$ is a distribution with a compact support lying in $\partial\Omega$. By virtue of continuity of the linear operator L_k in the space $\mathcal{D}'(\mathbb{R})$ we obtain the condition (L').

Necessity. Let the condition (L) be fulfilled. At first, we allocate a part which depend only on u , i.e. we find a polynomial $R_0(u)$ such that $Q_0 = Q - R_0$ has the property $Q_0(u, 0, \dots, 0) = 0$. Then for any function $\psi \in C^\infty(\mathbb{R})$ and $u \equiv 1$ there exists the limit

$$I[\psi] = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \left\{ P \left[\chi \left(\frac{x}{\epsilon} \right) \right] - R_0 \left(\chi \left(\frac{x}{\epsilon} \right) \right) \right\} \psi(x) dx.$$

In particular, for $\psi(x) = x^r$ for each $r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ there exists the limit

$$\begin{aligned} I[x^r] &= \lim_{\epsilon \rightarrow 0} \sum_{\|\alpha\| > 0} \frac{a_\alpha}{\epsilon^{\|\alpha\|}} \int_{\mathbb{R}} \left(\chi\left(\frac{x}{\epsilon}\right) \right)^{\alpha_0} \cdot \dots \cdot \left(\chi^{(l)}\left(\frac{x}{\epsilon}\right) \right)^{\alpha_l} x^r dx = \\ &= \lim_{\epsilon \rightarrow 0} \sum_{\|\alpha\| > 0} \frac{a_\alpha}{\epsilon^{\|\alpha\| - 1 - r}} \int_{\mathbb{R}} (\chi(y))^{\alpha_0} \cdot \dots \cdot (\chi^{(l)}(y))^{\alpha_l} y^r dy = \lim_{\epsilon \rightarrow 0} \sum_{\|\alpha\| > 0} \frac{a_\alpha \tilde{Q}_{\alpha r}}{\epsilon^{\|\alpha\| - 1 - r}}, \end{aligned}$$

where $\tilde{Q}_{\alpha r} = \int_{\mathbb{R}} (\chi(y))^{\alpha_0} \cdot \dots \cdot (\chi^{(l)}(y))^{\alpha_l} y^r dy$.

Because the limit exists then all numerators for each $m = \|\alpha\| > 1 + r$, $r \geq 0$ must vanish and we obtain for such m

$$\sum_{\|\alpha\|=m} a_\alpha \tilde{Q}_{\alpha r} = 0, \quad \forall \chi \in \mathbb{X}. \quad (3)$$

Let $r = 0$, at first. Terms of the order $\|\alpha\| \leq 1$ in $P[u]$ have the form $R_0(u) + R_1(u) u'$, where R_0 and R_1 are polynomials of one variable, i.e. such terms can be represented in the form (2). For these terms the condition (L) is fulfilled as we have seen in sufficiency therefore these terms may be removed from equality (3). For the rest

$$Q_{00}(\chi_0, \chi_1, \dots, \chi_l) := \sum_{\|\alpha\| \geq 2} a_\alpha \chi_0^{\alpha_0} \chi_1^{\alpha_1} \cdot \dots \cdot \chi_l^{\alpha_l}$$

for $u \equiv 1$ from the condition (L') we have

$$\int_{\mathbb{R}} Q_{00}(\chi, \chi', \chi'', \dots, \chi^{(l)}) dy = 0, \quad \forall \chi \in \mathbb{X}. \quad (4)$$

Now we will use the following

Lemma. Let $S = \sum a_\alpha x_0^{\alpha_0} x_1^{\alpha_1} \cdot \dots \cdot x_l^{\alpha_l}$ be a polynomial with the properties:

$$S(x_0, 0, \dots, 0) = 0; \quad \forall \chi \in \mathbb{X}, \quad \int_{-\infty}^{+\infty} S(\chi, \chi', \dots, \chi^{(l)}(y)) dy = 0. \quad (5)$$

Then there exists a polynomial $T(x_0, x_1, \dots, x_{l-1})$ such that for each smooth function ψ the equality

$$S(\psi(y), \psi'(y), \dots, \psi^{(l)}(y)) = \frac{d}{dy} T(\psi, \psi', \dots, \psi^{(l-1)})$$

is fulfilled.

Proof. (Lemma): Substitute in (5) instead of χ the function $\chi + t\psi$, where $\psi \in C_0^\infty(\mathbb{R})$, $t \in \mathbb{R}$. Differentiating on t and throwing derivatives we obtain the Euler-Lagrange equation

$$S'_\chi - \frac{d}{dy} S'_{\chi'} + \frac{d^2}{dy^2} S'_{\chi''} - \dots + (-1)^l \frac{d^l}{dy^l} S'_{\chi^{(l)}} = 0.$$

Proof of the lemma is completed by use of results of paper [1]. ■

Applying this lemma to the equality (4) we obtain there exists a polynomial $Q_1(x_0, x_1, \dots, x_{l-1})$ with the property

$$\forall u, \quad Q(u, u', \dots, u^{(l)}) - R_0(u) - R_1(u)u' = \frac{d}{dx}Q_1(u, u', \dots, u^{(l-1)}).$$

In the polynomial $Q_1(u, u', \dots, u^{(l-1)})$ we select terms of the form $R_2(u)u'$ which have a view of the type (2) which satisfy the condition (L') by above sufficiency, therefore the rest in $P[u]$ satisfies the condition (L') also and its terms have $\|\alpha\| \geq 2$. We will denote $Q_{10}(u, u', \dots, u^{(l-1)}) = Q_1 - R_2(u)u'$. Its first derivative satisfies the condition (3) with $r = 1$, i. e.

$$\int_{\mathbb{R}} y \frac{d}{dy} Q_{10}(\chi, \chi', \dots, \chi^{(l-1)}(y)) dy = 0.$$

We transfer the derivative and make sure the fulfillment of lemma conditions, which using gives existence of a polynomial $Q_2(x_0, x_1, \dots, x_{l-2})$ with property: for each smooth function u it is fulfilled

$$Q(u, u', \dots, u^{(l)}) - R_0(u) - R_1(u)u' - (R_2(u)u')' = \left(\frac{d}{dy}\right)^2 Q_2(u, \dots, u^{(l-2)}).$$

Decomposing

$$Q_2 = R_3(u)u' + Q_{20}(u, u', \dots, u^{(l-2)}),$$

we see that $\|\alpha\| \geq 2$ for summands Q_{20} , therefore terms in $\frac{d^2}{dy^2}Q_2$ have minimal order $\|\alpha\| \geq 4$ and for these terms we can again use relations (3) with $r \geq 2$.

In general, on k -th step we have

$$Q(u, \dots, u^{(l)}) - R_0(u) - \sum_{\kappa=1}^k (R_{\kappa}(u)u')^{(\kappa-1)} = \left(\frac{d}{dy}\right)^k Q_k(u, u', \dots, u^{(l-k)}).$$

Decompose the polynomial $Q_k : Q_k = R_{k+1}(u)u' + Q_{k0}(u, u', \dots, u^{(l-k)})$ and obtain for summands in Q_{k0} the order is $\|\alpha\| \geq 2$, therefore the expression

$$\left(\frac{d}{dy}\right)^k Q_{k0}(u, u', \dots, u^{(l-k)})$$

has terms of the minimal order $\|\alpha\| \geq 2 + k$ and we may use relations (3) with $r \geq k$ as far as the terms $R_{k+1}(u)u'$ may be removed. We transfer derivatives upon y^k and then, using that the derivatives of the function χ are compactly supported, we obtain

$$\int_{\mathbb{R}} Q_k(\chi, \chi', \dots, \chi^{(l-k)}(y)) dy = 0, \quad Q_k(x_0, 0, \dots, 0) = 0$$

and one may again apply the lemma. The proof is finished. ■

Transition to case of many variables is not in principal difficult and, practically only complicating above proof, we can prove the following statement.

Theorem 2. *A single nonlinear differential operator*

$$Pu = \sum_{\alpha\beta\ldots\gamma} a_{\alpha\beta\ldots\gamma} (D^\alpha u)^{k_\alpha} (D^\beta u)^{k_\beta} \cdot \ldots \cdot (D^\gamma u)^{k_\gamma}, \quad P0 = 0, \quad D^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$$

satisfies condition (L) if and only if there exist linear operators L_1, \ldots, L_m such that

$$Pu = \sum_{k=1}^m L_k(u^k) \quad (6)$$

where u^k is a power.

Proof. Proof of sufficiency can be led in just the same way as in the theorem 1 with evident modifications. Namely, apply the operator P from (6) to product $u\theta_{\Omega,j} \in C^\infty(\Omega)$. Then in the space $\mathcal{D}'(\Omega)$ $(u\theta_{\Omega,j})^k$ tends to $u^k\theta_\Omega$. From the continuity of the linear operator L_k in the space $\mathcal{D}'(\mathbb{R})$ we obtain the condition (L).

Proof of necessity will be passed by using the result of the theorem 1 on one-dimensional case. First, we will prove the necessity for an operator that acts on functions of the form $u(x_1, \ldots, x_n) = u_1(x_1) \cdot \ldots \cdot u_n(x_n)$, the set of such functions on appropriated rectangular domain \mathcal{R} will be called \mathbb{Y} . Indeed, in expression for Pu with $u \in \mathbb{Y}$ we may take $x_2 = \text{const}$, ..., $x_n = \text{const}$ and apply the theorem 1 to the function u_1 of variable x_1 because condition (L') with one variables follows from the condition (L) for several variables. We obtain the derivatives with respect to x_1 are removed outside of powers: $Pu = \sum_{k=1}^m L_k^1(u_1^k)$. Constants coefficients of operators L_k^1 contains functions $u_2(x_2), \ldots, u_n(x_n)$ and their derivatives. Moreover, the term L_k^1 contains function $u_2(x_2)$ and their derivatives so that sum of its powers $\alpha_0 + \ldots + \alpha_l$ in expression $\tilde{a}_\alpha u_2^{\alpha_0} (u_2')^{\alpha_1} (u_2'')^{\alpha_2} \cdot \ldots \cdot (u_2^{(l)})^{\alpha_l}$ is equal to k because this term appears after substitution $u = u_1(x_1) \cdot \ldots \cdot u_n(x_n)$, where functions u_1, \ldots, u_n are incomming with the same powers. The same is valid for other functions $u_j(x_j)$, $j = 3, \ldots, n$.

Analogous expansion for variable x_2 has analogous form $Pu = \sum_{k=1}^m L_k^2(u_2^k)$. Besides, the term L_k^2 contains function $u_1(x_1)$ and their derivatives so that sum of its powers is equal to k , another terms don't contains such power, therefore $L_k^1(u_1^k) = L_k^2(u_2^k)$, i.e. the derivatives with respect to x_2 in this term are removed outside of powers. It means that $L_k^1(u_1^k) = L_k^2(u_2^k) = L_k^{12}(u_1^k u_2^k)$ with some linear differential operator L^{12} . Continuing so we obtain the necessity on functions from \mathbb{Y} .

The second step of the proof is to make sure of correctness on arbitrary functions. Let remember a well-known formula $L_2(\Omega_1 \times \Omega_2) = L_2(\Omega_1) \otimes L_2(\Omega_2)$ from what we include that each smooth function $u(x_1, \ldots, x_n) \in L_2(\mathcal{R})$ may be approximated by products of smooth functions of view $u_1(x_1) \cdot \ldots \cdot u_n(x_n)$ in the topology L_2 . Therefore the expression Pu may be approximated by products of smooth functions of view $u_1(x_1) \cdot \ldots \cdot u_n(x_n)$ in the topology $\mathcal{D}'(\mathcal{R})$. ■

Note also that in the case of systems of the equations one should in the beginning to write each equation of the system as single, substituting instead of each components of amplitude vector the same function, but then for each single equation to use the foregoing condition (L).

Examples 1. The operator $(u^2)''/2 = (u')^2 + uu''$ satisfies the condition (L) but its summands are not. The same may be said on operators $(u^2)'''/2 = u'''u + 3u'u''$ and $(u^3)''/3 = 2(u')^2u + u^2u''$.

2. Well-known nonlinear physical equations with observed amplitudes satisfy the condition (L). These are the Navier-Stokes, Korteweg-de Vries, Yang-Mills, Einstein of gravitation, Kadomtsev-Petviashvili, Burgers, Bussinesq equations and many others.

3. Many equations of not nature origin don't satisfy the condition (L). These are some geometrical equations: the Monge-Ampere, Liouville, mean curvature equations and also many invented equations, for example, the second equation from Lax hierarchy ([2]): $q_t + 6q_{xxxxx} + 5\alpha^2 q^2 q_x + 20\alpha q_x q_{xx} \mp 10\alpha q q_{xxx}$.

If we add the requirement of relativistic invariance to the condition (L) of physical localization and condition of space homogeneity (of coefficients constancy) and if we will consider only scalar equations of the second order then we will have remained only the following type of equations

$$\square Q(u) + uR(u) = f,$$

where Q and R are some polynomials (functions, generally speaking) with constant coefficients.

References

- [1] GELFAND I M and DIKII L A, Asymptotic behaviour of the resolvent of Schurm-Liouville equations and the algebra of the Korteweg-De Vries equations, *Russ. math. Surv.* **30** (1975), No.5, 77–113.
- [2] LAX P D, Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.* **21** (1968), 467–490.