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# On Pauli graded contractions of $sl(3,\mathbb{C})$

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**Abstract**

We consider a special fine grading of $sl(3,\mathbb{C})$, where the grading subspaces are generated by $3 \times 3$ generalized Pauli matrices. This fine grading decomposes $sl(3,\mathbb{C})$ into eight one–dimensional subspaces. Our aim is to find all contractions of $sl(3,\mathbb{C})$ which preserve this grading. We have found that the symmetry group of this grading is isomorphic to the group of $2 \times 2$ matrices with entries from the cyclic group $Z_3$ and determinant $\pm 1 \pmod 3$. It is used to simplify the set of the associated nonlinear contraction equations as well as to identify its 186 classes of equivalent solutions.

## 1 Introduction

Gradings of simple Lie algebras $L$ over $\mathbb{C}$ (or $\mathbb{R}$) belong to the basic structural properties of $L$. Especially the fine gradings [1], recently classified [2, 3], yield distinguished bases of $L$. Of course, symmetries of fine gradings of simple Lie algebras also belong to important characteristics of these bases [4]. However, they are of interest not only in this respect. In the study of graded contractions, one expects that symmetries can be exploited as a tool to simplify the system of nonlinear equations determining the contractions which preserve given grading [5] and to classify their solutions.

It was shown in [2] that the classical Lie algebras $A_{n-1} = sl(n,\mathbb{C})$, $n = 2, 3, 4, ...$ possess, among others, an interesting class of fine gradings defined by $n \times n$ generalized Pauli matrices [6]. In this contribution we consider $A_2 = sl(3,\mathbb{C})$ and approach the problem of finding all contractions which preserve this grading. Contractions of the 3–dimensional $A_1 = sl(2,\mathbb{C})$ were thoroughly investigated in [7]. Also the set of so called toroidal contractions of $sl(3,\mathbb{C})$ was described in [8, 9]; 32 contracted non–isomorphic Lie algebras were identified there.

Here we apply the theory of graded contractions to the very nontrivial case of the Pauli grading of $sl(3,\mathbb{C})$. We show that the symmetry group of the grading significantly simplifies the system of 48 nonlinear contraction equations. In this short contribution we are able to display but one case out of 186 nontrivial classes of equivalent solutions and see that it corresponds to an 8-dimensional solvable Lie algebra.

The decomposition $\Gamma: \; L = \bigoplus_{i\in I} L_i$ is called a *grading* if, for any pair of indices $i,j \in I$, there exists an index $k \in I$ such that $0 \neq [L_i, L_j] \subseteq L_k$. If $\Gamma : L = \bigoplus_{i\in I} L_i$ is a grading and $g$ is an automorphism of $L$, then $\tilde{\Gamma} : L = \bigoplus_{i\in I} g(L_i)$ is also a grading; two such gradings are called *equivalent*. A grading $\Gamma : L = \bigoplus_{i\in I} L_i$ is a *refinement* of the grading $\overline{\Gamma} : L = \bigoplus_{i\in J} \overline{L}_j$ if for any $i \in I$ there exists $j \in J$ such that $L_i \subseteq \overline{L}_j$. A grading which cannot be properly refined is called *fine*.

To construct a grading of $L$ one can use a diagonalizable automorphism $g$ from $Aut\, L$. It is easy to see that the resulting decomposition of $L$ into eigenspaces of $g$ is a grading. Furthermore, if one takes a set of *commuting automorphisms* from $Aut\, L$, and decomposes $L$ according to all automorphisms into their common eigenspaces, a finer grading of $L$ is obtained in general. This procedure can be followed until a maximal Abelian group of diagonalizable automorphisms — a *MAD–group* — of $L$ is obtained. Now the importance of the MAD-groups for the theory of fine gradings of simple Lie algebras stems from the following fact proven in [1].

**Theorem 1.**

1. *If $L$ is a finite–dimensional simple Lie algebra over an algebraically closed field of characteristic zero (e.g. $\mathbb{C}$), then a grading $\Gamma$ of $L$ is fine if and only if there is a MAD-group $\mathcal{G}$ such that $\Gamma$ is the decomposition into eigenspaces of automorphisms from $\mathcal{G}$.*

2. *Two fine gradings $\Gamma_1$ and $\Gamma_2$ are equivalent if and only if the corresponding MAD-groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are conjugate in* $\mathrm{Aut}\, L$.

3. *If $L$ is simple, the index set $I$ can be embedded in a finite Abelian group $G$; hence if the multiplication in $G$ is written additively, then $0 \neq [L_i, L_j] \subseteq L_{i+j}$.*

Thus any fine grading of a simple Lie algebra over $\mathbb{C}$ is a decomposition $\Gamma$ of $L$ into eigenspaces of automorphisms belonging to some MAD-group $\mathcal{G} \subset \mathrm{Aut}\, L$. For any subspace $L_i$ of the fine grading and any element $g \in \mathcal{G}$ we have $g\, L_i = L_i$, *i.e.* the elements of a MAD-group preserve each subspace of the corresponding grading ($\mathcal{G}$-grading).

Here we are interested in *symmetries of a given grading*, *i.e.* in automorphisms of $L$ which preserve a $\mathcal{G}$-grading of $L$ but not each of its subspaces separately. In other words, we are looking for those elements of $\mathrm{Aut}\, L$ which *permute the grading subspaces.* Of course, the corresponding permutations form a rather small subgroup of the symmetric group.

We can see that these automorphisms are just the elements of the *normalizer* $\mathcal{N}(\mathcal{G})$ of the MAD-group $\mathcal{G}$ in $\mathrm{Aut}\, L$. Namely, the normalizer $\mathcal{N}(\mathcal{G})$ is defined as the set $\{h \in \mathrm{Aut}\, L \mid h^{-1}\mathcal{G}h \subset \mathcal{G}\}$. Let $h \in \mathcal{N}(\mathcal{G})$ and $L_i$ be a subspace of the fine grading $\bigoplus_{i\in I} L_i$ corresponding to the MAD-group $\mathcal{G}$. Since $h^{-1}fh \in \mathcal{G}$ for arbitrarily chosen $f \in \mathcal{G}$, we can find $g \in \mathcal{G}$ such that $h^{-1}fh = g$. Applying this automorphism to $L_i$ and using that $gL_i = L_i$, we obtain $f(hL_i) = hL_i$, i.e. $hL_i$ is an eigenspace for any automorphism $f \in \mathcal{G}$, which means that $hL_i = L_j$ for some index $j \in I$. Any $h \in \mathcal{N}(\mathcal{G})$ thus generates some permutation $\pi$ on the grading indices $\pi : I \to I$. Note that the same permutation $\pi$ of $I$ is induced by the elements of the whole coset $h\mathcal{G}$. Hence the symmetry transformations

— the permutations of grading subspaces — are in one–to–one correspondence with the elements of the quotient group $\mathcal{N}(\mathcal{G})/\mathcal{G}$.

Complete classification of MAD–groups for classical Lie algebras over $\mathbb{C}$ was given in [2]. Let us recall the particular case of the Lie algebras $sl(n,\mathbb{C})$ we are interested in. The group of automorphisms $\mathrm{Aut}\, sl(n,\mathbb{C})$ consists of inner automorphisms $Ad_A X$ and the outer ones $Out_A X$:

$$\begin{aligned} Ad_A X &:= A^{-1}XA, & A &\in SL(n,\mathbb{C}) = \{B \in \mathbb{C}^{n\times n}\,, \det B = 1\} \\ Out_A X &:= -(A^{-1}XA)^T\,, & A &\in SL(n,\mathbb{C}) \end{aligned}$$

The classification of MAD–groups for complex semisimple Lie algebras was given in [2].

## 2 Pauli grading of $sl(3,\mathbb{C})$

Explicit form of Pauli gradings of $sl(n,\mathbb{C})$ and of their symmetries for arbitrary $n = 2, 3, ...$ can be found in [10]. For the Pauli grading the MAD–group $\mathcal{G}$ is finite and consists of $n^2$ elements. Here we will describe only the case $n = 3$, noting that the transition to matrices generating the Pauli grading for arbitrary $n$ is straightforward. Let us note that in $\mathrm{Aut}\, sl(3,\mathbb{C})$ there are four non-conjugate MAD-groups [2], hence there are also *four inequivalent fine gradings* [8, 9, 4].

For the Pauli grading of $sl(3,\mathbb{C})$ we have:
The MAD–group

$$\mathcal{G} = \left\{ Ad_{Q^iP^j} \mid Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},\ P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ \omega = e^{i\frac{2\pi}{3}},\ i,j = 0,1,2 \right\}.$$

The grading decomposition $\Gamma:\ sl(3,\mathbb{C}) = \bigoplus_{(i,j)\neq(0,0)} L_{ij}$, *i.e.* $sl(3,\mathbb{C})$ decomposes into 8 one–dimensional grading subspaces $L_{ij} = \mathbb{C}X_{ij}$, $X_{ij} = Q^iP^j$, $i,j = 0,1,2$, $(i,j) \neq (0,0)$. The indices $(i,j)$ belong to the additive grading group $G = Z_3 \otimes Z_3$.

The symmetry group $\mathcal{N}(\mathcal{G})/\mathcal{G} = Z_2 \otimes SL(2, Z_3)$ has 48 elements generated by $Out_I$, $Ad_S$, $Ad_D$, where $S = (\omega^{-ij})$, $D = \mathrm{diag}(1,1,\omega)$ [10].

## 3 Graded contractions of Pauli graded $sl(3,\mathbb{C})$

In this section we want to exploit our explicit knowledge of the symmetry group $\mathcal{N}(\mathcal{G})/\mathcal{G}$ of the Pauli grading to simplify and, indeed, bring further insight into the structure of the problem of finding all contractions preserving this grading of $sl(3,\mathbb{C})$.

Let $L = \bigoplus_{i\in I} L_i$ be a grading decomposition of a Lie algebra with the commutator $[\,\cdot\,,\cdot\,]$. *Graded contraction* is defined in terms of a contracted commutator of the algebra. It involves the contraction parameters $\varepsilon_{ij}$ for $i,j \in I$ and the old commutator. The bilinear mapping of the form

$$[x,y]_{\mathrm{new}} := \varepsilon_{ij}[x,y] \qquad \text{for all } x \in L_i,\ y \in L_j$$

is a new Lie product on the same vector space $L$. To satisfy *antisymmetry* of the new commutator we have to choose $\varepsilon_{ij} = \varepsilon_{ji}$. The *Jacobi identity* for any triple of elements $x \in$

$L_i,\ y \in L_j, z \in L_k$ leads to a system of quadratic equations for the unknown contraction parameters $\varepsilon_{ij}$.

For the Pauli graded Lie algebra $sl(3,\mathbb{C}) = \bigoplus_{(i,j)} \neq (0,0) L_{ij}$, with 8 one-dimensional grading subspaces $L_{ij} = \mathbb{C}X_{ij}$, $X_{ij} = Q^i P^j$, this problem is quite formidable. Let us start with the commutators following from the basic identity $PQ = \omega QP$:

$$[X_{ij}, X_{i'j'}] = (\omega^{ji'} - \omega^{ij'}) X_{i+i'\,(\mathrm{mod}\ 3), j+j'\,(\mathrm{mod}\ 3)}, \quad X_{00} = 0, \tag{3.1}$$

hence $0 \neq [L_{ij}, L_{i'j'}] \subseteq L_{i+i'\,(\mathrm{mod}\ 3), j+j'\,(\mathrm{mod}\ 3)}, \quad L_{00} = 0$. They clearly satisfy the grading property with the index set $I$ embedded in the Abelian group $Z_3 \otimes Z_3$: the binary operation $((i,j),(i',j')) \mapsto (i+i'\,(\mathrm{mod}\ 3), j+j'\,(\mathrm{mod}\ 3))$ is the additive group law in $Z_3 \otimes Z_3$.

To obtain the contraction equations, let us first take the triple of vectors $X_{01}$, $X_{02}$ and $X_{10}$. The Jacobi identity for the new Lie product has the form

$$[X_{01}, [X_{02}, X_{10}]_{\mathrm{new}}]_{\mathrm{new}} + [X_{02}, [X_{10}, X_{01}]_{\mathrm{new}}]_{\mathrm{new}} + [X_{10}, [X_{01}, X_{02}]_{\mathrm{new}}]_{\mathrm{new}} = 0. \tag{3.2}$$

The commutation relations (3.1) give us

$$\varepsilon_{02,10)}\varepsilon_{01,12)}(\omega - 1)(\omega^2 - 1)X_{10} + \varepsilon_{10,01}\varepsilon_{02,11}(1 - \omega)(\omega^2 - 1)X_{10} = 0 \tag{3.3}$$

and therefore

$$\varepsilon_{02,10}\varepsilon_{01,12} = \varepsilon_{10,01}\varepsilon_{02,11}. \tag{3.4}$$

The Jacobi identity for another choice of vectors $X_{01}$, $X_{10}$ and $X_{11}$ yields the equality

$$\varepsilon_{10,11}\varepsilon_{01,21} - \varepsilon_{11,01}\varepsilon_{10,12} = 0\ . \tag{3.5}$$

The triple of indices 01, 10 and 11 is distinguished by the property that the indices of any $\varepsilon$ appearing in (3.5) are linearly independent over the field $Z_3$. Quite different are the indices in (3.4); there the pair of indices 01 and 02 is linearly dependent over the field $Z_3$. It will be described below that these two triples essentially exhaust distinct possibilities for the choice of representatives in the two classes of equations following from the Jacobi identity.

For all possible triples of basis elements $X_{ij}$ one always gets similar two–term equations, i.e. three–term equations do not appear in our system. Simple counting gives $\binom{8}{3} = 56$ triples. Since 8 triples of the form $X_{ij}$, $X_{i'j'}$ and $X_{i''j''}$, with the property $i + i' + i'' = 0\ (\mathrm{mod}\ 3)$ and $j + j' + j'' = 0\ (\mathrm{mod}\ 3)$ satisfy $[X_{ij}, [X_{i'j'}, X_{i''j''}]] = 0$, there remain in fact only 48 equations to be solved.

The number of unknown contraction parameters is also reduced from $\binom{8}{2} = 28$ to 24, since there are exactly 4 commutators (3.1) which are equal to zero, namely those in which $i + i' = 0\ (\mathrm{mod}\ 3)$, $j + j' = 0\ (\mathrm{mod}\ 3)$. These contraction parameters $\varepsilon_{01,02}$, $\varepsilon_{10,20}$, $\varepsilon_{11,22}$, $\varepsilon_{12,21}$ are irrelevant, since they do not appear in the equations.

The quotient group $\mathcal{N}(\mathcal{G})/\mathcal{G} = Z_2 \otimes SL(2, Z_3)$ is the *symmetry group* of our system [10]. The finite group $SL(2, Z_3)$ consists of matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with entries from} \quad Z_3 \quad \text{and} \quad ad - bc = 1\ (\mathrm{mod}\ 3). \tag{3.6}$$

It acts on grading subspaces $L_{ij}$ via permutations $L_{ij} \to L_{i'j'}$ with $(i'j') = \pi(ij) = (ij)M$. Note that $Z_3$ forms a field since 3 is prime. Now for $p$ prime the order $|SL(2, Z_p)| = p(p^2-1)$, hence $SL(2, Z_3)$ has 24 elements. The symmetry group $Z_2 \otimes SL(2, Z_3)$ of order 48 still contains matrices corresponding to outer automorphisms, *i.e.* matrices from $SL(2, Z_3)$ multiplied by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ with determinant $-1$.

It turns out that there are just two orbits of the symmetry group among our 48 quadratic equations. The symmetry transformations are the mappings on the index set $I$ defined by $(i, j) \mapsto (i, j)M$, where $M \in SL(2, Z_3)$. Applying such a mapping with a fixed matrix $M$ to the indices occurring in equation (3.4) we obtain a new equation corresponding to the Jacobi identity for another triple of grading subspaces. If we gradually apply all 24 matrices from $SL(2, Z_3)$ to the equations (3.4) and (3.5), we obtain two sets of quadratic equations which should be satisfied, each containing 24 equations. There is no transformation in the symmetry group which would transform (3.4) into (3.5); the transformations induced by the outer automorphisms are ineffective. In this way the symmetries of the system of equations are directly seen.

## 4 Example of a graded contraction

As an example of a nontrivial solution of our big system of nonlinear equations we present an eight-dimensional solvable (non-nilpotent) Lie algebra $L_s$. Its commutation relations are determined by the normalized solution $\varepsilon_{ij,i'j'}$ which we present in the form of a symmetric $8\times 8$ matrix. If the rows and columns are ordered according to $01, 02, 10, 20, 11, 22, 12, 21$, then we can write the matrix representation of our solution

$$\begin{pmatrix} \square & \square & 1 & 1 & 1 & 1 & 1 & 1 \\ \square & \square & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \square & \square & 0 & 0 & 0 & 0 \\ 1 & 1 & \square & \square & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & \square & \square & 0 & 0 \\ 1 & 1 & 0 & 1 & \square & \square & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & \square & \square \\ 1 & 1 & 0 & 1 & 0 & 1 & \square & \square \end{pmatrix},$$

where $\square$ denotes irrelevant elements. We can display the structure of $L_s$ in terms of its proper subalgebras

$$L_{s2} \subset L_{s1} \subset L_s;$$

$L_{s2}$ has basis elements $X_{10} = a_1$, $X_{11} = a_2$, $X_{12} = a_3$ and is the center of $L_{s1}$; $L_{s1}$ has, besides $a_1$, $a_2$, $a_3$, the basis elements $X_{22} = p_1$, $X_{21} = p_2$, $X_{20} = p_3$; $L_s$ still contains $X_{01} = r_1$ and $X_{02} = r_2$. In this notation the commutation relations of $L_s$ get the simple form

$$[a_i, a_j] = 0, \quad [a_i, p_j] = 0, \quad [p_i, p_j] = \varepsilon_{ijk} a_k,$$

$$[r_1, a_i] = a_{i+1}, \quad [r_2, a_i] = a_{i-1}, \quad [r_1, p_i] = p_{i-1}, \quad [r_2, p_i] = p_{i+1}.$$

We have presented here only one solution [11], in order to give feeling of the size and complexity of the classification task for the full set of 186 solutions to the contraction equations. Our complete results will be published elsewhere. They will supplement the set of 32 graded contractions obtained earlier for the toral fine grading [9, 8].

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