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Statistical mechanics of a Class of Anyonic Systems. The Rigorous Approach

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Abstract

A class of involutive Wick algebras (called anyonic-type Wick algebras) is selected and some its elementary properties are described. In particular, the Fock representations of the selected anyonic-type commutation relations are described. For the class of so-called r-yonic systems the question of the existence of the limiting thermodynamic functions is being addressed. The corresponding r-yonic partition functions are described in terms of interacting bosonic systems. Methods originally developed for a purely bosonic system are being adopted and successfully applied to this class of anyonic systems under considerations.

1 Introduction

The basic homotopy group on which the notion of statistics of a system of indistinguishable particles is based depends critically on whether the system under consideration in the two space dimensions or in higher. If the system is two–dimensional then the corresponding group $\mathbf{B}_n$ (called the braid group with n generators $\{e_i\}_{i=0}^{n-1}$ obeying the following defining relations $e_ie_j = e_je_i$ for $|i-j| > 1$, $e_ie_{i+1}e_i = e_{i+1}e_ie_{i+1}$ for $i = 1,\ldots,n-2$ and the unit e_0) is infinite for $n \geq 2$. This is in contrast to higher space dimensions, where the underlying group is being identified easily with the symmetric group $\mathbf{S}_n$. Although at the level of defining relations both groups differs only by the additional idempotency relation $e_i^2 = e_0$ for $i \in \{1,\ldots,n-1\}$, the physical and mathematical consequences of this fact are very deep.

As is well known, several very interesting new quantum phenomena (integer and fractional Hall effects, high temperature superconductivity etc.) have been experimentally detected in systems confined to live essentially on two dimensional space surfaces.

Most of the formulated in the current literature theoretical explanations of them exploit intensively the fact that collective excitations in the (dense) two–dimensional electronic gas may obey some nonstandard statistical rules. Presumably, the best known examples of that sort are anyons of Leinass–Myrheim type [1, 2] and composite fermions [1, 3].

However there are only few papers treating such questions with full mathematical physics rigor. Let us mention among them: the functional integral approach to the quantization of anyonic systems [4], and the Wick algebra approach [5, 6]. The present contribution contains some ideas related to the recent developments in the Wick algebra approach to the physics of anyons.

2 Wick algebras and their second quantizations

Let $\mathfrak{H}$ be a separable Hilbert space and let $T \in B(\mathfrak{H}\otimes\mathfrak{H})$ and such that for some complete ON system $\{e_i\}$ in $\mathfrak{H}$ the corresponding matrix representation of T in the basis $\{e_i\otimes e_j\}_{i,j}$, given by $T_{ij}^{kl} =< e_k \otimes e_l|Te_i \otimes e_j >$ has the property that for each pair (i,j) the number of those (k,l) : that $T_{ij}^{kl} \neq 0$ is finite.

A Wick algebra $W(\mathfrak{H},T)$ is defined as an abstract linear algebra generated by the system $\{a_f, a_g^+ :\ f,g \in h\}$ which obeys the following relations: $a_{f+g}^{\#} = a_f^{\#} + a_g^{\#}$, $a_{\alpha f} = \bar{\alpha} a_f, a_{\alpha f}^+ = \alpha a_f^+$, and defining $a_k^{\#} \equiv a(e_k)^{\#}$ the following commutation relations hold:

$$a_k a_l^+ - \sum_{i,j} T_{kl}^{ij} a_i^+ a_j = \delta_{kl} \tag{2.1}$$

It is easy to find a concrete realization of just defined algebra $W(\mathfrak{H},T)$ in the full tensor algebra of $\mathfrak{H}\otimes\mathfrak{H}^*$, see e.g., [5]. Given $W(\mathfrak{H},T)$, we define its corresponding Fock module $\Gamma_T(h)$ as a cyclic left module generated by the cyclic vector Ω. It follows easily that there exists uniquely defined sesquilinear form $< \cdot,\cdot >$ on $\Gamma_T(\mathfrak{H})$ with respect to which $a^+(f)$ becomes adjoint to $a(f)$. As we know from [5, 7]:

$$< \Phi,\Psi >_T = < \Phi|P(T)\Psi > \tag{2.2}$$

where $P(T) = \bigoplus_{n=0}^{\infty} P_n(T)$ in $\bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n}$ is defined recursively: $P_0(T) = 1, P_1(T) = 1$ and

$$P_{n+1}(T) = (1\otimes P_n(T))(1 + T^1 + T^2T^1 + \cdots + T^nT^{n-1}\dots T^1) \tag{2.3}$$

(where T^k denotes an operator acting in $\mathfrak{H}^{\otimes n+1}$ by $\underbrace{1_{\mathfrak{H}}\otimes\ldots\otimes 1_{\mathfrak{H}}}_{k-1 \text{ times}}\otimes T\otimes 1_{\mathfrak{H}}\otimes\ldots\otimes 1_{\mathfrak{H}}$).

Of major importance is the question of positivity of the considered metric operator $P(T)$. The following sufficient conditions for this to hold might be useful, [5, 7]:

P1 $T \geq 0$

P2 $\|T\| < \frac{1}{2}$ and if $\|T\| < \frac{1}{2}$ then $P(T) > 0$

P3 $\|T\| \leq 1$ and T obeys the Yang–Baxter $(Y-B)$ relations on $\mathfrak{H}^{\otimes 3}$: $T^1T^2T^1 = T^2T^1T^2$, and if $\|T\| < 1$ then $\ker P(T) = \{0\}$.

Let $h = h^+$ be a selfadjoint, nonnegative operator acting in $\mathfrak{H}$ and such that for any $\beta > 0$, $Tre^{-\beta h} < \infty$. The second quantization of the one–particle Gibbs semigroup $\Gamma_T(e^{-\beta h})$ is defined in the space $\Gamma_T(\mathfrak{H})$ by $\Gamma_T(e^{-\beta h}) = \bigoplus_{n=0}^{\infty}(e^{-\beta h})^{\otimes n}$ (the mathematical meaning of this is discussed in [6]). The main questions we are interested in are the following ones:

Question 1: When $Z_T(\beta) \equiv Tr_{\Gamma_T(\mathfrak{H})}\Gamma_T(e^{-\beta h}) < \infty$?

Question 2: Providing $Z_T(\beta) < \infty$, how to control the corresponding thermodynamic formalism, especially, the thermodynamic limit.

Observation. *Let $T \in B(\mathfrak{H} \otimes \mathfrak{H})$ obey one of the conditions P(1)–P(3) and let P(T) be the corresponding metric operator in $\Gamma_T(\mathfrak{H})$. If $\ker P(T) = \{0\}$ then for any $h = h^+ \geq 0$ such that $tr_{\mathfrak{H}}e^{-\beta h} < \infty$, the partition function $Z_T(h) \equiv Tr_{\Gamma_T(\mathfrak{H})}\Gamma_T(e^{-\beta h})$ does exist for any $\beta > 0$, but $Z_T(h) = Z_{T=0}(h)$!*

However the triviality of the kernel of the metric form $P(T)$ only is not sufficient to guarantee that the partition function $Z_T(h)$ depends on T in an interesting way.

Example. Let $\{a_i^+, a_j\}_{i,j=1,\ldots,N}$ be the twisted commutation relations (or anticommutation) algebra generators [8, 9], and let $h = h^+$ be a hermitian matrix in $\mathbb{C}^N$. The corresponding Fock modules will be denoted by $\Gamma_{TB(F)}(\mathbb{C}^N)$. Then

$$Tr_{\Gamma_{TB(F)}}\Gamma(e^{-\beta h}) = Tr_{\Gamma_{B(F)}}\Gamma(e^{-\beta h}),$$

where $\Gamma_{TB(F)}$ denotes the (standard) Boson (resp. Fermion) Fock module.

Proof. By explicit computations [6]. ■

3 Anyonic-type Wick algebras

A Wick algebra $W(\mathfrak{H}, T)$ is called an anyonic-type Wick algebra if the operator $T \in B(\mathfrak{H} \otimes \mathfrak{H})$ obeys the following conditions:

1. $T^1T^2T^1 = T^2T^1T^2$ in $\mathfrak{H}^{\otimes 3}$
2. $(1 - T)(1 + \hat{T}) = 0$,

where $\hat{T}$ is the partial (with respect to the first factor) adjoint to T in $\mathfrak{H} \otimes \mathfrak{H}$.

Let $\Gamma_T(\mathfrak{H})$ be the corresponding Fock module and let $P(T)$ be the corresponding metric form. It seems to be a special feature an the anyonic-type Wick algebra when the explicit form of $P(T)$ and, in particular, $\ker(P(T))$ can be easily given.

Proposition 1 ([6]). *Let $W(\mathfrak{H}, T)$ be an anyonic-type Wick algebra. Then the metric form $P(T)$ is an orthogonal projector in $\Gamma_T(\mathfrak{H})$ and*

$$\ker P(T) = P(T)^{\perp},$$

where $P(T)$ is by definition equal to the range of $P(T)$, i.e., $P(T) = P(T)\Gamma_T(\mathfrak{H})$.

3.1 Examples:

The standard bosons and fermions are of course, the best known examples of anyonic-type Wick algebras and the corresponding kernels are well known. In the case of bosons the corresponding kernel is equal to the orthogonal complement of symmetric tensors in $\Gamma(\mathfrak{H})$ and in the case of fermions this kernel is equal to the complement of antisymmetric tensors in $\Gamma(\mathfrak{H})$. The introduced class of anyonic-type commutation relations provides examples, where the corresponding kernels interpolate between bosonic and fermionic ones.

An interesting example of this type is that introduced by Faddeev and Zamolodchikov [10] and called an "exchange algebra" and further studied in [11].

For this goal let $\mathfrak{H} = L^2(\mathbb{R}^d)$, $d \geq 1$ and $r : \mathbb{R}^{2d} \mapsto \mathbb{R}$ such that $r(x,y)+r(y,x) = 0$. Then we define $T_r(f)(x,y) = e^{ir(x,y)} f(y,x)$. Next one easily checks that T_r defines anyonic-type commutation relations. In the corresponding Fock module $\Gamma_r(\mathfrak{H})$ the following algebra of r–commutation relations is represented by:

$$[r-CR] \begin{cases} a_r(x)a_r(y)^+ & - & e^{ir(x,y)}a_r(y)^+a_r(x) & = & \delta(x-y) \\ a_r(x)a_r(y) & - & e^{ir(x,y)}a_r(y)a_r(x) & = & 0 \\ a_r(x)^+a_r(y)^+ & - & e^{ir(x,y)}a_r(y)^+a_r(x)^+ & = & 0 \end{cases}$$

In particular, taking $d = 2$; $r(x,y) = -\Theta arg(x-y, \mathbf{n})$ where Θ is a so–called statistical parameter and $\mathbf{n}$ is a fixed unit vector in $\mathbb{R}^2$ we obtain exactly the Leinaas–Myrheim commutation relations [2].

4 Regularized, Leinaas–Myrheim anyons

Let $\Lambda \subset \mathbb{R}^d$, $d \geq 1$ be a bounded region with the boundary $\partial\Lambda$ of a class C^3 (at least piecewise). For $\sigma \in C^3(\partial\Lambda)$, $\sigma(x) \geq 0$ one can define a selfadjoint operator Δ^σ_Λ, being a suitable extension of the Laplace operator defined on $C^\infty_0(\Lambda)$. Then σ denotes a corresponding (classical) boundary condition. Each Laplace operator Δ^σ_Λ is nonpositive with discrete spectrum and such that for any $\beta > 0$ (inverse temperature), $\mu > 0$ (chemical potential) the corresponding one particle density operator $e^{-\beta h(\mu)}$, $h \equiv -\Delta^\sigma_\Lambda + \mu 1$ is of trace class:

$$Z^\sigma_\Lambda \equiv Tr_{L^2(\Lambda)} e^{-\beta h(\mu)} < \infty$$

Now let $r : \mathbb{R}^2 \mapsto \mathbb{R}$ be of class (at least) C^1 and such that $r(x,y) + r(y,x) = 0$. The restriction of r to $\Lambda \times \Lambda$ is denoted by r_Λ. The local one–particle Hilbert space $\mathfrak{H}_\Lambda = L^2(\Lambda, dx)$ and the corresponding $T_r f(x,y) = e^{ir(x,y)} f(y,x)$. The pair $\mathfrak{H}_\Lambda, T_r$ leads to the anyonic-type Wick algebra $W(\mathfrak{H}_\Lambda, T_r)$ with the corresponding Fock module $\Gamma_r(\mathfrak{H}_\Lambda)$ and creation and annihilation operators $a^+_r(x), a_r(y)$ acting there (in the sense of operator valued distributions) and obeying the (rCR) algebra.

An explicate description as well as other elementary properties of the Fock representation of the algebra (rCR) can be extracted from [11] (see also [10], see also the general construction of **section 2** above).

Our main interest will be focused on the problem of the existence of the thermodynamical formalism (in the so called thermodynamical limit) for the system of particles obeying (rCR) algebra. For this goal we define:

- the finite volume, grand canonical Gibbs ensemble partition function

$$Z^r_\Lambda(\beta,\mu) \equiv Tr_{\Gamma_r(\mathfrak{H}_\Lambda)}\Gamma_r(e^{-\beta h(\mu)}) \tag{4.1}$$

- the finite volume energy density

$$p^r_\Lambda(\beta,\mu) \equiv \frac{1}{|\Lambda|}\ln Z^r_\Lambda(\beta,\mu) \tag{4.2}$$

As it is well known, for the existence of the thermodynamic formalism (in the thermodynamical limit $\lim\limits_{\Lambda\nearrow\mathbb{R}^d}$), i.e., it is sufficient to control $\lim\limits_{\Lambda\nearrow\mathbb{R}^d} p^r_\Lambda(\beta,\mu) \equiv p^r_\infty(\beta,\mu)$, i.e., the infinite volume free energy density.

Let $\Gamma_1(\mathfrak{H}_\Lambda)$ be the bosonic Fock module built over $\mathfrak{H}_\Lambda$. It is proved in [11] that $Z^r_\Lambda(\beta,\mu)$ can be exposed in purely bosonic terms:

$$Z^r_\Lambda(\beta,\mu) = Tr_{\Gamma_1(\mathfrak{H}_\Lambda)}e^{-\beta H_\lambda(\beta,\mu)}, \tag{4.3}$$

where

$$H_\lambda(\beta,\mu) = \int_\Lambda dx\, a^+(x)(-\Delta^\sigma_\Lambda + \mu 1)a(x) + \iint\limits_{\Lambda^{\times 2}} dx\, dy\, a^+(x)a^+(y)V^r_2(x,y)a(y)a(x)$$
$$+ \iiint\limits_{\Lambda^{\times 3}} dx\, dy\, dz\, a^+(x)a^+(y)a^+(z)V^r_3(x,y,z)a(z)a(y)a(x),$$

where $a^+(x), a(y)$ are canonical bosonic creation, and resp. annihilation operators:

$$V^r_2(x,y) = \frac{1}{8}(\nabla_x r(x,y))^2 + \frac{1}{8}(\nabla_y r(x,y))^2 \tag{4.4}$$

∇ is the two–body interaction and

$$V^r_3(x,y,z) = \nabla_x r(x,y)\nabla_y r(y,z) + \text{ c. terms} \tag{4.5}$$

Equality (4.3) is a starting point of our approach.

It is well known that any operator $-\Delta^\sigma_\Lambda$ as above generates a Markovian semigroup [12] the kernel of which can be expressed by the Wiener integral representation formula:

$$e^{\beta\Delta^\sigma_\Lambda}(x,y) = \int\limits_{C([0,\beta],\hat{\Lambda})} dW^{\sigma,\beta}_{x|y}(\omega), \tag{4.6}$$

where $dW^{\sigma,\beta}_{x|y}$ is the corresponding conditional Wiener bridge measure.

Proposition 2. *The following Poisson–Wiener representation formula does hold:*

$$Z^r_\Lambda(\beta,\mu) = \sum_{n=0}^{\infty}\sum_{j_1,\ldots,j_n\geq 1}\frac{z^{n+|j|}}{n!\prod\limits_{k=1}^{n} j_k}\int \otimes^n_{i=1} dW^{j_1\beta}_{x_i|x_i}(\omega_i)e^{-E^\beta(\omega_1,\ldots,\omega_n)} \tag{4.7}$$

where we have denoted $j \equiv (j_1, \ldots, j_n)$, $|j| \equiv \sum_{k=1}^{n} j_k$ *and if* $\omega = (\omega_1, \ldots, \omega_n)$ *is a sequence of* n*–loops of length* $j_i\beta$ *respectively, then the self–energy of it is given explicitly by*

$$E^{\beta}(\omega) = \sum_{1\le\alpha\le\gamma\le n} \sum_{i_\alpha=1}^{j_\alpha} \sum_{i_\gamma=1}^{j_\gamma} \int_0^{\beta} d\tau\, V_2^r(\omega_\alpha(\tau + (i_\alpha - 1)\beta), \omega_\gamma(\tau + (i_\gamma - 1)\beta))$$

$$+ \sum_{1\le\alpha\le\delta\le\gamma\le n} \sum_{i_\alpha=1}^{j_\alpha} \sum_{i_\delta=1}^{j_\delta} \sum_{i_\gamma=1}^{j_\gamma} \int_0^{\beta} d\tau$$

$$\times V_3^r(\omega_\alpha(\tau + (i_\alpha - 1)\beta), \omega_\delta(\tau + (i_\delta - 1)\beta), \omega_\gamma(\tau + (i_\gamma - 1)\beta)) \quad (4.8)$$

Our main result is the following

Theorem 1. *Let us suppose that* V_2^r, V_3^r *obey the following conditions:*

1. $\exists B > 0\, \forall N \ge 0 \;\; E^{\beta}(x_1, \ldots, x_N) \ge -NB$ *(stability)*
2.
   - $\sup_x \left\{ \int dy\, |e^{-\beta V_2^r(x,y)} - 1| \equiv C_2(\beta) \right\} < \infty$
   - $\sup_{x, x_1, \ldots, x_n} \left\{ \int dy\, |e^{-\beta \sum_{j=1}^{n} V_3^r(x, x_j, y)} - 1| \right\} = C_3(n, \beta) < \infty$
   - $\sup_n \left\{ \sum_{p=0}^{\infty} \frac{1}{p!} C_3(n+p, \beta)\xi^p \right\} < \infty$ *for any* $\xi \ge 0$.

Then there exists $z_0 \in \mathbb{C}$ *depending on* β *(small) and* μ *(big) such that a unique* $\lim_{\Lambda \nearrow \mathbb{R}^d} p_\Lambda^r(\beta, \mu) = p_\infty^r(\beta, \mu)$ *does exist. This limit does not depends on choice of an increasing sequence of compact sets* $\Lambda \subset \mathbb{R}^d$ *and on a particular choice of* σ_Λ*. A limiting free energy density* $p_\infty^r(\beta, \mu)$ *is holomorphic in* μ*, providing* $|e^{\beta\mu}| < z_0$*. In particular the virial expansion, i.e.,*

$$\rho_\infty(\beta, \mu) \equiv \frac{\partial}{\partial \mu} p_\infty(\beta, \mu) \equiv \sum_{n=0}^{\infty} b_n(\beta) \frac{z^n}{n!}$$

is convergent if $|z| < z_0$.

Proof. By an extension of the classical methods of Ginibre [13] developed for the case of two–body interactions. For details we refer to [6]. ■

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