On the complexified affine metaplectic representation

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Abstract

We study exponentiability of the infinite polynomials with maximal degree 2 of creation and annihilation operators, which give a Fock Space-representation of the complexification of the affine symplectic group.

1 Introduction

In this paper, which is a continuation of [1], we study exponentiability of the infinite polynomials with maximal degree 2 of creation and annihilation operators, and obtain explicitly the local connection between the complexification of the affine metaplectic representation and the corresponding Lie algebra.

This type of operators has found their use in many areas. In [2] is considered the complexification of the Lie algebra of the affine symplectic group, and in Quantum Probability we find as special examples the fundamental noises and the Weyl Representation of the Euclidean group (cf. [3] and [4]).

The metaplectic representation in Bose Fock space is a projective unitary representation of the restricted symplectic group (cf. [5]). Corresponding to the metaplectic representation there is a projective skew-adjoint representation of the restricted symplectic Lie algebra consisting of skew-adjoint quadratic infinite polynomials of creation and annihilation operators. Also the unitary representation of CCR (Weyl relations) correspond to a skew-adjoint representation of creation and annihilation operators.

2 The Boson Fock algebra

To a given complex separable Hilbert space $\mathcal{H}, \langle , \rangle$ attach an algebra $\Gamma_0\mathcal{H}$ generated by $\mathcal{H}$ (*one-particle space*) and a unity which will be denoted by ø and called the *vacuum*. We call $\Gamma_0\mathcal{H}$ a *Fock algebra* if the scalar product from $\mathcal{H}$ is extended over $\Gamma_0\mathcal{H}$ in such a way that for every $x \in \mathcal{H}$ the operator $\mathbf{a}^+(x)$ of multiplication by x admits the adjoint $\mathbf{a}(x)$ defined on the whole $\Gamma_0\mathcal{H}$ and if the adjoint fulfills the Leibniz rule, i.e. $\langle xf, g\rangle = \langle f, \mathbf{a}(x)\, g\rangle$,

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$[\mathbf{a}(x), \mathbf{a}^+(y)] = \langle x, y\rangle I$ and $\mathbf{a}(x)\emptyset = 0$, where $[A,B] := AB - BA$. Finally, we assume that $\emptyset$ is a unit vector.

The algebras $\Gamma_0\mathcal{H}$ are commutative and are called *Bose algebras* (cf. [6]). We shall write $\Gamma\mathcal{H}$ for the completion of $\Gamma_0\mathcal{H}, <,>$. For $f \in \Gamma\mathcal{H}$, we often write briefly f^* instead of $\mathbf{a}(f)$ and f instead of $\mathbf{a}^+(f)$, and we denote by $\mathcal{H}^n$ the closed span of n-fold products of vectors from $\mathcal{H}$. By $\Gamma_1\mathcal{H}$ we shall denote the linear span of all $b \exp a$ for $a \in \mathcal{H}$ and $b \in \Gamma_0\mathcal{H}$.

3 Main Result

Let $L : \mathcal{H} \to \mathcal{H}$ be a bounded conjugate linear symmetric Hilbert Schmidt operator. Given an orthonormal basis $\{e_n\}_{n\in\mathbf{N}}$ in $\mathcal{H}$, we consider on $\Gamma_0\mathcal{H}$ the operators $\mathbf{a}(h_L) = \sum_{n=1}^{\infty} \mathbf{a}(Le_n)\mathbf{a}(e_n)$ and $\mathbf{a}^+(h_L) = \sum_{n=1}^{\infty} \mathbf{a}^+(e_n)\mathbf{a}^+(Le_n)$. Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. We consider $d\Gamma A = \sum_{n=1}^{\infty} \mathbf{a}^+(Ae_n)\mathbf{a}(e_n)$ on $\Gamma_0\mathcal{H}$, called the second quantization of the operator A. Operators of the form $d\Gamma A + \mathbf{a}^+(h_L) + \mathbf{a}(h_K)$ will be called *quadratic forms*.

Definition 1. For $L, K : \mathcal{H} \to \mathcal{H}$ bounded conjugate linear symmetric Hilbert Schmidt operators, $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator and $x, y \in \mathcal{H}$, we define the mapping

$$d\Phi \begin{pmatrix} A & L & x \\ K & -A^* & y \\ 0 & 0 & 0 \end{pmatrix} = d\Gamma A - \frac{1}{2}\mathbf{a}^+(h_L) + \frac{1}{2}\mathbf{a}(h_K) + \mathbf{a}^+(x) + \mathbf{a}(y)$$

of $\Gamma_0\mathcal{H}$ into $\Gamma\mathcal{H}$.

For a conjugate linear operator $L : \mathcal{H} \to \mathcal{H}$, symmetric and Hilbert Schmidt, we consider on $\Gamma_0\mathcal{H}$ the operator $\mathbf{a}(\delta_L) = \exp\left(-\frac{1}{2}\mathbf{a}(h_L)\right)$ and for $\|L\| < 1$ the operator $\mathbf{a}^+(\delta_L) = \exp\left(-\frac{1}{2}\mathbf{a}^+(h_L)\right)$ into $\Gamma\mathcal{H}$. For a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$, we consider on $\Gamma_0\mathcal{H}$, $\Gamma\left(e^A\right) = \exp(d\Gamma A)$.

Definition 2. We define on $\Gamma_0\mathcal{H}$

$$\Phi \begin{pmatrix} T & v \\ 0 & 1 \end{pmatrix} = e^{\mathbf{a}^+(x) + \mathbf{a}(y)}\mathbf{a}^+(\delta_L)(\Gamma A)\mathbf{a}(\delta_K),$$

where $v = \begin{pmatrix} x \\ y \end{pmatrix}$, $x, y \in \mathcal{H}$, A is bounded linear and invertible and L, K are bounded conjugate linear and Hilbert Schmidt, such that

$$T = \begin{pmatrix} I & L \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -K & I \end{pmatrix}$$

We can now state our main result, realizing the explicit connection between the Lie group and algebra structure. We refer to [7] for the proof and further details.

Theorem 1. *Let* $S = \begin{pmatrix} A & L \\ K & -A^* \end{pmatrix}$, $x, y \in \mathcal{H}$, $v = \begin{pmatrix} x \\ y \end{pmatrix}$ *and* $z \in B\left(0, \frac{\log \frac{3}{2}}{\|S\|}\right)$. *Then* $zd\Phi \begin{pmatrix} S & v \\ 0 & 0 \end{pmatrix}$ *is exponentiable on* $\Gamma_0\mathcal{H}$ *and*

$$\exp\left(zd\Phi\begin{pmatrix} S & v \\ 0 & 0 \end{pmatrix}\right) = e^{C(z)}\Phi\left(\exp\left(\widehat{z}\begin{pmatrix} S & v \\ 0 & 0 \end{pmatrix}\right)\right),$$

where $C(z)$ *is a holomorphic function, which can be explicitly given.*

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