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Wigner Quantization on the Circle and $\mathbf{R}_+$

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Abstract

We construct a deformation quantization for two cases of configuration spaces: the multiplicative group of positive real numbers $\mathbf{R}_+$ and the circle S^1. In these cases we define the momenta using the Fourier transform. Using the identification of symbols of quantum observables — real functions on the phase space — with classical observables, we introduce a non–commutative $\star$–product between a pair of them.

1 Introduction

In this contribution we proceed to study the relationship between classical mechanics and quantum mechanics, and to clarify mathematical connections between them for two simple examples of configuration spaces. Such attempts originally have led to the discipline called quantization methods [1]. The traditional methods contained in the works of Dirac, Heisenberg and Schrödinger are usually called canonical quantization. The subsequent development of this branch brought a general concept of quantization [2] which contains alternative quantization methods, among them the quantization as a deformation of an algebra of classical observables.

In classical mechanics and quantum mechanics, there are two basic concepts: states and observables. While in classical mechanics pure states are points in a phase space and observables are functions on the phase space, in quantum mechanics pure states are one–dimensional subspaces of a separable Hilbert space $\mathcal{H}$, generally of infinite dimension, and observables are selfadjoint operators on the Hilbert space.

In both theories the observables form an associative algebra which is commutative in classical mechanics and non–commutative in quantum mechanics. So the quantization can also be understood as a procedure replacing a commutative algebra by a non–commutative one [3, 4, 5], called a deformation quantization.

In order to perform the deformation, it is useful to describe the observables in quantum and classical mechanics by the objects of the same mathematical nature [1, 6]. For this purpose the Wigner correspondence can be used, which associates real functions on phase space (Wigner symbols) with selfadjoint operators on $\mathcal{H}$. Then the multiplication of operators on $\mathcal{H}$ corresponds to a non–commutative multiplication (so called $\star$–product) of the associated Wigner symbols.

If the symbols $W_F(p,q)$ and $W_G(p,q)$ are associated with the operators $\hat{F}$ and $\hat{G}$, then the multiplication $(W_F \star W_G)(p,q)$ corresponds to $\hat{F}\hat{G}$ according to the diagram

$$\begin{array}{ccc} \hat{F}, \hat{G} & \longrightarrow & \hat{F} \circ \hat{G} \\ \downarrow & & \downarrow \\ W_F(p,q), W_G(p,q) & \longrightarrow & (W_F \star W_G)(p,q) \end{array}$$

2 The Wigner symbol

For the construction of the Wigner symbols we are going to apply our method introduced in [7] for the case of compact groups.

- Let the configuration space $\mathcal{M}$ be a compact unimodular group, quantum Hilbert space $L^2(\mathcal{M}, dx)$, where dx is an invariant measure.
- Let $\hat{H}$ be a selfadjoint integral operator acting on $L^2(\mathcal{M}, dx)$ with the Hilbert–Schmidt kernel $H(x,y)$, i.e.

  $$(\hat{H}\psi)(x) = \int_{\mathcal{M}} H(x,y)\psi y dy,$$

  where

  $$H(x,y) = \overline{H(y,x)}.$$

- Let $\{\pi_i(\mathcal{M}), i \in I\}$ be the set of all irreducible representations of $\mathcal{M}$. According to the Peter–Weyl theorem, any L^2 function on a compact group $\mathcal{M}$ admits a Fourier expansion into the complete orthogonal basis of all matrix elements $\{\phi_k(x) = C^i_{mn}(x), k = (i,m,n) \in U)\}$ of all representations $\pi(\mathcal{M})$. We assume that the basis $\{\phi_k(x)\}$ of Hilbert space $\mathcal{H}$ is normalized.
- Let the operator $\hat{\mathcal{T}}$ act on $L^2(\mathcal{M} \times \mathcal{M})$

  $$\hat{\mathcal{T}} : f(x,y) \to f(xy, xy^{-1}),$$

  and let the inverse operator $\hat{\mathcal{T}}^{-1}$ exist (this is the case, e.g. for Lie groups for which the exponential map is onto).

Then the Wigner symbol of the operator $\hat{H}$ is a function on $\mathcal{M} \times U$ with the first variable in the group $x \in \mathcal{M}$ and the second in the set of indices $k = (i,m,n) \in U$

$$W_H(x,k) = \int_{\mathcal{M}} (\hat{\mathcal{T}}(H(x,y)))\phi_k(y) dy.$$

It can be written

$$W_H(k,x) = \hat{\mathcal{F}}(\hat{\mathcal{T}}(H(x,y)))$$

where $\hat{\mathcal{F}}$ is the Fourier transform in the second variable.

3 Wigner Quantization

We consider the Wigner quantization as quantization defined by the multiplication law between the Wigner symbols – the $\star$–product. Assuming the existence of the inverse operators $\hat{\mathcal{F}}^{-1}$ and $\hat{\mathcal{T}}^{-1}$, the general scheme is : let us have Wigner symbols W_F, W_G and W_{FG}, we define $\star$–product between them

$$W_{FG}(x,k) = (W_F \star W_G)(x,k)$$

$$W_{FG}(x,k) = \hat{\mathcal{F}}_k(\hat{\mathcal{T}}\langle x|\hat{F}\hat{G}|y\rangle) = \hat{\mathcal{F}}_k(\hat{\mathcal{T}}(\int_{\mathcal{M}} dz\langle x|\hat{F}|z\rangle\langle z|\hat{G}|y\rangle)) =$$

$$\hat{\mathcal{F}}_k(\hat{\mathcal{T}}(\int_{\mathcal{M}} dz(\hat{\mathcal{T}}^{-1}\hat{\mathcal{F}}^{-1}W_F)(x,z)(\hat{\mathcal{T}}^{-1}\hat{\mathcal{F}}^{-1}W_G)(z,y))).$$

Thus the $\star$–product is expressed by an integral over the manifold $\mathcal{M}$. The function $H(x,y)$ can be expanded in a double Fourier series

$$H(x,y) = \sum_{mn\in U} h_{m,n}\phi_m(x)\overline{\phi_n(y)},$$

where $h_{m,n} = \overline{h_{n,m}}$, and in some cases one can use the orthogonality relation of the Fourier basis to simplify the relations.

3.1 Wigner quantization on a circle

An arbitrary L^2 function on the circle can be expanded in the Fourier series

$$f(x) = \sum_k f_k\phi_k(x) = \sum_k f_k e^{ikx},$$

where $k = 0, 1, 2, \ldots$ and $x \in (-\pi, \pi)$. The function corresponding to a selfadjoint operator $\hat{H}$ is

$$H(x,y) = \sum_{k,l} h_{k,l}e^{ikx}e^{-ily},$$

where $x, y \in (-\pi, \pi)$, $k, l = 0, \pm1, \pm2, \ldots$, and $h_{k,l} = \overline{h_{l,k}}$. The Wigner symbols of the selfadjoint operators $\hat{H}$ and $\hat{G}$ are

$$W_H(x,k) = \sum_l h_{l,k-l}e^{ix(2l-k)},$$

$$W_G(z,m) = \sum_n g_{n,m-n}e^{iz(2n-m)},$$

and the symbol corresponding to the product of the operators $\hat{H}$ and $\hat{G}$ is

$$W_{GH}(x,l) = \sum_{k,m} g_{m,k}h_{k,l-m}e^{ix(2m-l)}.$$

Let us now start with the symbols W_H and W_G. To determine the $\star$–product, we have to determine the coefficients

$$h_{m,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_H(x, m-n)e^{ix(m+n)}dx.$$

The result is

$$W_{HG}(x,l) = (W_H \star W_G)(x,l) = \sum_{k,m} e^{ix(2m-l)}\frac{1}{2\pi} \times$$

$$\int_{-\pi}^{\pi} dz \frac{1}{2\pi}\int_{-\pi}^{\pi} dy W_H(z, k-m)W_G(y, l-m-k)e^{iz(k+m)}e^{iy(l+k+m)}.$$

Hence for any two real functions on phase space their $\star$–product is given by

$$(h \star g)(x,l) = \sum_{k,m} e^{ix(2m-l)}\frac{1}{2\pi} \times$$

$$\int_{-\pi}^{\pi} dz \frac{1}{2\pi}\int_{-\pi}^{\pi} dy h(z, k-m)g(y, l-m-k)e^{iz(k+m)}e^{iy(l+k+m)}.$$

3.2 Wigner quantization on $\mathbf{R}_+$

Our approach is based on the idea that $\mathbf{R}_+$ is an Abelian multiplicative group with the Haar measure $d\eta = \frac{dy}{y}$ and it is isomorphic with the additive group $\mathbf{R}$ via $y = e^{\eta}$. Hence the characters are related by $e^{ip\ln y} = e^{ip\eta}$ and labeled by $p \in \mathbf{R}$.

Let us consider a selfadjoint operator $\hat{H}$ on $L^2(\mathbf{R}_+, dx)$. The corresponding kernel of an integral operator $H(x,y) = \langle x|\hat{H}|y\rangle = \overline{H(y,x)}$ is also selfadjoint. In order to get a real symbol of $\hat{H}$ on phase space $\mathbf{R}_+ \times \mathbf{R}$, we introduce the following transformations:

- The operator $\hat{\mathcal{T}}$ acts on $L^2(\mathbf{R}_+ \times \mathbf{R}_+, dx\, dy)$ by

$$\hat{\mathcal{T}} : H(x,y) \mapsto h(x,y) = H(xy, xy^{-1}),$$

  and $\hat{\mathcal{T}}^{-1}$ is its inverse

$$\hat{\mathcal{T}}^{-1} : h(x,y) \mapsto H(x,y) = h(\sqrt{xy}, \sqrt{xy^{-1}}),$$

- The operator $\hat{\mathcal{F}}$ acts from $L^2(\mathbf{R}_+ \times \mathbf{R}_+, dx\, dy)$ to $L^2(\mathbf{R}_+ \times \mathbf{R}, dx\, dp)$

$$\hat{\mathcal{F}} : h(x,y) \mapsto W_H(x,p) = \int_0^{\infty} h(x,y)\mathrm{e}^{\imath p\ln y}\frac{\mathrm{d}y}{y}$$

  and the operator $\hat{\mathcal{F}}^{-1}$ is its inverse

$$\hat{\mathcal{F}}^{-1} : W_H(x,p) \mapsto h(x,y) = \int_0^{\infty} W_H(x,p)\mathrm{e}^{-\imath p\ln y}\mathrm{d}p.$$

The Wigner symbol of $\hat{H}$ is defined as

$$W_H(x,p) = \hat{\mathcal{F}}(\hat{\mathcal{T}}(H(x,y)))$$

and it is a real function on $\mathbf{R}_+ \times \mathbf{R}$.

A non–commutative multiplication law — the $\star$–product between the Wigner symbols W_F and W_G on phase space $\mathbf{R}_+ \times \mathbf{R}$, is

$$(W_F \star W_G)(x,p) = W_{FG}(x,p),$$

where

$$(W_F \star W_G)(x,p) = \hat{\mathcal{F}}(\hat{\mathcal{T}}(\int_0^\infty (\hat{\mathcal{T}}^{-1}\hat{\mathcal{F}}^{-1}W_F)(x,z)(\hat{\mathcal{T}}^{-1}\hat{\mathcal{F}}^{-1}W_G)(z,y)))dz$$

It is straightforward to convert the quantization into integration over $\mathbf{R}_+ \times \mathbf{R}$: let $f(x,p)$ and $g(x,p)$, be real functions on phase space, then their $\star$–product is given by the integral

$$(f \star g)(x,l) = \frac{1}{4\pi^2} \int_{\mathbf{R}} \mathrm{d}k \int_{\mathbf{R}} \mathrm{d}p \int_{\mathbf{R}_+} \mathrm{d}z \int_{\mathbf{R}_+} \mathrm{d}y\, f(\sqrt{xyz},p)\, g(\sqrt{\frac{xy}{z}},k) \times$$

$$\mathrm{e}^{-\imath p \ln\sqrt{\frac{xz}{y}}}\, \mathrm{e}^{-\imath k \ln\sqrt{\frac{y}{xz}}}\, \mathrm{e}^{\imath l \ln z}.$$

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