

# Calculating Methods for the Irregular Hermite Interpolation Polynomial

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**Abstract.** Based on the theory of the regular Hermite interpolation polynomial, several calculating methods including basic function, multiple difference quotients, etc., have been proposed to solve the complex irregular Hermite interpolation polynomial.

## Theory of the Hermite interpolation

Set  $y = f(x)$  has definition in interval  $[a, b]$ , and has corresponding values  $y_0, y_1, \dots, y_n$  on the points of  $a \leq x_0 < x_1 < \dots < x_n \leq b$ , if there is a simple function  $P(x)$  satisfied the following formula:

$$P(x_i) = y_i \quad (i = 1, 2, \dots, n) \quad (1)$$

where  $P(x)$  is the interpolation function of  $f(x)$ ,  $x_0, x_1, \dots, x_n$  are the interpolation nodes, interval  $[a, b]$  is the interpolation interval<sup>[1]</sup>.

If  $P(x)$  is the polynomial with the number not more than  $n$ , namely

$$P(x) = a_0 + a_1x + \dots + a_nx^n \quad (2)$$

where  $a_i (i = 0, 1, \dots, n)$  are the real numbers, we can say  $P(x)$  is a interpolation polynomial.

Suppose  $y_j = f(x_j), m_j = f'(x_j) (j = 0, 1, \dots, n)$  on the nodes  $a \leq x_0 < x_1 < \dots < x_n \leq b$ , if there is an interpolation polynomial  $H(x)$  satisfied the conditions (3):

$$H(x_j) = y_j, H'(x_j) = m_j \quad (j = 0, 1, \dots, n) \quad (3)$$

then  $H(x)$  is the Hermite interpolation polynomial<sup>[2]</sup>.

If  $y_i$  and  $m_j (i = 0, 1, \dots, n)$  are all known on the nodes  $a \leq x_0 < x_1 < \dots < x_n \leq b$ , we can say  $H(x)$  is the regular Hermite interpolation polynomial, correspondingly, the methods of seeking  $H(x)$  is the regular Hermite interpolation; If only some of  $m_j (i = 0, 1, \dots, n)$  are known on the nodes  $a \leq x_0 < x_1 < \dots < x_n \leq b$ , we can say  $H(x)$  is the irregular Hermite interpolation polynomial, the methods of seeking  $H(x)$  is the irregular Hermite interpolation<sup>[3-4]</sup>.

The rest of this paper is proposes several plans for solving the irregular Hermite interpolation polynomial.

## Several plans for solving the irregular Hermite interpolation polynomial

If  $y_i (i = 0, 1, \dots, n)$  and  $m_j (j = 0, 1, \dots, k; k < n)$  are all known on the points  $a \leq x_0 < x_1 < \dots < x_n \leq b$ , and  $m_j (j = k + 1, \dots, n)$  are unknown. In this paper, we want to seek an irregular Hermite interpolation polynomial to meet the following equations:

$$\begin{cases} H(x_i) = y_i & (i = 0, 1, \dots, n) \\ H'(x_j) = m_j & (j = 0, 1, \dots, k; k < n) \end{cases} \quad (4)$$

### Using the method of regular Hermite interpolation to resolve the irregular Hermite interpolation

Using the known conditions which satisfied the regular Hermite interpolation polynomial to find out corresponding Hermite interpolation polynomial. Then supposed the form of  $H_{n+k+1}(x)$

and used the surplus conditions determining the unknown part.

Based on the known conditions  $H(x_i) = y_i$  ( $i = 0, 1, \dots, k; k < n$ ) and  $H'(x_j) = m_j$  ( $j = 0, 1, \dots, k; k < n$ ), we can find out the regular Hermite interpolation polynomial  $H_k(x)$ . The process is as follows:

Using the basis function of Lagrange interpolation polynomial to seek interpolation basis function  $\alpha_j(x)$  and  $\beta_j(x)$

$$\begin{cases} \alpha_j(x_i) = \delta_{ji} = \begin{cases} 0, & j \neq i, \\ 1, & j = i, \end{cases} & \alpha'_j(x_i) = 0 \\ \beta_j(x_i) = 0, & \beta'_j(x_i) = \delta_{ji}. \end{cases} \quad (i, j = 0, 1, \dots, k; k < n) \quad (5)$$

Polynomial  $H(x) = H_{2k+1}(x)$  which satisfied the conditions of  $H(x_i) = y_i$  ( $i = 0, 1, \dots, k; k < n$ ) and  $H'(x_j) = m_j$  ( $j = 0, 1, \dots, k; k < n$ ) can be written as follows:

$$H_{2k+1}(x) = \sum_{j=0}^n [y_j \alpha_j(x) + m_j \beta_j(x)] \quad (6)$$

By (5), we can concluded

$$H_{2k+1}(x_i) = y_i, \quad H'_{2k+1}(x_i) = m_i \quad (i = 0, 1, \dots, k; k < n)$$

$$\text{Suppose } \alpha_j(x) = (ax + b)l_j^2(x) \quad (7)$$

where  $l_j(x)$  are the basis functions. We can derived the following conclusions base on (5)

$$\alpha_j(x_j) = (ax_j + b)l_j^2(x_j) = 1,$$

$$\alpha'_j(x_j) = l_j(x_j) [(al_j(x_j) + 2(ax_j + b)l'_j(x_j))] = 0$$

$$\begin{cases} ax_j + b = 1 \\ a + 2l'_j(x_j) = 0 \end{cases}$$

the results is  $a = -2l'_j(x_j), b = 1 + 2x_j l'_j(x_j)$ .

Taking logarithm and derivation at both ends of  $l_j(x) = \frac{(x-x_0)\dots(x-x_{j-1})(x-x_{j+1})\dots(x-x_k)}{(x_j-x_0)\dots(x_j-x_{j-1})(x_j-x_{j+1})\dots(x_j-x_k)}$

( $j = 0, 1, \dots, k; k < n$ ), we can obtain

$$l'_j(x_j) = \sum_{\substack{i=0 \\ i \neq j}}^k \frac{1}{x_j - x_i} \quad (j = 0, 1, \dots, k; k < n) \text{ so}$$

$$\alpha_j(x) = \left[ 1 - 2(x - x_j) \sum_{\substack{i=0 \\ i \neq j}}^k \frac{1}{x_j - x_i} \right] l_j^2(x) \quad (j = 0, 1, \dots, k; k < n) \quad (8)$$

Similarly, we can obtain

$$\beta_j(x) = (x - x_j)l_j^2(x) \quad (j = 0, 1, \dots, k; k < n) \quad (9)$$

We can also proof the interpolation polynomial which satisfied the conditions  $H(x_i) = y_i (i = 0, 1, \dots, k; k < n)$  and  $H'(x_j) = m_j (j = 0, 1, \dots, k; k < n)$  is unique.

Suppose  $H_{2k+1}(x)$  and  $\overline{H_{2k+1}(x)}$  all satisfied the above conditions, then

$$\varphi(x) = H_{2k+1}(x) - \overline{H_{2k+1}(x)} \quad (10)$$

has double roots on each nodes  $x_i (i = 0, 1, \dots, k; k < n)$ , namely  $\varphi(x)$  has  $2k + 2$  double roots. But  $\varphi(x)$  is a polynomial with the number not more than  $2k + 1$ , so  $\varphi(x) \equiv 0$ . The Unique is proofed.

$$\text{Supposed } H_{n+k+1}(x) = H_{2k+1}(x) + Q_{n+k+1}(x) \quad (11)$$

set the highest coefficients of  $Q_{n+k+1}(x)$  as 1, and bring  $(x_0, y_0)$  into (11), through  $H_{2k+1}(x_0) = y_0$ , we can obtain  $Q_{n+k+1}(x_0) = 0$  (12)

namely  $x_0$  is one of the zero point of  $Q_{n+k+1}(x)$ . Then bring  $(x_1, y_1)$  into (11), through  $H_{2k+1}(x_1) = y_1$ , we can obtain

$$Q_{n+k+1}(x_1) = 0 \tag{13}$$

namely  $x_1$  is one of the zero point of  $Q_{n+k+1}(x)$ . Similarly, bring  $x_i (i = 2, 3, \dots, k; k < n)$  into (11), through  $H_{2k+1}(x_i) = y_i (i = 2, 3, \dots, k; k < n)$ , we can obtain

$$Q_{n+k+1}(x_i) = 0 \quad (i = 2, 3, \dots, k; k < n) \tag{14}$$

namely  $x_i (i = 2, 3, \dots, k; k < n)$  are the zero points of  $Q_{n+k+1}(x)$ . Because the number of  $Q_{n+k+1}(x)$  is  $n+k+1$ , so we can suppose

$$Q_{n+k+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_k)A_n(x) \tag{15}$$

where  $A_n(x)$  is a polynomial with the number of  $n$ .

Through  $H_{n+k+1}(x) = H_{2k+1}(x) + Q_{n+k+1}(x)$ , we can obtain

$$H'_{n+k+1}(x) = H'_{2k+1}(x) + Q'_{n+k+1}(x) \tag{16}$$

bring  $x_i (i = 0, 1, \dots, k; k < n)$  into (16), by  $H'_{2k+1}(x_i) = y_i (i = 0, 1, \dots, k; k < n)$ , we can obtain

$$Q'_{n+k+1}(x_i) = 0 \quad (i = 0, 1, \dots, k; k < n) \tag{17}$$

namely  $x_i (i = 0, 1, \dots, k; k < n)$  are the double roots of  $Q_{n+k+1}(x)$ , so

$$Q_{n+k+1}(x) = (x-x_0)^2(x-x_1)^2\cdots(x-x_k)^2B_{n-k-1}(x) \tag{18}$$

where the number of  $Q_{n+k+1}(x)$  is  $n-k-1$ , bringing  $(x_i, y_i) (i = k+1, \dots, n)$  into (11), by  $H_{2k+1}(x_i) = y_i (i = k+1, \dots, n)$ , we can obtain

$$Q_{n+k+1}(x_i) = 0 \quad (i = k+1, \dots, n) \tag{19}$$

namely  $(x_i, y_i) (i = k+1, \dots, n)$  is the zero points of  $Q_{n+k+1}(x)$ , and  $Q_{n+k+1}(x)$  can be expressed as

$$Q_{n+k+1}(x) = (x-x_0)^2(x-x_1)^2\cdots(x-x_k)^2(x-x_{k+1})\cdots(x-x_n)B(x) \tag{20}$$

Owing to the number of  $Q_{n+k+1}(x)$  is  $n+k+1$ , and the number of polynomial

$$(x-x_0)^2(x-x_1)^2\cdots(x-x_k)^2(x-x_{k+1})\cdots(x-x_n)$$

is  $n+k+2$ , so the number of  $B(x)$  is  $-1$ , namely

$$B(x) = x^{-1}$$

### Basic function

Changing  $H_{n+k+1}(x)$  into the form of interpolation base function, then by the known condition, using the method of undetermined coefficient determining its representation.

$$\text{Suppose } H_{n+k+1}(x) = \sum_{j=0}^n \alpha_j(x)y_j + \sum_{j=0}^k \beta_j(x)m_j \tag{21}$$

$$\text{where } \begin{cases} \alpha_j(x_i) = \delta_{ji} = \begin{cases} 0, & j \neq i, \\ 1, & j = i, \end{cases} & \alpha'_j(x_i) = 0 \quad (i, j = 0, 1, \dots, n) \\ \beta_j(x_i) = 0, & \beta'_j(x_i) = \delta_{ji}. \quad (i, j = 0, 1, \dots, k; k < n) \end{cases} \tag{22}$$

$$\text{by (21) } H'_{n+k+1}(x) = \sum_{j=0}^n \alpha'_j(x)y_j + \sum_{j=0}^k \beta'_j(x)m_j \tag{23}$$

bring  $(x_0, y_0)$  into (21), we can obtain

$$\begin{cases} \alpha_0(x_0) = 1, \alpha_i(x_0) = 0 \quad (i = 1, 2, \dots, n) \\ \beta_i(x_0) = 0 \quad (i = 0, 1, \dots, k; k < n) \end{cases} \tag{24}$$

bring  $(x_0, y_0)$  into (23), we can obtain

$$\begin{cases} \alpha'_i(x_0) = 0 \quad (i = 0, 1, \dots, n) \\ \beta'_0(x_0) = 1, \beta'_i(x_0) = 0 \quad (i = 1, 2, \dots, k; k < n) \end{cases} \tag{25}$$

By (24) and (25), we can obtain

$$\alpha_0(x) = (x-x_1)^2(x-x_2)^2\cdots(x-x_k)^2(x-x_{k+1})\cdots(x-x_n)(A_0x+B_0) \tag{26}$$

and

$$\beta_0(x) = (x - x_0)(x - x_1)^2(x - x_2)^2 \cdots (x - x_k)^2(x - x_{k+1}) \cdots (x - x_n)C_0 \quad (27)$$

In (43), by  $\begin{cases} \alpha_0(x_0) = 1 \\ \alpha'_0(x_0) = 0 \end{cases}$ , we can obtain the following equations:

$$\begin{cases} (x_0 - x_1)^2(x_0 - x_2)^2 \cdots (x_0 - x_k)^2(x_0 - x_{k+1}) \cdots (x_0 - x_n)(A_0x_0 + B_0) = 1 \\ 2(x_0 - x_1)(x_0 - x_{k+1}) \cdots (x_0 - x_n)(A_0x_0 + B_0) \prod_{i=2}^k (x_0 - x_i)^2 + \cdots + (x_0 - x_1)^2(x_0 - x_2)^2 \cdots (x_0 - x_k)^2(x_0 - x_{k+1}) \cdots (x_0 - x_n)A_0 = 0 \end{cases}$$

Solving these equations, we can obtain  $A_0, B_0$ . Bring  $x_0$  into (44), by  $\beta_0(x_0) = 0$ , we can obtain  $C_0$ .

Similarly, bring  $(x_j, y_j)$  ( $j = 1, 2, \dots, k; k < n$ ) into (21) and (23), we can obtain:

$$\begin{cases} \alpha_i(x_j) = 1 & (i = 1, 2, \dots, n; j = 1, 2, \dots, k; k < n; j = i) \\ \alpha_i(x_j) = 0 & (i = 1, 2, \dots, n; j = 1, 2, \dots, k; k < n; j \neq i) \\ \beta_i(x_j) = 0 & (i, j = 1, 2, \dots, k; k < n) \end{cases} \quad (28)$$

and

$$\begin{cases} \alpha'_i(x_j) = 0 & (i = 1, 2, \dots, n; j = 1, 2, \dots, k; k < n) \\ \beta'_i(x_j) = 1 & (i, j = 1, 2, \dots, k; k < n; j = i) \\ \beta'_i(x_j) = 0 & (i, j = 1, 2, \dots, k; k < n; j \neq i) \end{cases} \quad (29)$$

By (28) and (29), we can obtain

$$\alpha_i(x) = (x - x_0)^2(x - x_1)^2 \cdots (x - x_{i-1})^2(x - x_{i+1})^2 \cdots (x - x_k)^2(x - x_{k+1}) \cdots (x - x_n)(A_i x + B_i) \quad (30)$$

$(i = 1, 2, \dots, k, k + 1, \dots, n)$

and

$$\beta_i(x) = (x - x_0)^2(x - x_1)^2 \cdots (x - x_{i-1})^2(x - x_{i+1})^2 \cdots (x - x_k)^2(x - x_{k+1}) \cdots (x - x_n)C_i \quad (31)$$

$(i = 1, 2, \dots, k; k < n)$

To sum up, we can determined  $H_{n+k+1}(x)$  through (21).

### Multiple difference quotients (Universal formula)

We can obtain the representation of  $H_{n+k+1}(x)$  according to the given known conditions  $H(x_i) = y_i$  ( $i = 0, 1, \dots, k; k < n$ ) and  $H'(x_j) = m_j$  ( $j = 0, 1, \dots, k; k < n$ ) by difference quotient.

The formula of Lagrange interpolation can be seen as the promotion of linear equation of two points, if we started from the following equation

$$P_1(x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) \quad (32)$$

and extended it to the situation of the  $n + 1$  interpolation points  $(x_0, f_0), (x_1, f_1), \dots, (x_n, f_n)$ , the interpolation polynomial can be expressed as:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \quad (33)$$

where  $a_0, a_1, \dots, a_n$  are the undetermined coefficients, which can be determined by the interpolation conditions  $P_n(x_j) = f_j$  ( $j = 0, 1, \dots, n$ ).

When  $x = x_0$ ,  $P_n(x_0) = a_0 = f_0$ ; when  $x = x_1$ ,  $P_n(x_1) = a_0 + a_1(x_1 - x_0) = f_1$ , we can concluded

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} \quad (34)$$

When  $x = x_2$ ,  $P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2$ , we can concluded

$$a_2 = \frac{\frac{f_2 - f_0}{x_2 - x_0} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_1} \quad (35)$$

So on, we can conclude  $a_3, a_4, \dots, a_n$ . In order to write the general expression of coefficient  $a_k$ , we introduced the definition of difference quotient.

Usually, we called  $f[x_0, x_k] = \frac{f(x_k) - f(x_0)}{x_k - x_0}$  is the first order difference quotient of  $f(x)$

appoint to  $x_0, x_k$ , correspondingly, we Called  $f[x_0, x_1, x_k] = \frac{f[x_0, x_k] - f[x_0, x_1]}{x_k - x_1}$  is the second order

difference quotient of  $f(x)$  appoint to  $x_0, x_1, x_k$ . Generally, we called

$$f[x_0, x_1, \dots, x_k] = \frac{f[x_0, x_1, \dots, x_{k-2}, x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_{k-1}} \tag{36}$$

is the  $k$ -order difference quotient of  $f(x)$  appoint to  $x_0, x_1, x_k$ .

The process of using multiple difference quotients solving the irregular Hermite interpolation polynomial can be displayed by the table of difference quotient, the detail is shown in table 1.

**Table 1: Table of difference quotient**

$x_k$	$y_k$	$x - x_0$	$x - x_0$	$x - x_1$	$x - x_1$	...
$x_0$	$y_0$					
$x_0$	$y_0$	$f[x_0, x_0]$				
$x_1$	$y_1$	$f[x_0, x_1]$	$f[x_0, x_0, x_1]$			
$x_1$	$y_1$	$f[x_1, x_1]$	$f[x_0, x_1, x_1]$	$f[x_0, x_0, x_1, x_1]$		
$x_2$	$y_2$	$f[x_1, x_2]$	$f[x_1, x_1, x_2]$	$f[x_0, x_1, x_1, x_2]$	$f[x_0, x_0, x_1, x_1, x_2]$	
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

By the property of difference quotient: If  $f(x)$  exists  $n$ -order derivative in  $[a, b]$ , and the nodes  $x_0, x_1, \dots, x_n \in [a, b]$ , the relationship between  $n$ -order difference quotient and derivative is as follows:

$$f[x_0, x_1, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}, \xi \in [a, b] \tag{37}$$

$$f[x_i, x_i] = \frac{f'(x_i)}{1!}, (i = 0, 1, \dots, n) \tag{38}$$

$$f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i}, (0 \leq i < j \leq n) \tag{39}$$

By the given conditions, we can calculate

$$f(x_i) = y_i, (i = 0, 1, \dots, n) \text{ and } f'(x_i) = m_i, (i = 0, 1, \dots, k; k < n)$$

Thus, the required irregular Hermite interpolation polynomial is

$$H_{n+k+1}(x) = f(x_0) + f[x_0, x_0](x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 + f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1) + \dots + f[x_0, x_1, \dots, x_n](x - x_0)^2 \dots (x - x_{n-1}) \tag{40}$$

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