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On Poisson Realizations of Transitive Lie Algebroids

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Abstract

We show that every transitive Lie algebroid over a connected symplectic manifold comes from an intrinsic Lie algebroid of a symplectic leaf of a certain Poisson structure. The reconstruction of the corresponding Poisson structures from the Lie algebroid is given in terms of coupling tensors.

1 Introduction

The correspondence between Poisson structures and Lie algebroids plays an important role in various problems in Poisson geometry (see, for example, [1, 2, 3, 4]). As is well known [5], the cotangent bundle of an arbitrary Poisson manifold carries a natural Lie algebroid structure compatible with the symplectic foliation. Let (M, Ψ) be a Poisson manifold with Poisson tensor Ψ. Then the Poisson bracket on M admits the natural extension to the bracket for 1-forms on M:

$$\{\alpha, \beta\}_{T^*M} = \Psi^\#(\alpha)\rfloor d\beta - \Psi^\#(\beta)\rfloor d\alpha - d\langle\alpha, \Psi^\#(\beta)\rangle,$$

here $\Psi^\# : T^*M \to TM$ is the vector bundle morphism associated with Ψ. This structure makes the cotangent bundle T^*M into a Lie algebroid $\big(T^*M, \Psi^\#, \{\,,\,\}_{T^*M},\big)$ called the Lie algebroid of the Poisson manifold (M, Ψ). Given a symplectic leaf (B, ω) of M one can restrict the bracket $\{\,,\,\}_{T^*M}$ to a Lie bracket on smooth sections of the restricted cotangent bundle $T^*_B M$ [3, 6, 7]. The result is a *transitive* Lie algebroid $(T^*_B M, \Psi^\#_B, \{\,,\,\}_{T^*_B M})$ over B called the *Lie algebroid of the symplectic leaf B*.

So, every symplectic leaf of a Poisson manifold carries an intrinsic transitive Lie algebroid structure which controls the infinitesimal Poisson geometry around the leaf. One can ask the natural question: is there also a connection in the reverse direction between transitive Lie algebroids over a symplectic base and Poisson structures? In this paper we give an affirmative answer to this question. The reconstruction of the Poisson structure from a transitive Lie algebroid is based on the contravariant version [8] of the minimal

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coupling procedure due to Sternberg [9, 10]. The corresponding Poisson structure is called a *coupling tensor* [8] and represents the result of coupling the symplectic base structure and the fiberwise Lie-Poisson structure on a certain vector bundle via a *connection* of the transitive Lie algebroid [6, 7]. Remark that connection-dependent Poisson structures of such a type were studied in [11, 12] in the context of a Hamiltonian formulation of Wong's equation. As an application, we discuss the passage between transitive Lie algebroids and coupling tensors from the viewpoint of Hamiltonian formalism.

2 Reconstruction of Poisson Structures

Recall that a *Lie algebroid* over a manifold B is a triple $(A, \rho, \{\, , \,\}_A)$ consisting of a vector bundle $A \to B$ together with a bundle map $\rho : A \to TB$, called the *anchor*, and a Lie algebra structure $\{\, , \,\}_A$ on the space of smooth sections $\Gamma(A)$ such that: ρ is a Lie algebra homomorphism and the Leibniz identity holds, $\{a_1, f a_2\}_A = f\{a_1, a_2\}_A + (L_{\rho(a_1)} f)a_2$ for any $a_1, a_2 \in \Gamma(A)$, $f \in C^\infty(B)$. An isomorphism between two Lie algebroids is defined as vector bundle morphism compatible with the anchors and the Lie brackets in a natural way. The kernel $\mathfrak{g}_B := \ker \rho \subset A$ of the anchor is called the *isotropy* of a Lie algebroid. If ρ is a fiberwise surjection, then the the Lie algebroid is said to be *transitive* [6, 7]. In this case, the isotropy $\mathfrak{g}_B$ is a *Lie algebra bundle.* For every $\xi \in B$ the fiber $\mathfrak{g}_\xi$ carries a Lie bracket $[,]^{\mathrm{fib}}_\xi$ induced from $\{\, , \,\}_A$, which varies smoothly with ξ. Then the dual $\mathfrak{g}^*_\xi$ of $\mathfrak{g}_\xi$ is equipped with the Lie-Poisson bracket which makes $\mathfrak{g}^*_B$ into a bundle of Lie-Poisson manifolds (for more detail, see [1, 2]).

For example, in the case of the Lie algebroid $(T^*_B M, \Psi^\#_B, \{\, , \,\}_{T^*_B M})$ of a symplectic leaf (B, ω) of a Poisson manifold (M, Ψ) , the isotropy coincides with the annihilator $TB^0 = \ker \Psi^\#_B$ of TB in $T_B M$. The dual of the isotropy is identified with the normal bundle $E = T_B M/TB$ to B. The fiberwise Lie-Poisson structure of E is just the *linearized transverse Poisson structure* of Ψ at B [13].

Now we formulate our main result.

Theorem 1. *Every transitive Lie algebroid* $(A, \rho, \{\, , \,\}_A)$ *over a connected symplectic manifold* (B, ω) *admits a Poisson realization in the following sense. In a neighborhood of the zero section* B *of the dual* $\mathfrak{g}^*_B$ *of the isotropy there exists a Poisson tensor* Π *such that: (i)* (B, ω) *is a symplectic leaf of* Π*, and (ii) the Lie algebroid of the leaf* (B, ω) *is isomorphic to* $(A, \rho, \{\, , \,\}_A)$.

To prove this theorem we give an explicit description of the Poisson structure Π in terms of the algebroid A. As we will see the main features of Π are completely different from the properties of the dual Lie-Poisson structure on A^* [14] uniquely determined by the homogeneity condition: the bracket of fiberwise linear functions on A^* is fiberwise linear.

Suppose we are given a transitive Lie algebroid $(A, \rho, \{\, , \,\}_A)$ over a connected symplectic manifold (B, ω). Then there is an exact sequence of vector bundles $\mathfrak{g}_B \to A \xrightarrow{\rho} TB$. Fix a vector bundle morphism $\gamma : TB \to A$ such that $\rho \circ \gamma = \mathrm{id}$. Such a mapping is called a *connection* of the transitive Lie algebroid [6, 7]. The *curvature* of γ is the $\mathfrak{g}_B$-valued valued 2-form $\mathcal{R} \in \Omega^2(B, \mathfrak{g}_B)$ determined by $\mathcal{R}(u_1, u_2) := \{\gamma(u_1), \gamma(u_2)\}_A - \gamma([u_1, u_2])$ for $u_1, u_2 \in \mathcal{X}(B)$. One can associate to γ a linear connection ∇ on the isotropy $\mathfrak{g}_B$, called an *adjoint connection*, and defined as follows $\nabla_u \eta = \{\gamma(u), \eta\}_A$ for $u \in \mathcal{X}(B), \eta \in \Gamma(\mathfrak{g}_B)$. From the Lie algebroid axioms we get the following information [7]. The adjoint connection ∇ preserves the fiberwise Lie structure on $\mathfrak{g}_B$,

$$\nabla([\eta_1, \eta_2]^{\text{fib}}) = [\nabla\eta_1, \eta_2]^{\text{fib}} + [\eta_1, \nabla\eta_2]^{\text{fib}}. \tag{2.1}$$

and the curvature form $\text{Curv}^\nabla : TB \oplus TB \to \text{End}(\mathfrak{g}_B)$ of ∇ is related with the 2-form $\mathcal{R}$ by the adjoint operator,

$$\text{Curv}^\nabla = \text{ad} \circ \mathcal{R}, \tag{2.2}$$

where $\text{ad} \circ \eta := [\eta, \cdot]^{\text{fib}}$ for $\eta \in \Gamma(\mathfrak{g}_B)$. Moreover, the modified Bianchi identity holds

$$\nabla\mathcal{R} = 0. \tag{2.3}$$

Using the symplectic structure ω on B and $\mathcal{R}$, let us introduce the following 2-form on the total space $\mathfrak{g}_B^*$: $\mathcal{F} := \pi^*\omega - \ell \circ \pi^*\mathcal{R}$. Here $\pi : \mathfrak{g}_B^* \to B$ is the projection and $\ell : \Gamma(\mathfrak{g}_B) \to C^\infty_{\text{lin}}(\mathfrak{g}_B^*)$ is the natural identification of $\Gamma(\mathfrak{g}_B)$ with the space $C^\infty_{\text{lin}}(\mathfrak{g}_B^*)$ of the smooth *fiberwise linear* functions on $\mathfrak{g}_B^*$, $\ell(\eta)(x) =< x, \eta(\xi) >$ $(x \in \mathfrak{g}_B^*, \xi = \pi(x))$.

Fix a basis $\{\eta_\sigma\}$ of local sections of $\mathfrak{g}_B$. Let $x = (x^\sigma)$ be the associated coordinates on the fiber of $\mathfrak{g}_B^*$ and $\xi = (\xi^i)$ are local coordinates on B. In coordinates, we have $\mathcal{F} = \frac{1}{2}\mathcal{F}_{ij}(\xi, x) d\xi^i \wedge d\xi^j$, where $\mathcal{F}_{ij}(\xi, x) = \omega_{ij}(\xi) - x^\sigma \mathcal{R}_{\sigma,ij}(\xi)$. It is clear that the 2-form $\mathcal{F}$ is nondegenerate in a neighborhood N_γ of B in $\mathfrak{g}_B^*$. Let us define (local) vector fields on the total space $\mathfrak{g}_B^*$ by $\text{hor}_i := \frac{\partial}{\partial\xi^i} - \theta^\sigma_{i\nu}(\xi) x^\nu \frac{\partial}{\partial x^\sigma}$, where $\nabla_{\frac{\partial}{\partial\xi^i}} \eta_\nu = -\theta^\sigma_{i\nu}(\xi)\eta_\sigma$. The vector fields hor_i form the *horizontal distribution* on $\mathfrak{g}_B^*$ of the dual connection ∇^*. Next, the fiberwise Lie-Poisson structure on $\mathfrak{g}_B^*$ induces the Poisson tensor $\Lambda = \frac{1}{2}\lambda^{\alpha\beta}_\nu(\xi)\, x^\nu \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}$ on the total space, where $\lambda^{\alpha\beta}_\nu(\xi)$ are the structure constants of $\mathfrak{g}_\xi$ with respect to the basis $\{\eta_\sigma(\xi)\}$. Finally, we define the following bivector field

$$\Pi_\gamma := -\frac{1}{2}\mathcal{F}^{ij}\, \text{hor}_i \wedge \text{hor}_j + \Lambda, \tag{2.4}$$

where $\mathcal{F}^{is}\mathcal{F}_{sj} = \delta^i_j$. Although Π_γ is defined in terms of coordinates and a basis, one can show that it is independent of these choices. Thus, Π_γ is a well-defined bivector field on N_γ depending on the connection γ.

Proposition 1. *Bivector field* Π_γ (2.4) *is a Poisson tensor satisfying the properties (i),(ii) in Theorem 1.*

The Jacoby identity for Π_γ follows from the invariance property (2.1), the curvature identity (2.2) and the modified Bianchi identity (2.3), [8]. By construction (B, ω) is a symplectic leaf of Π_γ. Taking into account splitting $A = \gamma(TB) \oplus \mathfrak{g}_B$, we identify A with $T^*_B\mathfrak{g}_B^* = TB \oplus \mathfrak{g}_B$. Then, looking at the infinitesimal part of Π_γ at B, we see that the property (ii) in Theorem 1 holds.

The Poisson tensor Π_γ is called the *coupling tensor* [8] associated with a pair (A, γ). Observe that Π_γ is the sum of two bivector fields Π^H_γ and Π^V_γ called the horizontal and vertical parts, respectively. The vertical part $\Pi^V_\gamma = \Lambda$ is a Poisson tensor completely determined by the fiberwise Lie-Poisson structure of $\mathfrak{g}_B^*$. The horizontal part Π^H_γ satisfies the Jacobi identity if and only if the curvature $\mathcal{R}^\gamma$ vanishes, or equivalently, the subspace $\pi^* C^\infty(B)$ is closed with respect to the Poisson bracket. In this case, Π_γ^H is just the horizontal lift of the Poisson tensor on (B, ω).

For a given A the Poisson tensor Π_γ depends on the choice of γ. Suppose we are given other connection $\tilde{\gamma}$ of A and let $\Pi_{\tilde{\gamma}}$ be the corresponding coupling tensor. We say that Π_γ and $\Pi_{\tilde{\gamma}}$ are *neighborhood equivalent* if there exists a diffeomorphism $\mathbf{f} : U \to \tilde{U}$ between two neighborhoods $U \subset N_\gamma$ and $\tilde{U} \subset N_{\tilde{\gamma}}$ of B in $\mathfrak{g}_B^*$ such that $\mathbf{f}^* \Pi_{\tilde{\gamma}} = \Pi_\gamma$.

Proposition 2. Π_γ *is independent of the choice of a connection* γ *up to neighborhood equivalence.*

So, one can speak on an intrinsic coupling tensor of the transitive Lie algebroid A. The proof of this Proposition is based on a contravariant version [8] of the Moser homotopy method. The key observation here is that there exists a smooth homotopy between any two connections of A which induces a homotopy between corresponding coupling tensors.

3 Linear Hamiltonian Vector Fields

One can associate a Poisson algebra to a given a transitive Lie algebroid A over a connected symplectic manifold (B, ω) in the following way. Let $C_{\mathrm{aff}}^\infty(\mathfrak{g}_B^*)$ be the space of smooth *fiberwise affine* functions on $\mathfrak{g}_B^*$. Then

$$C_{\mathrm{aff}}^\infty(\mathfrak{g}_B^*) \approx C^\infty(B) \oplus C_{\mathrm{lin}}^\infty(\mathfrak{g}_B^*) \tag{3.1}$$

and every fiberwise affine function ϕ is represented as $\phi = \pi^* f + \ell(\eta) \approx f \oplus \eta$, where $f \in C^\infty(B)$ and $\eta \in \Gamma(\mathfrak{g}_B)$. Fix a connection γ of A and consider the corresponding data $(\nabla, \mathcal{R})$. Then one can define a Lie bracket on $C_{\mathrm{aff}}^\infty(\mathfrak{g}_B^*)$ by

$$\{\phi_1, \phi_2\}_{\mathrm{aff}} := \{f_1, f_2\}_B \oplus \left(\nabla_{v_{f_1}} \eta_2 - \nabla_{v_{f_2}} \eta_1 + [\eta_1, \eta_2]^{\mathrm{fib}} + \mathcal{R}(v_{f_1}, v_{f_2}) \right), \tag{3.2}$$

for $\phi_1 = f_1 \oplus \eta_1$ and $\phi_2 = f_2 \oplus \eta_2$. Here $\{,\}_B$ is the Poisson bracket and v_f is the Hamiltonian vector field of f on (B, ω). Define also the *linearized pointwise product* $\circ$ for affine functions by $\phi_1 \circ \phi_2 := (f_1 \cdot f_2) \oplus (f_1 \cdot \eta_2 + f_2 \cdot \eta_1)$. This makes $C_{\mathrm{aff}}^\infty(\mathfrak{g}_B^*)$ into a *commutative associative algebra* with unit $(1 \oplus 0)$. One can show that the bracket $\{,\}_{\mathrm{aff}}$ and the linearized pointwise product are compatible by the Leibniz identity and hence the triple $(C_{\mathrm{aff}}^\infty(\mathfrak{g}_B^*), \circ, \{,\}_{\mathrm{aff}})$ defines a *Poisson algebra* associated with (A, γ). This algebra is independent of the choice of γ up to an isomorphism.

A vector field $\mathcal{V}$ on the total space $\mathfrak{g}_B^*$ is called *linear* if the Lie derivative $L_\mathcal{V} : C^\infty(\mathfrak{g}_B^*) \to C^\infty(\mathfrak{g}_B^*)$ along $\mathcal{V}$ sends $C_{\mathrm{lin}}^\infty(\mathfrak{g}_B^*)$ into $C_{\mathrm{lin}}^\infty(\mathfrak{g}_B^*)$. This implies that $\mathcal{V}$ *descends* to a vector field v on B, $d\pi \circ \mathcal{V} = v \circ \pi$. The Lie algebra of linear vector fields is denoted by $\mathcal{X}_{\mathrm{lin}}(\mathfrak{g}_B^*)$. By the analogy with Poisson manifolds, we say that a linear vector field $\mathcal{V}$ is *Hamiltonian* relative to bracket (3.2) if there exists a fiberwise affine function $\phi = f \oplus \eta$ such that

$$L_\mathcal{V} = \mathrm{ad}_\phi \ \text{ on } C_{\mathrm{aff}}^\infty(\mathfrak{g}_B^*), \tag{3.3}$$

where $\mathrm{ad}_\phi := \{\phi, \cdot\}_{\mathrm{aff}}$ is the adjoint operator of ϕ. The Hamiltonian vector field of ϕ is denoted by $\mathcal{V} = \mathcal{V}_\phi$. Clearly, $\mathcal{V}_\phi$ descends to v_f. In the contrast to the usual situation, one can not say that every ϕ admits a linear Hamiltonian vector field. Indeed, condition (3.3) says that ad_ϕ as a derivation of the associative algebra $(C^\infty_{\mathrm{aff}}(\mathfrak{g}^*_B), \circ)$ admits an extension to a derivation of $C^\infty(\mathfrak{g}^*_B)$. But it is not true for an arbitrary ϕ. To see that, for every $\phi = f \oplus \eta$ we define the *torsion* of ad_ϕ as a $\mathbb{R}$-linear operator $\mathcal{T}_\phi : C^\infty(B) \to C^\infty_{\mathrm{lin}}(\mathfrak{g}^*_B)$ letting $\mathcal{T}_\phi(g) := \mathcal{R}(v_f, v_g) - \nabla_{v_g}\eta \ \forall\, g \in C^\infty(B)$. The space of all torsion free functions is denoted by $\mathcal{A}_\gamma := \{\phi \in C^\infty_{\mathrm{aff}}(\mathfrak{g}^*_B) \mid \mathcal{T}_\phi = 0\}$. In fact, $\mathcal{A}_\gamma$ is a Lie subalgebra in $(C^\infty_{\mathrm{aff}}(\mathfrak{g}^*_B), \{,\}_{\mathrm{aff}})$. Since $C^\infty_{\mathrm{lin}}(\mathfrak{g}^*_B)$ is an ideal in the Poisson algebra, splitting (3.1) is invariant with respect to ad_ϕ only if its torsion vanishes. Finally, we observe that $\phi \in C^\infty_{\mathrm{lin}}(\mathfrak{g}^*_B)$ admits a linear Hamiltonian vector field in the sense of (3.3) if and only if $\phi \in \mathcal{A}_\gamma$. Moreover, the correspondence

$$\mathcal{A}_\gamma \ni \phi \mapsto \mathcal{V}_\phi \in \mathcal{X}_{\mathrm{lin}}(\mathfrak{g}^*_B) \tag{3.4}$$

is a Lie algebra homorphism.

Proposition 3. *Let Π_γ be the coupling tensor associated with (A, γ). Then the Lie algebra of all linear vector fields on $\mathfrak{g}^*_B$ which are Hamiltonian relative to the Poisson structure Π_γ, coincides with the image of $\mathcal{A}_\gamma$ under homorphism* (3.4)*.*

So, the complement $C^\infty_{\mathrm{aff}}(\mathfrak{g}^*_B) \setminus \mathcal{A}_\gamma$ consists of all affine functions ϕ whose Hamiltonian vector fields $\Pi^\#_\gamma d\phi$ are not linear. One can show that $C^\infty_{\mathrm{aff}}(\mathfrak{g}^*_B) \setminus \mathcal{A}_\gamma$ is nonempty. This observation is related with the phenomenon: the linearization of Hamiltonian systems at a given symplectic leaf of a Poisson manifold may destroy the Hamiltonian property.

References

[1] Cannas da Silva A and Weinstein A, Geometric models for noncommutative algebras, Berkeley Mathematics Lecture Notes, AMS, Providence, 1999.

[2] Fernandes R L, Lie algebroids, holonomy and characteristic class, *Adv.in Math.* **170** (2002), 119–179.

[3] Itskov V M, Karasev M V and Vorobjev Yu M, Infinitesimal Poisson geometry, *Amer. Math. Soc. Transl.* (2) **187** (1998), 327–360.

[4] Vaisman I, Lectures on the Geometry of Poisson manifolds, Progress in Mathematics, **118**, Birkhäuser, Berlin,1994.

[5] Weinstein A, Symplectic groupoids and Poisson manifolds, *Bull. Amer.Math.Soc.***16** (1987),101-104.

[6] Kubarski J, The Chern-Weil homorphism of regular Lie algebroids, Publ.Dep.Math. Nouvelle Ser.,Univ. Claude-Bernaard Lyon 1,1991,1-69.

[7] Mackenzie K C H, Lie Groupoids and Lie Algebroids in Differential Geometry, LMS Lecture Note Ser., **124**, Cambridge Univ. Press, Cambridge, 1987.

[8] Vorobiev Yu M, Coupling tensors and Poisson geometry near a single symplectic leaf, Lie Algebroids, Banach Center Publ. **54** (2001) 249-274.

[9] Guillemin V, Lerman E and Sternberg S, Symplectic Fibrations and Multiplicity Diagrams, Cambridge Univ. Press, Cambridge, 1996.

[10] Sternberg S, Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Young–Mills field, *Proc. Nat. Acad. Sci . U.S.A.* **74** (1977), 5253–5254.

[11] Montgomery R, Canonical formalism of a classical particle in a Yang–Mills field and Wong's equations, *Let. Math. Phys.* **8** (1984), 59–67.

[12] Motgomery R, Marsden J E and Ratiu T, Gauged Lie–Poisson structures, *Cont. Math. AMS*, **28** (Boulder Proceedings on Fluids and Plasmas) (1984), 101–114.

[13] Weinstein A, The local structure of Poisson manifolds, *J. Diff. Geom.* **18** (1983), 523–557.

[14] Courant T J, Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 631–661.