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New Geometrical Applications of the Elliptic Integrals: The Mylar Balloon

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Abstract

An explicit parameterization in terms of elliptic integrals (functions) for the Mylar balloon is found which then is used to calculate various geometric quantities as well as to study all kinds of geodesics on this surface.

1 Introduction

Elliptic integrals and functions are subjects which nowadays often are relegated to the hinterlands of the college mathematics curriculum. We can only guess at the reason for this, but one possible explanation is the advent of efficient computational means for integrals. While standard integration techniques allow us to obtain closed form expressions (in terms of trigonometric functions, exponentials and logarithms) for any integral of the form

$$\int \mathcal{R}(x, \sqrt{P(x)})\mathrm{d}x$$

where $\mathcal{R}(x, \sqrt{P(x)})$ is a rational function and $P(x)$ is a linear or quadratic polynomial, we are forced to expand our dictionary of "elementary" functions if we wish to handle polynomials of higher degree. In particular, when $P(x)$ is cubic or quartic, then the required functions are called *elliptic functions*. For a brief history of the development of elliptic functions, see [1]. For a straightforward exposition of their properties and applications, see [2] and [3]. An alternative approach based on the so called *symmetric elliptic integrals* along respective computational issues and algorithms are discussed in [4]. Finally, for a neat recent approach in terms of dynamical systems, see [5].

The main point of this article is that elliptic functions provide an effective (albeit underused) tool for describing geometric objects. Here, due to the shortage of space we shall focus mainly on a single illustrative example — the so called Mylar balloon — where elliptic integrals and functions are essential tools for obtaining interesting geometric information beyond simple numerical calculation and depiction.

2 Elliptic Functions and Elliptic Integrals

The easiest way to understand the elliptic functions is to consider them as analogies of the ordinary trigonometric functions. From freshman calculus, we know that

$$\arcsin(x) = \int_0^x \frac{\mathrm{d}u}{\sqrt{1-u^2}}.$$

Of course, if $x = \sin(t)$ $(-\pi/2 \le t \le \pi/2)$, then we have

$$t = \arcsin(\sin(t)) = \int_0^{\sin(t)} \frac{\mathrm{d}u}{\sqrt{1-u^2}}.$$

In this way, we may view $\sin(t)$ as an inverse function for the integral. Now, fixing some k with $0 \le k \le 1$ (called the *modulus*), we make the

Definition 1. The *Jacobi sine function* $\mathrm{sn}(u,k)$ is the inverse function of the following integral. Namely,

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}. \tag{2.1}$$

More generally, we write

$$F(z,k) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} \tag{2.2}$$

and call $F(z,k)$ an *elliptic integral of the first kind*.

An *elliptic integral of the second kind* is defined by

$$E(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

When $z = 1$ in $F(z,k)$ and $E(z,k)$, then these integrals are respectively denoted by $K(k)$ and $E(k)$ and called the *complete* elliptic integrals of the first and second kind.

Finally, the *elliptic integral of the third kind with a parameter* n is defined by

$$\Pi(z,n,k) = \int_0^z \frac{\mathrm{d}t}{(1-nt^2)\sqrt{1-t^2}\sqrt{1-k^2t^2}}. \tag{2.3}$$

The *Jacobi cosine function* $\mathrm{cn}(u,k)$ may be defined in terms of $\mathrm{sn}(u,k)$;

$$\mathrm{sn}^2(u,k) + \mathrm{cn}^2(u,k) = 1.$$

A third *Jacobi elliptic function* $\mathrm{dn}(u,k)$ is defined by the equation

$$\mathrm{dn}^2(u,k) + k^2\,\mathrm{sn}^2(u,k) = 1.$$

3 Some Differential Geometry

We are interested in explicitly describing geometric objects by parameterization in terms of elliptic functions. Of course, one reason we want to do this is that we have an array of classical tools with which to study such a parameterized surface. These tools are the heart and soul of differential geometry and we now recall some of its basics. Modern expositions of the subject can be found in, for instance, [6], [7], [8], [9] and [10]. A parameterized surface $\mathcal{S}$ is determined almost uniquely by its first and second fundamental forms

$$I = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2 \qquad \text{and} \qquad II = L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\mathrm{d}v + N\,\mathrm{d}v^2 \tag{3.1}$$

where the coefficients are given by

$$\begin{aligned} E = E[u,v] &= \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = F[u,v] = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = G[u,v] = \mathbf{x}_v \cdot \mathbf{x}_v, \\ L = L[u,v] &= \mathbf{x}_{uu} \cdot \mathbf{n}, \quad M = M[u,v] = \mathbf{x}_{uv} \cdot \mathbf{n}, \quad N = N[u,v] = \mathbf{x}_{vv} \cdot \mathbf{n}. \end{aligned} \tag{3.2}$$

Here $\mathbf{n}$ is the unit normal vector to $\mathcal{S}$

$$\mathbf{n} = \mathbf{n}[u,v] = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}. \tag{3.3}$$

Intuitively, the *metric coefficients* E, F and G describe the stretching necessary to map a piece of the plane up to the surface under the parameterization, while the coefficients L, M and N of II have more to do with acceleration and, hence, curvature. Indeed, there are classical formulas which describe two types of curvatures at every point of the surface. These are the *Gaussian* and the *mean* (meaning "average") curvatures, denoted by K and H respectively. The formulas are

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

For a unit tangent vector $\mathbf{t}$ (at a point p), the *normal curvature* in the $\mathbf{t}$-direction, $k(\mathbf{t})$, is given by slicing the surface with the plane determined by $\mathbf{t}$ and the unit normal $\mathbf{n}$ and taking the curvature (at p) of the intersection curve. (In some sense, this is the most fundamental type of curvature associated to a surface.) This process defines a continuous function $k\colon S^1 \to \mathbb{R}$ (where we identify unit vectors in $\mathbb{R}^2$ with the circle S^1). Because S^1 is compact, there exist a max k_1 and a min k_2 for k. These are called *principal curvatures* and it is known that $K = k_1k_2$ and $H = (k_1 + k_2)/2$ (see [9]). From these equations, it is easy to derive the relations

$$k_1 = H + \sqrt{H^2 - K} \qquad \text{and} \qquad k_2 = H - \sqrt{H^2 - K}. \tag{3.4}$$

We will deal only with a surface of revolution which has a parameterization of the general form (up to permutation of coordinates)

$$\mathbf{x}(u,v) = (h(u)\cos(v),\ h(u)\sin(v),\ g(u)). \tag{3.5}$$

It is easy to compute that, for such a surface, we always have $F = 0 = M$, so the formulas for Gauss and mean curvatures reduce accordingly. The principal curvatures for a surface of revolution are given by

$$k_\mu = \frac{g''h' - g'h''}{(g'^2 + h'^2)^{3/2}} \qquad \text{and} \qquad k_\pi = \frac{g'}{h\sqrt{g'^2 + h'^2}}. \tag{3.6}$$

The subscripts μ and π stand for the tangent directions along the meridian $(h(u), g(u))$ and parallel circle respectively.

But we are interested in understanding finer details of a surface. Namely, we shall be interested in understanding something about the geodesics on a surface. Intuitively, *geodesics* are the straight lines of a surface; the shortest distances between points. Any curve $\alpha(t)$ which lies on a surface $\mathcal{S}$ parameterized by $\mathbf{x}(u,v)$ may be written as $\alpha(t) = \mathbf{x}(u(t), v(t))$, with $u(t)$ and $v(t)$ determining the curve.

Definition 2. Geodesics are completely determined as solutions of a set of second order differential equations (once initial conditions are specified) called the *geodesic equations*:

$$u'' + \frac{E_u}{2E}u'^2 + \frac{E_v}{E}u'v' - \frac{G_u}{2E}v'^2 = 0 \tag{3.7}$$

$$v'' - \frac{E_v}{2G}u'^2 + \frac{G_u}{G}u'v' + \frac{G_v}{2G}v'^2 = 0. \tag{3.8}$$

Here we have taken $F = 0$ since this will be true for the surface we consider. There is also a special feature about geodesics on a surface of revolution which will allow us to predict geodesic behavior and then verify it pictorially. This feature is called the *Clairaut relation*.

Definition 3. Suppose a parameterization $\mathbf{x}(u,v)$ has metric coefficients E and G which only depend on the parameter u and $F = 0$. Then $\mathbf{x}(u,v)$ is said to be *u-Clairaut*.

Proposition 1. *For a u-Clairaut parameterization, geodesics are characterized by the integral relation*

$$v = \pm \int \frac{c\sqrt{E}}{\sqrt{G}\,\sqrt{G - c^2}}\,\mathrm{d}u. \tag{3.9}$$

Proposition 2. *Let ϕ be the angle between the tangent vector of a geodesic and $\mathbf{x}_u$, the tangent vector of the u-parameter curve given by fixing a v-value in the parameterization $\mathbf{x}(u,v)$. Then the* Clairaut relation *holds:*

$$\sqrt{G}\,\sin(\phi) = c, \qquad \text{where } c \text{ is a constant.} \tag{3.10}$$

We will see that the Clairaut relation restricts geodesics in fundamental ways.

4 The Mylar Balloon

The Mylar[1] balloon is constructed by taking two circular disks of Mylar, sewing them along their boundaries and then inflating with either air or helium. Somewhat surprisingly, these balloons are not spherical as one naïvely might expect from the well-known fact that the sphere possesses the maximal volume for a given surface area. This experimental fact suggests the following mathematical problem: given a circular Mylar balloon of deflated radius a, what will be the shape of the balloon when it is fully inflated? This question

[1] According to Webster's New World Dictionary, *Mylar* is a trademark for a polyester made of the extremely thin sheets of great tensile strength.

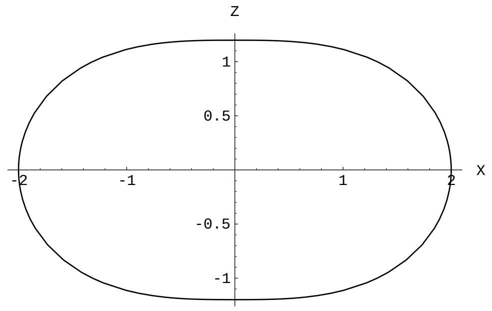

Figure 1. The profile of the mylar balloon in XOZ plane.

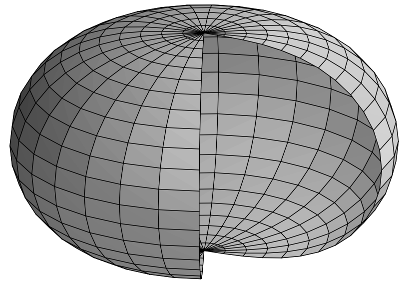

Figure 2. An open part of the mylar balloon surface drawn using the parameterization (4.2).

was first raised by Paulsen [11] who succeeded in determining the radius, thickness and volume of the inflated balloon. Paulsen's answers were in terms of the gamma function. Elsewhere [12], we have shown that elliptic functions are equally as effective in answering these questions. Moreover, we achieve a deeper understanding of the geometry of the Mylar balloon because our approach gives also:

- calculations of the Gaussian and the mean curvatures of the balloon
- a (surprising!) formula for the surface area of the balloon and
- characterization of the balloon in terms of the principal curvatures.

Furthermore, combining our results with Paulsen's, we have found some interesting relationships between the gamma function and elliptic integrals. However, until the description in terms of elliptic functions, more refined geometric qualities of the Mylar balloon were out of reach. Now we have the opportunity to apply the tools of differential geometry to truly understand a beautiful example of a physical principle constraining the shape.

So, let us start with the mathematical model of the balloon. When the Mylar disk is inflated, the radius deforms to a curve $z = z(x)$ which we take to be in the first quadrant of the xz-plane. Of course, the curve proceeds from its highest point on the z-axis to a point of intersection with the x-axis. This is the right hand side of the curve which, when revolved about the z-axis, produces the top half of the balloon. The bottom half is just a reflection of the upper through the xy-plane. Let r be the radius of the inflated balloon. Because of its physical properties, the Mylar does not stretch significantly so that the arclength of the curve $z(x)$ from $x = 0$ to $x = r$ is equal to the initial radius a. That is, we have

$$\int_0^r \sqrt{1 + z'(x)^2}\, \mathrm{d}x = a. \tag{4.1}$$

The basic shape of the balloon was determined by this constraint and the requirement that the enclosed volume is maximal [12]. There, we have proved

Theorem 1. *The surface of revolution $\mathcal{S}$ which models the Mylar balloon is parameterized by $\mathbf{x} = \mathbf{x}[u, v] = (x(u, v),\ y(u, v),\ z(u, v))$ where, for $u \in [-K(1/\sqrt{2}), K(1/\sqrt{2})]$ and*

$v \in [0\,, 2\pi]$,

$$\begin{aligned} x(u,v) &= r\,\mathrm{cn}\left(u, \frac{1}{\sqrt{2}}\right)\cos v, \qquad y(u,v) = r\,\mathrm{cn}\left(u, \frac{1}{\sqrt{2}}\right)\sin v, \\ z(u,v) &= r\sqrt{2}\left[E\left(\mathrm{sn}\left(u, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\mathrm{sn}\left(u, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)\right]. \end{aligned} \tag{4.2}$$

One can put this parameterization into a computer algebra system like *Maple* or *Mathematica* and plot. We then see the familiar profile and shape of a Mylar balloon in Fig. 1 and Fig. 2.

Having the explicit parameterizations of the profile curve ($v \equiv 0$ in (4.2)) and the surface of the mylar balloon we now turn to the study of their geometries. Of principal importance is the relation between the respective radii of the deflated and inflated balloons. By (4.2) we have (where we shorten $\mathrm{sn}(u, 1/\sqrt{2})$ to $\mathrm{sn}(u)$, $K(1/\sqrt{2}) = K$ etc.),

$$\begin{aligned} \int_0^K \sqrt{x'(u)^2 + z'(u)^2}\,\mathrm{d}u &= \int_0^K r\sqrt{\mathrm{sn}^2(u)\mathrm{dn}^2(u) + \frac{1}{2}\mathrm{cn}^4(u)}\,\mathrm{d}u \\ &= \int_0^K r\sqrt{\mathrm{sn}^2(u)(1 - \frac{1}{2}\mathrm{sn}^2(u)) + \frac{1}{2}(1 - \mathrm{sn}^2(u))^2}\,\mathrm{d}u \\ &= r\int_0^K \sqrt{\mathrm{sn}^2(u) - \frac{1}{2}\mathrm{sn}^4(u) + \frac{1}{2} - \mathrm{sn}^2(u) + \frac{1}{2}\mathrm{sn}^4(u)}\,\mathrm{d}u \\ &= \frac{r}{\sqrt{2}}\int_0^K \mathrm{d}u = \frac{r}{\sqrt{2}}K(1/\sqrt{2}) = a. \end{aligned}$$

With 4-digit-accuracy, the numerical relations between a and r are:

$$a \approx 1.3110\,r \qquad \text{and} \qquad r \approx 0.7627\,a. \tag{4.3}$$

Regarding the thickness τ of the balloon, we have to take $2\,z(\pi/2)$ as given by (4.2) in order to obtain

$$\tau = 2z(\pi/2) = 2\sqrt{2}[E(1/\sqrt{2}) - \frac{1}{2}K(1/\sqrt{2})]\,r. \tag{4.4}$$

The numerical calculations are as follows: $\tau \approx 1.1981\,r \approx 0.9139\,a$. In order to answer the volume question, we notice first that the relevant integral can be put into the form

$$V = \pi\sqrt{2}\,r^3\int_0^K \mathrm{cn}^4\left(u, \frac{1}{\sqrt{2}}\right)\mathrm{d}u \tag{4.5}$$

and therefore to obtain

$$V = \frac{\pi\sqrt{2}}{3}K(1/\sqrt{2})\,r^3. \tag{4.6}$$

Numerical analysis in this case gives $V \approx 2.7458\,r^3 \approx 1.2185\,a^3$. Besides, truly geometrical relations such as the following

$$V = \frac{4}{3}a^2\tau = \frac{2\pi}{3}ar^2 \tag{4.7}$$

and

$$\pi\, r^2 = 2\, a\, \tau. \tag{4.8}$$

may be derived. Of no less importance is the observation that the thickness to diameter ratio of the inflated balloon is scale invariant (i.e. independent of the actual size of the balloon). We can calculate $\tau/d \approx 0.599$, where $d = 2\, r$.

Of course, the point of this article is to show how elliptic functions provide more information about an object than numerical calculations alone. We have seen this a bit already from an analytic point of view, but now we will see it in full force geometrically.

Namely, we shall derive the differential geometric characteristics of the balloon (4.2) in terms of elliptic functions and then apply these results to study the curvatures and geodesics of the balloon. From the relation $\mathrm{dn}^2(u,k) + k^2\mathrm{sn}^2(u,k) = 1$, and the choice $k = 1/\sqrt{2}$, we calculate the following for the surface (4.2):

$$\begin{aligned} &E = \frac{r^2}{2}, \qquad F = 0, \quad G = r^2\, \mathrm{cn}^2\Big(u, \tfrac{1}{\sqrt{2}}\Big) \\ &L = r\, \mathrm{cn}\Big(u, \tfrac{1}{\sqrt{2}}\Big), \quad M = 0, \quad N = r\, \mathrm{cn}^3\Big(u, \tfrac{1}{\sqrt{2}}\Big). \end{aligned} \tag{4.9}$$

Our first application of these calculations gives us something which is quite surprising. The formula for the volume of the Mylar balloon involves either the complete elliptic integral of the first kind [12] or the gamma function [11], so we might expect that a formula for surface area would be equally as complicated. Nevertheless, we have

Theorem 2. *The surface area of the Mylar balloon $\mathcal{S}$ of inflated radius r is given by $A(\mathcal{S}) = \pi^2 r^2$.*

Proof. The surface area element $\mathrm{d}A(\mathcal{S})$ is given by

$$\mathrm{d}A(\mathcal{S}) = \sqrt{E\,G - F^2}\; \mathrm{d}u\, \mathrm{d}v = \sqrt{E\,G}\; \mathrm{d}u\, \mathrm{d}v = \frac{r^2 \mathrm{cn}\Big(u, \frac{1}{\sqrt{2}}\Big)}{\sqrt{2}}\, \mathrm{d}u\, \mathrm{d}v.$$

Now it is quite easy to find the total surface area $A(\mathcal{S})$ of the Mylar balloon $\mathcal{S}$ by computing the following integral (where we denote again $K(1/\sqrt{2})$ by K):

$$\begin{aligned} A(\mathcal{S}) &= \int_{\mathcal{S}} \mathrm{d}A(\mathcal{S}) = \frac{r^2}{\sqrt{2}} \int_0^{2\pi}\int_{-K}^{K} \mathrm{cn}\Big(u, \frac{1}{\sqrt{2}}\Big)\, \mathrm{d}u\mathrm{d}v = 4\pi\, \frac{r^2}{\sqrt{2}} \int_0^{K} \frac{\mathrm{cn}\Big(u, \frac{1}{\sqrt{2}}\Big)\, \mathrm{dn}\Big(u, \frac{1}{\sqrt{2}}\Big)}{\mathrm{dn}\Big(u, \frac{1}{\sqrt{2}}\Big)}\, \mathrm{d}u \\ &= 4\pi\, \frac{r^2}{\sqrt{2}} \int_0^1 \frac{\mathrm{d}w}{\sqrt{1 - \frac{1}{2}\, w^2}} \qquad \text{for } w = \mathrm{sn}(u, 1/\sqrt{2}) \\ &= 4\pi\, \frac{r^2}{\sqrt{2}}\, \sqrt{2}\, \arcsin\left(\frac{w}{\sqrt{2}}\right)\Big|_0^1 = 4\pi\, \frac{r^2}{\sqrt{2}}\, \frac{\pi\sqrt{2}}{4} = \pi^2 r^2. \end{aligned}$$

■

Now let's focus on qualities of the balloon central to its shape. We can easily obtain the curvatures for the balloon from the coefficients of the first and second fundamental forms (4.9). The Gauss curvature K and the mean curvature H are computed to be:

$$\begin{aligned} K &= \kappa_1 . \kappa_2 = \frac{L}{E} . \frac{N}{G} = \frac{r\,\mathrm{cn}\left(u, \frac{1}{\sqrt{2}}\right) r\,\mathrm{cn}^3\left(u, \frac{1}{\sqrt{2}}\right)}{r^2/2 \cdot r^2 \mathrm{cn}^2\left(u, \frac{1}{\sqrt{2}}\right)} = \frac{2\,\mathrm{cn}^2\left(u, \frac{1}{\sqrt{2}}\right)}{r^2} \\ H &= \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) \\ &= \frac{r^2/2 \cdot r\,\mathrm{cn}^3\left(u, \frac{1}{\sqrt{2}}\right) + r^2 \mathrm{cn}^2\left(u, \frac{1}{\sqrt{2}}\right) r\,\mathrm{cn}\left(u, \frac{1}{\sqrt{2}}\right)}{2\left(r^2/2 \cdot r^2\mathrm{cn}^2\left(u, \frac{1}{\sqrt{2}}\right)\right)} = \frac{3\,\mathrm{cn}\left(u, \frac{1}{\sqrt{2}}\right)}{2r}. \end{aligned} \tag{4.10}$$

These formulas actually allow us to verify our intuition about one particular aspect of the balloon's geometry. When we look at the balloon, we "see" the North and South poles as being "flat", but it is difficult to make this precise. However, we can prove the following geometric result which tells us that the poles are very flat indeed.

Theorem 3. *The North and South pole of the Mylar balloon are planar points (i.e. points whose normal curvatures are zero in all tangent directions).*

Proof. The North pole of the balloon corresponds to $u = K(1/\sqrt{2})$ and we know that $\mathrm{cn}(K(1/\sqrt{2}), 1/\sqrt{2}) = 0$. Therefore, we see from the formulas for K and H above that both Gauss curvature K and mean curvature H are zero. Hence, we have $\kappa_1 = 0$ and $\kappa_2 = 0$. Since these are the maximal and minimal normal curvatures, we see that all normal curvatures are zero. The same is true for the South pole by symmetry. ■

The Gaussian and the mean curvatures satisfy $K = (8/9)H^2$. From (4.10), we also see that, for every u, the principal curvatures satisfy

$$\kappa_\mu = \kappa_1 = \frac{2\mathrm{cn}(u, 1/\sqrt{2})}{r} = 2\,\kappa_2 = 2\,\kappa_\pi . \tag{4.11}$$

Either of these relationships identify the Mylar balloon as a very special type of *Weingarten surface* (i.e. a surface whose principal curvatures satisfy a functional relation). *Surprisingly*, this relation between principal curvatures actually *characterizes* the balloon *uniquely* and leads to the following theorem (for the details of the proof see [12])

Theorem 4. *The only surface of revolution $\mathcal{M}$ for which $\kappa_\mu = 2\,\kappa_\pi$ is the Mylar balloon.*

Relying on this theorem we can state that the surface

$$\begin{aligned} x(u,v) &= \frac{r}{\sqrt{\cosh(2\,u)}}\cos(v), \qquad y(u,v) = \frac{r}{\sqrt{\cosh(2\,u)}}\sin(v), \\ z(u,v) &= \sqrt{2}\,r\left[E\left(\frac{\sqrt{2}\sinh(u)}{\sqrt{\cosh(2\,u)}}, \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\frac{\sqrt{2}\sinh(u)}{\sqrt{\cosh(2\,u)}}, \frac{1}{\sqrt{2}}\right)\right] \end{aligned} \tag{4.12}$$

where $u \in (-\infty, \infty)$, $v \in [0, 2\pi]$ and for which

$$I = \frac{r^2}{\cosh(2u)}\,(\mathrm{d}u^2 + \mathrm{d}v^2) \qquad \text{and} \qquad II = \frac{r}{\cosh(2\,u)^{\frac{3}{2}}}\,(2\,\mathrm{d}u^2 + \mathrm{d}v^2) \tag{4.13}$$

is just the Mylar balloon and that (4.12) provides its conformal representation.

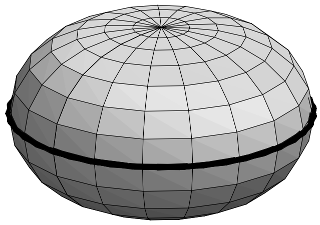

Figure 3. The "equator" is a geodesic

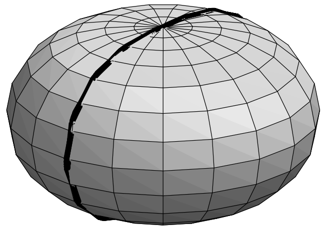

Figure 4. A u-parameter curve is a geodesic

5 Geodesics on the Mylar Balloon

Now let's understand a deeper quality of a surface – its geodesics. Again, we emphasize that this would simply be impossible without knowing the explicit parameterizations (4.2) or (4.12) – and *that* this would be impossible without using the elliptic functions. The balloon has metric coefficients $E = r^2/2$, $F = 0$ and $G = r^2 \operatorname{cn}^2(u, 1/\sqrt{2})$ only depending on the parameter u, so the parameterization $\mathbf{x}(u, v)$ is u-Clairaut. In particular, the Clairaut relation holds: $\sqrt{G}\,\sin(\phi) = c$ along any geodesic, where c is constant and ϕ is the angle between the geodesic's tangent vector and $\mathbf{x}_u$. For the mylar balloon, the Clairaut relation is $r \operatorname{cn}(u, 1/\sqrt{2})\,\sin(\phi) = c$. Notice that this means that

$$r^2 \operatorname{cn}^2(u, 1/\sqrt{2}) \geq c^2. \tag{5.1}$$

Therefore, a geodesic must always obey (5.1). This restricts geodesics in a meaningful way. Let's look at some examples of geodesics on the mylar balloon. Now, in the Clairaut relation (with $r = 1$)

$$\sqrt{G}\,\sin(\phi) = \operatorname{cn}(u, 1/\sqrt{2})\,\sin(\phi) = c,$$

let's take $\phi = \pi/2$. That is, we initially move in the $\mathbf{x}_v$-direction. If we also take $u = 0$ (i.e. the equator of the balloon), then $c = 1$. But $\operatorname{cn}(u, 1/\sqrt{2}) \leq 1$ and $\sin(\phi) \leq 1$, so the only way the Clairaut relation can hold is if $u = 0$ and $\phi = \pi/2$ for all time along the geodesic. We predict that the geodesic stays on the equator. The plot Fig. 3 verifies this.

What if we start in the $\mathbf{x}_u$-direction? Then $\phi = 0$, so $c = 0$ also. Then we see that $\phi = 0$ always along the geodesic. This means that u-parameter curves are geodesics (a well-known classical fact). A closed geodesic starting in the $\mathbf{x}_u$-direction is calculated and depicted in Fig. 4. If we keep $\phi = \pi/2$, but we take say $u = 0.4$, then we know from the Clairaut relation and the fact that the elliptic cosine decreases from 0 to K that, along the geodesic, $u \leq 0.4$. But this confines the geodesic to a predictable strip on the balloon (see Fig. 5).

All these considerations can be put in more analytical form. Making use of the conformal parameterization (4.12) which leads to the metric (4.13) and performing the integration in (3.9) one gets

$$v(u) = \pm \frac{c}{\sqrt{r^2 - c^2}} \Pi \left(\frac{\sinh(u)}{\sqrt{\cosh 2u}},\ 2,\ \frac{r\sqrt{2}}{\sqrt{r^2 - c^2}} \right) \tag{5.2}$$

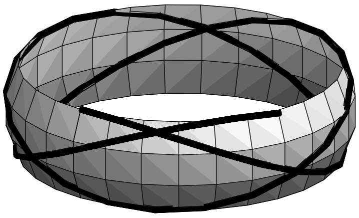

Figure 5. Geodesic confined by the Clairaut relation

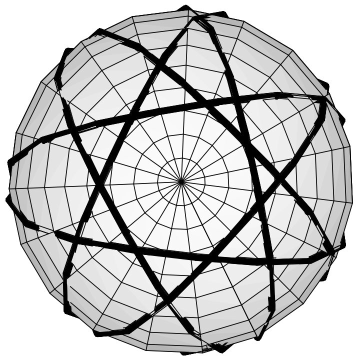

Figure 6. Periodic geodesic

and therefore an *explicit* parameterization of all geodesics! In particular one can easily derive the condition for drawing the periodic geodesics as the one shown in Fig. 6.

We have seen that the understanding of elliptic functions provides interesting insights into the geometry of variational problems such as that of the mylar balloon. Another problem which is amenable to such an "elliptical" analysis is that of surfaces of constant curvature and we hope to say more about this elsewhere.

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