Hamiltonian dynamics of planar affinely-rigid body

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Abstract

We discuss the dynamics of an affinely-rigid body in two dimensions. Translational degrees of freedom are neglected. The special stress is laid on completely integrable models solvable in terms of the separation of variables method.

1   Introduction

The general formulation of the mechanics of an affinely-rigid body in n dimensions was presented in [1, 2, 3]. Obviously, it is the special case $n = 3$ that is directly physically applicable, if a proper potential model is chosen. For realistic potentials, the three-dimensional problem is very difficult. The reason is that the group $SO(3, \mathbb{R})$ (and generally $SO(n, \mathbb{R})$ for $n > 2$) is semisimple and because of this the deformative degrees of freedom in kinetic energy form are mixed in a very malicious, non-separable way. Some general aspects of the two-dimensional model were investigated in [4]. The two-dimensional study may be also useful as a preliminary step towards the analysis of realistic three-dimensional problems. Here we consider some isotropic dynamical models in two dimensions. They are both physically reasonable (e.g. from the point of view of macroscopic elasticity) and analytically treatable in terms of the separation of variables method (Stäckel theorem). Some expressions for the action-angle variables are derived, in particular, the dependence of energy on the action parameters is discussed. Our calculations are based on the method of complex integration, elaborated in this context by Max Born [5]. The degeneracy of these models is explicitly described and Bohr-Sommerfeld quantization is performed.

2   Some integrable two-dimensional problems

Discussed is the two-dimensional affinely-rigid body. Namely, we consider discrete or continuous system of material points for which the configuration space may be identified with the affine group. In other words, all affine relations between its constituents are frozen, i.e. during the affine motion such fundamental affine relationships like the parallelism of straight lines, ratio of segments on the same straight line, and of course the very manifold of

all straight lines are preserved. When translational motion is neglected, the configuration space becomes identical with the linear group $Q = GL(2, \mathbb{R})$. We are interested in qualitative properties of the particular dynamical model rather than in geometric foundations of the theory analyzed in [3]. That is why we prefer to use here the matrix notation. The motion of the affinely-rigid body (without translations) has the form: $x^i(t, a) = \Phi^i_A(t)a^A$, where $x^i(t, a)$ is the position of the a-th material point at the time instant t and Φ is the internal configuration of the body, which connects Lagrangian coordinates a^A with Eulerian ones x^i. The most adequate description of degrees of freedom is that based on the following decomposition of matrices: $\Phi = ODR^T$, where $O, R \in SO(2, \mathbb{R})$ are orthogonal matrices, D is diagonal and positive, and the orthogonal group $SO(2, \mathbb{R})$ is a commutative group of plane rotations. Spatial rotations are described by the action of $SO(2, \mathbb{R})$ on $GL(2, \mathbb{R})$ through the left regular translations. Similarly, material rotations are represented by the action of the rotation subgroup through the right multiplication. It leads to the natural parameterization of the problem:

$$O = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right], \quad D = \left[\begin{array}{cc} D_1 & 0 \\ 0 & D_2 \end{array} \right], \quad R = \left[\begin{array}{cc} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{array} \right]$$

where $0 \leq \theta \leq 2\pi, \quad D_1 > 0 , \quad D_2 > 0, \quad 0 \leq \psi \leq 2\pi$.

Matrix D describes the scalar state of deformation, i.e. tells us how strongly the body is deformed, but does not contain any information as to how the deformation is oriented in physical space and in the body. The orthogonal matrices O and R describe the space and body orientation of the strain. This decomposition is unique up to the permutation of diagonal elements of D accompanied by the simultaneous multiplying of O, R on the right by the appropriate special orthogonal matrix. This implies that the potential energy of doubly isotropic models depends only on D and is invariant with respect to the permutation of its non-vanishing matrix elements [2]. We consider the affinely-rigid body with the double dynamical isotropy: spatial and material. This means that the system is invariant under both physical and material rigid rotations. Restricting the usual multi-particle kinetic energy to the manifold of affine constraints we obtain [2, 3]: $T = \frac{1}{2}\mathrm{Tr}(\frac{d\Phi}{dt}J\frac{d\Phi^T}{dt})$, where J denotes the co-moving quadrupole moment of inertia. We shall consider only highly symmetric model, where J is isotropic, i.e., its matrix has the form μI; μ denoting a positive constant and I the 2×2 identity matrix. Then the isotropic kinetic energy has the following form

$$T = \frac{\mu}{2} \left[\left(D_1^2 + D_2^2\right) \left(\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\psi}{dt}\right)^2 \right) - 4D_1 D_2 \frac{d\theta}{dt}\frac{d\psi}{dt} + \left(\frac{dD_1}{dt}\right)^2 + \left(\frac{dD_2}{dt}\right)^2 \right].$$

We can notice that the matrices O and R do not enter into this equation, hence the angles θ, ψ are cyclic variables. In these coordinates the Hamilton-Jacobi equation is non-separable even in the interaction-free case. However, the separability becomes possible in new variables, obtained by the $\frac{\pi}{4}$-rotation in the plane of deformation invariants D_1, D_2 and by an appropriate modification of angular variables. Thus, we introduce new coordinates: $\alpha = \frac{1}{\sqrt{2}}(D_1 + D_2)$, $\beta = \frac{1}{\sqrt{2}}(D_1 - D_2)$, $\eta = \theta - \psi$, $\gamma = \theta + \psi$. In macroscopic, phenomenological elasticity theory $D_1 > 0 , \quad D_2 > 0$, thus $\alpha > 0, \quad |\beta| < \alpha$. However, describing discrete or finite systems of material points (e.g. molecules), one can admit singular and mirror-reflected configurations. Then, to some extent D_1, D_2, α, β may

be arbitrary. In these coordinates the kinetic energy has the form (both diagonal and separable)

$$T = \frac{\mu}{2} \left[\alpha^2 \left(\frac{d\eta}{dt} \right)^2 + \beta^2 \left(\frac{d\gamma}{dt} \right)^2 + \left(\frac{d\alpha}{dt} \right)^2 + \left(\frac{d\beta}{dt} \right)^2 \right]. \tag{2.1}$$

In our calculations we will use the following coordinates on the invariants plane, to separate the Hamilton-Jacobi equation:

1. Cartesian coordinates α, β (mentioned above).

2. Polar coordinates r, φ which are defined as follows: $\alpha = \sqrt{r} \cos \frac{\varphi}{2}$, $\beta = \sqrt{r} \sin \frac{\varphi}{2}$ ($r > 0$, $-\pi/2 < \varphi < \pi/2$).

The kinetic energy has the following form

$$T = \frac{\mu}{2} \left[r \cos^2 \frac{\varphi}{2} \left(\frac{d\eta}{dt} \right)^2 + r \sin^2 \frac{\varphi}{2} \left(\frac{d\gamma}{dt} \right)^2 + \frac{1}{4r} \left(\frac{dr}{dt} \right)^2 + \frac{r}{4} \left(\frac{d\varphi}{dt} \right)^2 \right]. \tag{2.2}$$

The Cartesian and polar variables are orthogonal, thus we can use the classical Stäckel theorem to determine a general form of potentials which are in some sense isotropic and admit analytical calculations based on the separation of variables method. The corresponding stationary Hamilton-Jacobi equation is separable for potentials of the shape, corresponding to the Cartesian and polar coordinates:

$$V(\theta, \psi, \alpha, \beta) = \frac{V_\eta (\theta - \psi)}{\alpha^2} + \frac{V_\gamma (\theta + \psi)}{\beta^2} + V_\alpha(\alpha) + V_\beta(\beta),$$

$$V(\theta, \psi, r, \varphi) = \frac{V_\eta (\theta - \psi)}{r \cos^2 \frac{\varphi}{2}} + \frac{V_\gamma (\theta + \psi)}{r \sin^2 \frac{\varphi}{2}} + V_r(r) + \frac{V_\varphi(\varphi)}{r}.$$

We consider doubly-isotropic models, in which the potential energy does not depend on variables θ, ψ (equivalently η, γ), i.e.: $V_\eta = 0$ and $V_\gamma = 0$. Performing the Legendre transformation we obtain the corresponding Hamiltonian in the form:

1. Cartesian coordinates $H = H_\alpha + H_\beta$:

$$H = \frac{1}{2\mu} \left(\frac{(p_\theta - p_\psi)^2}{4\alpha^2} + p_\alpha^2 \right) + \frac{1}{2\mu} \left(\frac{(p_\theta + p_\psi)^2}{4\beta^2} + p_\beta^2 \right) + V_\alpha(\alpha) + V_\beta(\beta), \quad (2.3)$$

where p_θ, p_ψ, p_α, p_β are the canonical momenta conjugate to θ, ψ, α, β, respectively, and $H_\alpha = \frac{1}{2\mu} \left(\frac{(p_\theta - p_\psi)^2}{4\alpha^2} + p_\alpha^2 \right) + V_\alpha(\alpha)$, $H_\beta = \frac{1}{2\mu} \left(\frac{(p_\theta + p_\psi)^2}{4\beta^2} + p_\beta^2 \right) + V_\beta(\beta)$. The quantities $H_\alpha, H_\beta, p_\theta, p_\psi$ form a Poisson-involutive system of constants of motion.

2. Polar coordinates $H = H_r + \frac{h_\varphi}{r}$:

$$H = \frac{2r}{\mu} p_r^2 + \frac{1}{2r\mu} \left(\frac{p_\theta^2 + p_\psi^2 + 2p_\theta p_\psi \cos \varphi}{\sin^2 \varphi} + 4p_\varphi^2 \right) + V_r(r) + \frac{V_\varphi(\varphi)}{r}, \tag{2.4}$$

where p_r, p_φ, p_θ, p_ψ are the canonical momenta conjugate to r, φ, θ, ψ, respectively, and $H_r = \frac{2r}{\mu} p_r^2 + V_r(r)$, $h_\varphi = \frac{1}{2\mu} \left(\frac{p_\theta^2 + p_\psi^2 + 2p_\theta p_\psi \cos \varphi}{\sin^2 \varphi} + 4p_\varphi^2 \right) + V_\varphi(\varphi)$. In this case, the quantities $H, h_\varphi, p_\theta, p_\psi$ form a Poisson-involutive system of constants of motion.

The stationary Hamilton-Jacobi equation is as follows:

1. Cartesian coordinates:

$$\left(\frac{1}{4\alpha^2} + \frac{1}{4\beta^2}\right)\left(\left(\frac{\partial S}{\partial\theta}\right)^2 + \left(\frac{\partial S}{\partial\psi}\right)^2\right) + \left(\frac{1}{2\beta^2} - \frac{1}{2\alpha^2}\right)\frac{\partial^2 S}{\partial\theta\partial\psi} \tag{2.5}$$

$$+ \left(\frac{\partial S}{\partial\alpha}\right)^2 + \left(\frac{\partial S}{\partial\beta}\right)^2 = 2\mu\left(E - (V_\alpha(\alpha) + V_\beta(\beta))\right),$$

where E is a fixed value of energy. Due to the fact that the variables θ, ψ have the cyclic character, we may write: $S = S_\theta(\theta) + S_\psi(\psi) + S_\alpha(\alpha) + S_\beta(\beta) = a\theta + b\psi + S_\alpha(\alpha) + S_\beta(\beta)$. Finally we obtain the action variables in the following form:

$$J_\theta = \oint p_\theta d\theta = 2\pi a, \quad J_\alpha = \pm\oint\sqrt{2\mu\left(E_\alpha - V_\alpha(\alpha)\right) - \frac{(J_\theta - J_\psi)^2}{16\pi^2\alpha^2}}d\alpha,$$

$$J_\psi = \oint p_\psi d\psi = 2\pi b, \quad J_\beta = \pm\oint\sqrt{2\mu\left(E_\beta - V_\beta(\beta)\right) - \frac{(J_\theta + J_\psi)^2}{16\pi^2\beta^2}}\,d\beta, \tag{2.6}$$

where E_α, E_β, a, b are separation constants.

2. Polar coordinates:

$$\frac{1}{r\sin^2\varphi}\left(\left(\frac{\partial S}{\partial\theta}\right)^2 + \left(\frac{\partial S}{\partial\psi}\right)^2\right) + \frac{2\cos\varphi}{r\sin^2\varphi}\frac{\partial^2 S}{\partial\theta\partial\psi} + 4r\left(\frac{\partial S}{\partial r}\right)^2 \tag{2.7}$$

$$+\frac{4}{r}\left(\frac{\partial S}{\partial\varphi}\right)^2 = 2\mu\left(E - V_r(r) - \frac{V_\varphi(\varphi)}{r}\right),$$

where E is a fixed value of energy and again the solutions are sought in the separated form: $S = S_\theta(\theta) + S_\psi(\psi) + S_r(r) + S_\varphi(\varphi) = a\theta + b\psi + S_r(r) + S_\varphi(\varphi)$. After some calculations the action variables are as follows:

$$J_\theta = \oint p_\theta d\theta = 2\pi a, \quad J_r = \pm\oint\sqrt{\frac{\mu}{2r}\left(E - V_r(r)\right) - \frac{\mu e_\varphi}{2r^2}}dr,$$

$$J_\psi = \oint p_\psi d\psi = 2\pi b, \quad J_\varphi = \pm\oint\sqrt{\frac{\mu}{2}\left(e_\varphi - V_\varphi(\varphi)\right) - \frac{J_\theta^2 + J_\psi^2 + 2J_\theta J_\psi\cos\varphi}{16\pi^2\sin^2\varphi}}\,d\varphi,$$
$$\tag{2.8}$$

where E, e_φ, a, b are separation constants.

3 Some rigorously solvable models

In (2.6) and (2.8) the expressions J_α, J_β, J_r, J_φ depend on potentials $V_\alpha(\alpha)$, $V_\beta(\beta)$, $V_r(r)$, $V_\varphi(\varphi)$, respectively. After specifying the form of these potentials we can obtain the Hamilton function H as some function of our action variables, i.e. 1. $H = E(J_\alpha, J_\beta, J_\theta, J_\psi)$ and 2. $H = E(J_r, J_\varphi, J_\theta, J_\psi)$. So we can find the explicit dependence of the energy E on the action variables and the possible further degeneracy. We will also perform the usual Bohr-Sommerfeld quantization procedure for all our models.

There exist potentials [2, 3] which could lead to Hamilton-Jacobi equations separable simultaneously in two mentioned above coordinate systems. As we know from analytical mechanics this simultaneous separability usually has to do with some degeneracy and additional constants of motion of completely integrable systems. General form of that potential is as follows: $V = \frac{A}{\alpha^2} + \frac{B}{\beta^2} + C\left(\alpha^2 + \beta^2\right)$, where A, B, C are constants. By an appropriate choice of A, B, C (more generally some arbitrary one-variable functions might be used instead of them), one can obtain the potential with a local minimum at the reference configuration $\Phi = I$. In a certain neighborhood of $\Phi = I$ some phenomenological conditions, known from elasticity theory, will be satisfied.

3.1 *Cartesian* model

Here we consider the model of the *Cartesian* potential i.e. $V(\alpha, \beta) = \frac{C}{4}\left(\alpha^2 + \frac{4}{\alpha^2}\right) + \frac{C}{4}\beta^2$, $C > 0$. After some calculations we obtain the dependence of the energy $E = E_\alpha + E_\beta$ on the action variables as follows:

$$E = \frac{\omega}{4\pi\sqrt{2}}\left(4J + |J_\theta + J_\psi| + \sqrt{32\mu\pi^2 C + (J_\theta - J_\psi)^2}\right), \quad \omega = \sqrt{\frac{C}{\mu}}, \qquad (3.1)$$

where $E_\alpha = \frac{\omega}{4\pi\sqrt{2}}\left(4J_\alpha + \sqrt{32\mu\pi^2 C + (J_\theta - J_\psi)^2}\right)$, $E_\beta = \frac{\omega}{4\pi\sqrt{2}}\left(4J_\beta + |J_\theta + J_\psi|\right)$. There exists a partial degeneracy of this system because the energy in (3.1) depends on a rational combination of action variables typical for degenerated systems, i.e. $J = J_\alpha + J_\beta$. The relationship between frequencies becomes $\nu_\alpha = \nu_\beta$. The energy E depends on the action variables J_θ, J_ψ through their integer linear combinations. However, there occur two different ones, thus there is no degeneracy. Then performing the Bohr-Sommerfeld quantization procedure, i.e. supposing that $J = nh$, $J_\theta = mh$, $J_\psi = lh$, where h is the Planck constant and $n = 0, 1, \dots$; $m, l = 0, \pm 1, \dots$, we obtain the energy spectrum in the form:

$$E = \frac{1}{2\sqrt{2}}\hbar\omega\left(4n + |m + l| + \sqrt{(m - l)^2 + \frac{8C\mu}{\hbar^2}}\right). \qquad (3.2)$$

3.2 *Polar* model

Next we consider the model of the *polar* potential, i.e. $V(r, \varphi) = \frac{C}{4}\left(r + \frac{4}{r}\right) + \frac{C}{r}tg^2\frac{\varphi}{2}$, $C > 0$. Now we obtain the following expressions for the dependencies of the constant e_φ and the energy E on the action variables: $E = \frac{J_r}{\pi}\sqrt{\frac{C}{2\mu}} + \sqrt{C^2 + Ce_\varphi}$, where

$$e_\varphi = \left[\sqrt{\tfrac{1}{2\mu\pi^2}} J_\varphi + \sqrt{\tfrac{1}{32\mu\pi^2} (J_\theta + J_\psi)^2} + \sqrt{C + \tfrac{1}{32\mu\pi^2} (J_\theta - J_\psi)^2} \right]^2 - C. \text{ Finally}$$

$$E = \frac{\omega}{4\pi\sqrt{2}} \left(4J + |J_\theta + J_\psi| + \sqrt{32\mu\pi^2 C + (J_\theta - J_\psi)^2} \right). \tag{3.3}$$

There exists also partial degeneracy of this system because the energy in (3.3) depends on an integer combination of action variables, i.e. $J = J_r + J_\varphi$. The relationship between frequencies becomes $\nu_r = \nu_\varphi$. The energy E depends on J_θ, J_ψ through two different integer-coefficients combinations, thus there is no degeneracy. The Bohr-Sommerfeld quantization conditions yield the following form of the energy spectrum:

$$E = \frac{1}{2\sqrt{2}} \hbar\omega \left(4n + |m + l| + \sqrt{(m - l)^2 + \frac{8C\mu}{\hbar^2}} \right). \tag{3.4}$$

It is interesting to compare this expression with (3.2). Separability of the Hamilton-Jacobi equation in two different coordinate systems has to do with hidden symmetries and degeneracy of the problem. So, our models are only one-fold degenerate and all trajectories are dense on some three-dimensional tori. Therefore, Bohr-Sommerfeld quantization involves three quantum numbers (in the sense of the old quantum theory).

The planar model is mathematically interesting in itself and effectively analytically treatable. However, some applications are also possible in macroscopic elasticity and hydrodynamics (homogeneous vibrations of cross-sections of elastic cylinders or cross-sections of fluid streams). Microscopic applications are also possible, e.g., vibrations of planar molecules (such as S_8, C_6H_6). Obviously, in microscopic problems the quantized models are not only physically admissible, but just desirable.

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