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Action-Angle Analysis of Some Geometric Models of Internal Degrees of Freedom

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Abstract

We derive and discuss equations of motion of infinitesimal affinely-rigid body moving in Riemannian spaces. There is no concept of extended rigid and affinely rigid body in a general Riemannian space. Therefore the gyroscopes with affine degrees of freedom are described as moving bases attached to the material point. This base is a remnant of extended rigid and affinely rigid body in a flat space. The special stress is laid on affinely rigid bodies in two-dimensional constant curvature spaces (sphere and pseudosphere). In particular, we consider incompressible affinely rigid bodies, like, e.g. fat spots on a water surface (e.g. petrol pollution). This is a two-dimensional analogue of three-dimensional incompressible objects like fluid droplets.

1 Introduction

In a generic Riemann space (M, g) it is rather typical that the isometry and affine groups are trivial, or at least, their dimensions are smaller than those for flat spaces. So there is no concept of extended rigid or affinely-rigid body. However, one can consider infinitesimal objects of this kind, so small that one can consider them as injected into tangent spaces. More precisely, such objects are structured material points, i.e., material points with attached linear bases describing internal degrees of freedom. These bases will be denoted by $e = (\ldots, e_A, \ldots)$, and their duals by $\tilde{e} = (\ldots, e^A, \ldots)$. Local coordinates x^i in M establish representation of e, $\tilde{e}$ as fields of matrices $e^i{}_A$, $e^A{}_i$, where $e^A{}_i \, e^i{}_B = \delta^A{}_B$. When the body is metrically-rigid, i.e., the frame is orthonormal, $g_{ij} e^i_A e^j_B = \delta_{AB}$ then the functions $(x^i,\, e^i{}_A)$, being functionally dependent cannot be used as generalized coordinates on the configuration space, i.e., the bundle $F(M, g)$ of g-orthonormal frames. Therefore, it is only the general form of balance laws for spin and linear momentum that may be explicitly written, but not an effective dynamical system. Obviously, for affinely rigid body $(x^i,\, e^i{}_A)$ are well defined generalized coordinates. The simplest way to escape the difficulty in the rigid body case and to establish effective generalized coordinates on $(FM,\, g)$ is to use some fixed g-orthonormal nonholonomic reference system $E = (\ldots, E_A, \ldots)$ (field of frames) on M; $g_{ij} E^i_A E^j_B = \delta_{AB}$. As usual, the inverse co-frame will be denoted by $\tilde{E} = (\ldots, E^A, \ldots)$.

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Usually E is somehow chosen on the basis of details of $(M,\ g)$-geometry. Now we can put: $e_A(t) = E_B\left(x(t)\right) R^B{}_A(t)$, where e_K, E_L are attached at $x(t)$, and R is time dependent orthogonal matrix, $\delta_{CD} R^C_A R^D_B = \delta_{AB}$. There are some standard methods for parameterizing $SO(n, R)$ (Euler angles, rotations vector etc.) and in this way some independent coordinates on $F(M, g)$ may be introduced. The co-moving components of angular velocity Λ are defined in a usual way,

$$\frac{De_B}{Dt} := e_A \Lambda^A_B,\ \Lambda^A_B = \Omega^A_B + \kappa^A_B,\ \kappa^A_B = \left[R^{-1}\right]^A_C \frac{dR^C_B}{dt},\ \Omega^A_B = \left[R^{-1}\right]^A_F \Gamma^F_{DC} R^D_B R^C_G v^G, \tag{1.1}$$

where v^G are the co-moving components of the translational velocity: $v^G = e^G_i (dx^i/dt)$ and Γ^F_{DC} are anholonomic components of the affine connection with respect to E ($\nabla_B E_A = E_C \Gamma^C{}_{AB}$). We discuss in some details the motion of infinitesimal gyroscopes moving on the two-dimensional constant curvature spaces like the sphere and pseudosphere [1].

2 Infinitesimal homogeneously-deformable gyroscope

An extended affinely rigid body is a system of material points with all affine relations kept frozen during any admissible motion [2, 3]. In a non-Euclidean physical space we can consider only infinitesimal objects of this kind, just as it was the case with rigid bodies in Riemannian space. There are no longer any constrains imposed on e^i_A, so in principle there is no reason to use the prescribed reference frame E_A any longer. But from the point of view of practical applications such a description is still useful. Namely, it is rather typical that we are interested in problems in which finite rotations interact with extra imposed small deformations. Therefore, in the affine case we continue to use some prescribed anholonomic orthonormal frame E.

Now we return to the general, non-restricted motion of infinitesimal affinely-rigid bodies. In a general Riemannian manifold (M, g) we have the following expression for the total kinetic energy ([4]-[6]):

$$T = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{De^i_A}{Dt} \frac{De^j_B}{Dt} J^{AB} = T_{\text{tr}} + T_{\text{int}}. \tag{2.1}$$

After putting $e_A = E_B \varphi^B_A$ (now φ^B_A is a general nonsingular matrix), we obtain for the internal part of the kinetic energy:

$$T_{\text{int}} = \frac{1}{2} \delta_{MN} \varphi^M_K \varphi^N_L \Lambda^K_A \Lambda^L_B J^{AB}, \tag{2.2}$$

where the co-moving affine angular velocity Λ is implicitly defined by the formula (1.1) and J^{AB} is the co-moving quadrupole of inertia . One can show that in analogy to (1.1): $\Lambda^A_B = \left(\varphi^{-1}\right)^A_F \Gamma^F_{DC} \varphi^D_B \varphi^C_G v^G + \left(\varphi^{-1}\right)^A_C \left(d\varphi^C_B/dt\right)$, $v^E = e^E_i (dx^i/dt)$. The parameterization of φ which we choose, depends on the structure of internal inertia J. When $J = JId_n$ (Id_n denoting the $n \times n$ identity matrix, J being a scalar constant), then it is convenient to use the two-polar decomposition [7]:

$$\varphi^A_B = U^A_C D^C_D \left[V^{-1}\right]^D_B, \tag{2.3}$$

where $U,\ V$ are orthogonal and D is diagonal. After some calculations we obtain then:

$$T = -\frac{J}{2}\mathrm{tr}[D^2\chi^2] - \frac{J}{2}\mathrm{tr}[D^2\vartheta^2] + J\mathrm{tr}[D\chi D\vartheta] + \frac{J}{2}\mathrm{tr}[(\dot{D})^2], \tag{2.4}$$

where D is the diagonal factor of the two-polar decomposition and the skew-symmetric matrices $\chi,\ \vartheta$ represent, respectively, the angular velocities related to U and V. More precisely: $\vartheta = V^{-1}(dV/dt)$, whereas χ is defined by the following formulae: $(Dw_A/Dt) = w_B\chi^B_A$, $w_A := E_B U^B_A$.

3 Motion on two-dimensional sphere and pseudosphere and action variables

We discuss the compressible and incompressible motion on two-dimensional spheres and pseudospheres. J are isotropic, so we use the two-polar decomposition. From now on all angular velocities become one-dimensional objects, denoted by scalar factors $\chi,\ \vartheta$; more precisely, they are equal to $\chi\epsilon$ and $\vartheta\epsilon$ where $\epsilon := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The two-polar decomposition (2.3) is based on: $U = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $V = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$, $D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$. The internal kinetic energy (2.2) is given by:

$$T_{\mathrm{int}} = \frac{J}{2}\left[\dot{\lambda}^2 + \dot{\mu}^2\right] + \frac{J}{2}\left[\lambda^2 + \mu^2\right]\chi^2 + \frac{J}{2}\left[\lambda^2 + \mu^2\right]\vartheta^2 - 2J\lambda\mu\chi\vartheta. \tag{3.1}$$

If we want to describe an incompressible body, like drop of water in three dimensions or spot of fat on two dimensional water surface then for planar problems we should use $\mu = 1/\lambda$; the internal kinetic energy is then given by the formula:

$$T_{\mathrm{int}} = \frac{J}{2}\left[1 + \frac{1}{\lambda^4}\right]\dot{\lambda}^2 + \frac{J}{2}\left[\frac{1}{\lambda^2} + \lambda^2\right]\chi^2 + \frac{J}{2}\left[\frac{1}{\lambda^2} + \lambda^2\right]\vartheta^2 - 2J\chi\vartheta, \tag{3.2}$$

where now χ and ϑ are the only independent components of angular velocity matrices denoted previously by the same symbols. All expressions for sphere and pseudosphere look similarly so we can use common symbols S, C for denoting the functions sin, cos in the spherical case and sh, ch, respectively in the pseudospherical case. Explicitly: $\vartheta = d\beta/dt$ and $\chi = d\alpha/dt + C(r/R)\,d\varphi/dt$. It is convenient to include the compressible case: $\gamma := \alpha + \beta\ \ \delta := \alpha - \beta,\ x := (1/2)(\lambda - \mu),\ y := (1/2)(\lambda + \mu)$. For the incompressible case: $\gamma := \alpha + \beta\ \ \delta := \alpha - \beta$. If $\mu = 1/\lambda \Rightarrow x := (1/2)(\lambda - 1/\lambda),\ y := (1/2)(\lambda + 1/\lambda)$ the deformation variable are interdependent, so for the incompressible case we will use λ alone. The metric elements are given by the following expressions [1]:

$$ds^2 = dr^2 + R^2 S^2(r/R)\,d\varphi^2;\ \text{on sphere } r \in [0, \pi R) \text{ and on pseudosphere } r \in [0, \infty]. \tag{3.3}$$

Then the kinetic energy may be written in the following form:

$$T = (m/2)G_{ij}(dq^i/dt)(dq^j/dt),$$

where $\{q^i\} = \{r, \varphi, \gamma, \delta, x, y\}$

We assume Lagrangians of the form $L = T - V$, V depending only on generalized coordinates. Now we use the "polar" coordinates on $S^2(0,R)$, and $H^2(0,R)$. In both cases they are denoted by the same symbols (r,φ). The translational energies have the form [1]: $T_{\mathrm{tr}} = (m/2)\left[(dr/dt)^2 + R^2S^2(r/R)(d\varphi/dt)^2\right]$.

In the canonical formalism: $T = (1/2m)G^{ij}p_ip_j$, $H = T + V$, where $G_{ij}G^{jk} = \delta_i{}^k$. In the spherical and pseudospherical case we use vector fields E_A which are mutually orthonormal and directed parallelly to "latitudes" and "meridians", E_φ, E_r. The dynamics in our models is doubly isotropic. After some calculations we obtain that the kinetic term of the Hamiltonian for the compressible case has the form:

$$T = \frac{p_r^2}{2m} + \frac{p_\varphi^2 - 2C\frac{r}{R}p_\varphi(p_\gamma+p_\delta) + \left(\frac{mR^2}{J}S^2\frac{r}{R} + C^2\frac{r}{R}\right)(p_\gamma+p_\delta)^2}{2mR^2S^2\frac{r}{R}} + \frac{p_x^2}{2J} + \frac{p_y^2}{2J} + \frac{p_\gamma^2}{2Jx^2} + \frac{p_\delta^2}{2Jy^2}, \quad (3.4)$$

and for the incompressible case

$$T = \frac{p_r^2}{2m} + \frac{\left((p_\gamma+p_\delta)C\frac{r}{R} - p_\varphi^2\right)^2}{2mR^2S^2\frac{r}{R}} + \frac{p_\gamma^2\lambda^2}{J(1-\lambda^2)^2} + \frac{p_\delta^2\lambda^2}{J(1+\lambda^2)^2} + \frac{p_\lambda^2\lambda^4}{2J(1+\lambda^4)}. \quad (3.5)$$

In pseudospherical case $\frac{mR^2}{J}S^2\left(\frac{r}{R}\right)$ may have two forms: $\pm\frac{mR^2}{J}\sinh^2\left(\frac{r}{R}\right)$; both of them may be physically applicable. Hamilton-Jacobi equation $H\left(q^a, \frac{\partial S_0}{\partial q^a}\right) = \frac{1}{2m}G^{ij}\frac{\partial S_0}{\partial q^i}\frac{\partial S_0}{\partial q^j} + V(q) = E$ is separable for potentials which consist of two terms: one depends on r (translational variable) and the other depends on variables which concern the deformation: $V(q) = V_r(r) + V_x(x) + V_y(y)$, $V(q) = V_r(r) + V_\lambda(\lambda)$. Then the action variables may be explicitly calculated. The resulting equations separate when φ, γ and δ are cyclic variables [8]. In the stationary case $H\left(q, \frac{\partial S_0}{\partial q}\right) = E$, $S_0 = S_r(r) + l\varphi + C_\gamma\gamma + C_\delta\delta + S_x(x) + S_y(y)$ and for incompressible model $S_0 = S_r(r) + l\varphi + C_\gamma\gamma + C_\delta\delta + S_\lambda(\lambda))$. Now we put $p_\gamma = \frac{1}{2}(p_\alpha + p_\beta)$ and $p_\delta = \frac{1}{2}(p_\alpha - p_\beta)$. Then: $S_0 = S_r(r) + l\varphi + C_\alpha\alpha + C_\beta\beta + S_x(x) + S_y(y)$, and in the second case $S_0 = S_r(r) + l\varphi + C_\alpha\alpha + C_\beta\beta + S_\lambda(\lambda)$. The Hamilton-Jacobi equation for the compressible body has the form:

$$E = \frac{1}{2m}\left(\frac{dS_r(r)}{dr}\right)^2 + \frac{\left(l - C_\alpha C\left(\frac{r}{R}\right)\right)^2}{2mR^2S^2\left(\frac{r}{R}\right)} + V(r) + \frac{1}{2I}\left(\frac{dS_x}{dx}\right)^2 + \frac{(C_\alpha+C_\beta)^2}{8Jx^2} + V_x(x) + \frac{1}{2J}\left(\frac{dS_y}{dy}\right)^2 + \frac{(C_\alpha-C_\beta)^2}{8Jy^2} + V_y(y), \quad (3.6)$$

and for the incompressible case:

$$E = \frac{1}{2m}\left(\frac{dS_r(r)}{dr}\right)^2 - \frac{\left(l - (C_\alpha+C_\beta)C\left(\frac{r}{R}\right)\right)^2}{2mR^2S^2\left(\frac{r}{R}\right)} + V(r) + \frac{C_\gamma^2\lambda^2}{J(1-\lambda^2)^2} + \frac{C_\delta^2\lambda^2}{J(1+\lambda^2)^2} + \left(\frac{dS_\lambda}{d\lambda}\right)^2\frac{\lambda^4}{2J(1+\lambda^4)} + V(\lambda), \quad (3.7)$$

where $J_q = \int_0^{2\pi} p_q dq$, $p_i = \frac{\partial S}{\partial q^i}$. Explicitly

$$J_\varphi = \oint p_\varphi d\varphi = l\int_0^{2\pi} d\varphi = 2\pi l, \quad J_\alpha = \oint p_\alpha d\alpha = C_\alpha\int_0^{2\pi} d\alpha = 2\pi C_\alpha.$$

Now we can calculate J_r, J_x, J_y for the compressible case. The constants of separation C_x and C_y have the same form for the sphere and pseudosphere:

$$C_x := \frac{1}{2J}\left(\frac{dS_x}{dx}\right) + \frac{(J_\alpha + J_\beta)^2}{32\pi^2 J x^2} + V_x(x), \quad C_y := \frac{1}{2J}\left(\frac{dS_y}{dy}\right) + \frac{(J_\alpha - J_\beta)^2}{32\pi^2 J y^2} + V_y(y),$$

$$J_r = \oint p_r dr = \oint \sqrt{2m\left(E - C_x - C_y - V_r(r)\right) - \frac{\left(J_\varphi - J_\alpha C\left(\frac{r}{R}\right)\right)^2}{4\pi^2 R^2 S^2\left(\frac{r}{R}\right)}}dr, \tag{3.8}$$

$$J_x = \oint\sqrt{2J\left(C_x - V_x(x)\right) - \frac{(J_\alpha + J_\beta)^2}{16\pi^2 x^2}}dx,\ J_y = \oint\sqrt{2J\left(C_y - V_y(y)\right) - \frac{(J_\alpha - J_\beta)^2}{16\pi^2 y^2}}dy. \tag{3.9}$$

Now let us return to the incompressible case. The constants of separation for sphere and pseudosphere are given by:

$$C_\lambda = \frac{J_\gamma^2 \lambda^2}{J 4\pi^2(1-\lambda^2)^2} + \frac{J_\delta^2\lambda^2}{J4\pi^2(1+\lambda^2)^2} + \left(\frac{dS_\lambda}{d\lambda}\right)^2 \frac{\lambda^4}{2J(1+\lambda^4)} + V_\lambda(\lambda),$$

$$J_r = \oint p_r dr = \oint \sqrt{2m\left(E - C_\lambda - V_r(r)\right) - \frac{\left(l - (J_\alpha + J_\beta)\, C\left(\frac{r}{R}\right)\right)^2}{4\pi^2 R^2 S^2\left(\frac{r}{R}\right)}}dr, \tag{3.10}$$

$$J_\lambda = \oint\sqrt{2I\left(C_\lambda - V_\lambda(\lambda)\right)\frac{2J(1+\lambda^4)}{\lambda^4} - \frac{J_\gamma^2 2J(1+\lambda^4)}{J4\pi^2(1-\lambda^2)^2\lambda^2} - \frac{J_\delta^2 2J(1+\lambda^4)}{J4\pi^2(1+\lambda^2)^2\lambda^2}}d\lambda. \tag{3.11}$$

As we see J_λ has a more complicated form than J_x and J_y, but J_r for both cases is the same.

If we want to calculate J_r explicitly, we should choose potentials which are related to the geometry of our manifold and thus simplify calculations. We can choose potentials in the form: $V_r(r) = f(r)\det[g^{ij}]$, because $\det[g^{ij}]$ it is a scalar density of weight -2, thus it is a well-defined geometric object. In the spherical case

$$\det[g^{ij}] = \frac{1}{R^2\sin^2(\frac{r}{R})} \to f(r) = R^2\cos^2(\frac{r}{R}) \to V_r(r) = \frac{\cos^2(\frac{r}{R})}{\sin^2(\frac{r}{R})} = \cot^2(\frac{r}{R});$$

for such a potential and for the compressible spherical case J_r (3.8) has the form:

$$\begin{aligned} J_r = &\sqrt{2m4\pi^2R^2(E - C_x - C_y + 1) + J_\alpha^2} \\ &- \frac{1}{2}\sqrt{2m4\pi^2R^2 + (J_\varphi + J_\alpha)^2} - \frac{1}{2}\sqrt{2m4\pi^2R^2 + (J_\varphi - J_\alpha)^2}, \end{aligned} \tag{3.12}$$

where C_x, C_y are constant. In an incompressible spherical case J_r has a similar form; we must only put $(-C_x - C_y)$ instead of $(-C_\lambda)$.

In the pseudospherical case we can choose:

$$\det[g^{ij}] = \frac{1}{R^2\sinh^2(\frac{r}{R})} \to f(r) = \cosh^2(\frac{r}{R}) \to V_r(r) = \frac{\cosh^2(\frac{r}{R})}{R^2\sinh^2(\frac{r}{R})} = \frac{1}{R^2}\coth^2(\frac{r}{R}).$$

For the compressible pseudospherical case (3.8):

$$J_r = -\sqrt{2m4\pi^2 R^2(\frac{1}{R^2} - E + C_x + C_y) + J_\alpha^2} + \frac{1}{2}\sqrt{2m4\pi^2 R^2 + (J_\varphi + J_\alpha)^2} - \frac{1}{2}\sqrt{2m4\pi^2 R^2 + (J_\alpha - J_\beta)^2}. \quad (3.13)$$

In the incompressible pseudospherical model there is a similar structure and again we must only put $(-C_\lambda)$ instead of $(-C_x - C_y)$ for the integration constants.

Now let us consider the motion in the plane of deformation variables x, y. There exist "universally separable" potentials [7] and they have the form: $V(x,y) = A/x^2 + B/y^2 + C\left(x^2 + y^2\right)$. If A, B and C are constants, then $V_F\left(x,y\right) = F/y^2 + (F/4)\left(x^2 + y^2\right)$, where F is constant. For the incompressible model there exists a natural counterpart of the above quoted potential. It has the form: $V_F\left(\lambda\right) = 4F\lambda^2/\left(\lambda^2 + 1\right)^2 + F(1 - \lambda^4)^2/16\lambda^2$. However, as yet we were unable to find either the explicit solution or perform a qualitative analysis. Taking into account the potential V_F we can calculate the action variables J_x, J_y for all discussed cases:

$$J_x = -\sqrt{\frac{(J_\alpha + J_\beta)^2}{16}} - \pi C_x\sqrt{\frac{2I}{F}}, \quad J_y = -\pi C_y\sqrt{\frac{2I}{F}} - \pi\sqrt{2IF + \frac{(J_\alpha - J_\beta)^2}{16\pi^2}}. \quad (3.14)$$

Remark. The affinely-rigid body is the simplest finite-dimensional generalization of the metrically rigid body, with non-trivial degrees of freedom. The mechanics of affinely rigid bodies provides a foundation of the theory of generalized media with internal degrees of freedom (like molecular crystals). Affinely-rigid bodies model may be convenient in various important procedures like the finite-elements for ordinary continuum.

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