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A new derivation of the plane wave expansion into spherical harmonics and related Fourier transforms

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Abstract

This article summarizes a new, direct approach to the determination of the expansion into spherical harmonics of the exponential $e^{i(x|y)}$ with $x, y \in R^d$. It is elementary in the sense that it is based on direct computations with the canonical decomposition of homogeneous polynomials into harmonic components and avoids using any integral identities. The proof makes also use of the standard representation theoretic properties of spherical harmonics and the explicit form of the reproducing kernels for these spaces by means of classical Gegenbauer polynomials. In the last section of the paper a new method of computing the Fourier transforms of $SO(d)$-finite functions on the unit sphere is presented which enables us to reobtain both the classical Bochner identity and generalizations of it due to one of the present authors and F. J. Gonzalez Vieli.

1 Introduction and Preliminaries

This paper aims at a short presentation of a new, direct approach to a chapter of classical harmonic analysis in Euclidean spaces. It is concerned with determining the most general representation of the Fourier transform for functions possessing the rotational symmetry of a finite-dimensional representation of the rotation group $\mathbf{SO}(d)$. This problem is usually presented in a way which relies on the various types of integral identities related to the Hecke-Funk identity as in [5], or the Poisson integral representation of Bessel functions as in [1, 9]. In this approach the expansion of the plane wave into spherical harmonics is deduced as a corollary from these integral identities. Our approach reverses the order and determines the expansion into spherical harmonics of the exponential $e^{i(x|y)}$, for $x, y \in R^d$ by direct computations involving only the canonical decomposition of homogeneous polynomials.

By a systematic application of the technique of harmonic projections the detailed decomposition of the homogeneous polynomial $(x \mid y)^l$ involving harmonics of lower orders is obtained. It is expressed in terms of the classical Gegenbauer polynomials and can also serve as a vehicle for obtaining some properties of these important classical orthogonal polynomials. With this expressions at hand the summation of the exponential series is a routine matter and produces the formula (2.1), usually called the plane wave expansion. The last part of this article demonstrates how one can use the expansion to obtain results on the Fourier transform of functions which are products of radial factor and homogeneous polynomial. The formulae we obtain encompass both the classical Bochner identity [1, 9, 5], as well as its generalizations obtained by one of the authors (A.S.) in [11] and more recently by F. J. Gonzalez Vieli in [6]. The detailed presentation of the results contained here is given in the forthcoming papers [2, 3] of the authors.

1.1 The notation

We consider the Euclidean space $\mathbb{R}^d$ of dimension $d \geq 3$ with inner product denoted by $(\, x \mid y \,)$ and $r^2 = |x|^2 = (\, x \mid x \,)$ the square of the Euclidean norm of x. For any nonzero vector x we set $x = |x|\xi$, where ξ is of unit length. The set of all vectors of the unit length, the unit sphere in $\mathbb{R}^d$ is denoted S^{d-1}. For any nonzero vectors $x = |x|\xi$ and $y = |y|\eta$ in $\mathbb{R}^d$ we shall write $u = |x| \cdot |v|$, so that $(\, x \mid v \,) = u \cos\theta$, where θ denotes the angle between the vectors x and v. By $d\sigma(\cdot)$ we denote the Euclidean surface measure on the unit sphere, normalized by the condition $\int_{S^{d-1}} d\sigma = 1$. Further we define a constant α by setting $d = 2\alpha + 2$.

1.2 Decomposition of homogeneous polynomials

First we recall basic facts of the spherical harmonics theory in the Euclidean space $\mathbb{R}^d$. By $\mathcal{P}^l = \mathcal{P}^l(\mathbb{R}^d)$ we denote the space consisting of complex valued polynomial functions defined on $\mathbb{R}^d$ which are homogeneous of degree l and introduce the inner product in $\mathcal{P}^l$ by setting

$$[\, P \mid Q \,] := \int_{S^{d-1}} P(\xi)\overline{Q(\xi)}\, d\sigma(\xi), \tag{1.1}$$

Let Δ denote the Laplacian in $\mathbb{R}^d$, $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$. By restriction to $\mathcal{P}^l$ with $l \geq 2$ it gives rise to surjection $\Delta : \mathcal{P}^l \to \mathcal{P}^{l-2}$ and the multiplication by r^2 yields a linear injection $\mathcal{P}^{l-2} \to \mathcal{P}^l$. The kernel of Δ in $\mathcal{P}^l$ is denoted $\mathcal{H}^l$ and consists of harmonic and homogeneous polynomials. The dimension of $\mathcal{H}^l$ is given by $\dim \mathcal{H}^l = \frac{2(l+\alpha)\Gamma(2\alpha+l)}{\Gamma(l+1)\Gamma(2\alpha+1)}$, where $\Gamma(z)$ denotes the Euler gamma function. Each of the spaces $\mathcal{P}^l$ is invariant under the natural action of the special orthogonal group $\mathbf{SO}(d)$ given by $\rho \cdot f(x) = f(\rho^{-1}x)$, for $\rho \in \mathbf{SO}(d)$, $x \in \mathbb{R}^d$ and it is known that spaces $\mathcal{H}^l$ are invariant and irreducible under this action. The elements of $\mathcal{H}^l$ are called solid harmonics and their restrictions to the unit sphere — spherical surface harmonics of order l — note that a homogeneous polynomial is completely determined by its restriction to the unit sphere. Restricting the action of $\mathbf{SO}(d)$ to functions on the unit sphere we see that the space of spherical surface harmonics of order l is also irreducible.

The following Theorem, which combines results of [8, 12, 13], summarizes what is needed for the sequel concerning the structure of homogeneous polynomials. As usual $[m]$ denotes the integer part of a real number m.

Theorem 1 (The canonical decomposition of homogeneous polynomials). *Every polynomial $P \in \mathcal{P}^l$ has a unique representation of the form*

$$P = \sum_{k=0}^{[l/2]} r^{2k} h_{l-2k}(P), \qquad \text{with} \qquad h_{l-2k}(P) = \sum_{j=0}^{[l/2]-k} e^l_j(k) r^{2j} \Delta^{k+j} P \in \mathcal{H}^{l-2k} \quad (1.2)$$

where the coefficients are given by $e^l_j(k) = (-1)^j \frac{(\alpha+l-2k)\Gamma(\alpha+l-2k-j)}{4^{k+j}k!j!\Gamma(\alpha+l+1-k)}$. The resulting direct sum decomposition $\mathcal{P}^l = \bigoplus_{k=0}^{[l/2]} r^{2k}\mathcal{H}^{l-2k}$ is orthogonal with respect to the inner product (1.1) and invariant under the action of $\mathbf{SO}(d)$.

We shall call the polynomials $h_{l-2k}(P)$ the harmonic components of P, while the component of the highest degree, $h_l(P)$, is called its harmonic projection.

1.3 The Bessel function

We shall also need the following basic properties of Bessel functions. The Bessel functions of the first kind and order ν with $\mathrm{Re}\,\nu > -1$ can be defined by the power series expansion

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k}. \quad (1.3)$$

Only the case of nonnegative integer and half-integer values of ν, in other words $\nu \in \frac{1}{2}\mathbb{Z}$, will be considered here. Note that the series defining $J_\nu(t)$ is absolutely convergent for all (real or complex) values of t. In the sequel it will be more convenient to use its "regularization" $j_\nu(t)$, called sometimes "the small Bessel function", defined by

$$j_\nu(t) = \Gamma(\nu+1)\left(\frac{t}{2}\right)^{-\nu} J_\nu(t), \quad \text{satisfying} \quad y'' + \frac{2\nu+1}{t}y' + y = 0, \quad j_\nu(0) = 1 \quad (1.4)$$

It fulfills the recurrence relations

$$\frac{2^{-\nu}}{\Gamma(\nu+1)}\left(\frac{1}{t}\frac{d}{dt}\right)^l j_\nu(t) = (-1)^l \frac{2^{-\nu-l}}{\Gamma(\nu+l+1)} j_{\nu+l}(t), \qquad l \in \mathbb{Z} \quad (1.5)$$

which allow to express $j_\nu(t)$ in terms of $j_{-1/2}$ or j_0. Since $j_{-1/2}(t) = \cos t$ it follows that $j_\nu(t)$ is an elementary function for $\nu = n + 1/2,\ n \in \mathbb{Z}_+$.

1.4 The Gegenbauer polynomials and zonal harmonics

The Gegenbauer polynomial $C^\alpha_p(\cdot)$ of degree p may be given by the explicit formula, cf. e.g. [1, 4]

$$C^\alpha_p(t) = \sum_{j=0}^{[p/2]} (-1)^j \frac{\Gamma(\alpha+p-j)}{\Gamma(\alpha)\Gamma(j+1)\Gamma(p-2j+1)} (2t)^{p-2j}. \quad (1.6)$$

As well known, the polynomials $C^\alpha_p(t)$ form the set of orthogonal polynomials on interval $[-1,\, 1]$ with respect to the measure $d\mu(t) = (1-t^2)^{\alpha-1/2}dt$.

Choose a unit vector $\eta \in S^{d-1}$ and let $K_\eta \simeq \mathbf{SO}(d-1)$ be the isotropy subgroup of η in $\mathbf{SO}(d)$. K_η is isomorphic with $\mathbf{SO}(d-1)$. It is known that the subspace of K_η-invariant vectors in $\mathcal{H}^l$, which will be denoted as $\big(\mathcal{H}^l\big)^{K_\eta}$, consists of scalar multiples of the harmonic projection of the invariant $(\,\cdot \mid \eta\,)^l \in \mathcal{P}^l$. We have the following lemma

Lemma 1. *The unique K_η-invariant vector $Z^l_\eta(x) \in \mathcal{H}^l$ normalized by the condition $Z^l_\eta(\eta) = 1$ is given by the formula*

$$Z^l_\eta(x) = |x|^l \frac{\Gamma(2\alpha)\Gamma(l+1)}{\Gamma(2\alpha+l)} C^\alpha_l((\,\xi \mid \eta\,)), \qquad \text{where} \quad x = |x|\xi. \tag{1.7}$$

The polynomial $Z^l_\eta(x)$ enjoys the following reproducing property: For each $P \in \mathcal{H}^l$

$$\dim \mathcal{H}^l [\, Z^l_\eta \mid \overline{P}\,] = \dim \mathcal{H}^l \int_{S^{d-1}} Z^l_\eta(\xi) P(\xi)\, d\sigma(\xi) = P(\eta). \tag{1.8}$$

$Z^l_\eta(x)$ are also called zonal polynomials, because of their invariance with respect to the isotropy group K_η.

2 Plane wave expansion

For many questions of applied analysis it is often necessary to expand the exponential function $e^{i(\,x|v\,)}$, which is a solution of the Helmholtz equation $(\Delta + |v|^2)\phi = 0$, into harmonic polynomials. This is used for example in describing quantum mechanical scattering of particles or scattering of waves in electrodynamics. Below we present the main steps of a new proof of this expansion which was obtained by the present authors in [2].

Theorem 2. *For arbitrary unit vectors ξ, $\eta \in S^{d-1} \subset \mathbb{R}^d$ and $r \in \mathbb{R}$ the following expansion holds*

$$e^{ir(\xi|\eta)} = \Gamma(\alpha)\Big(\frac{r}{2}\Big)^{-\alpha} \sum_{m=0}^{\infty} i^m (\alpha+m) J_{\alpha+m}(r) C^\alpha_m\big((\xi \mid \eta)\big) \tag{2.1}$$

$$= \sum_{l=0}^{\infty} i^m \dim \mathcal{H}^m \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} \Big(\frac{r}{2}\Big)^m j_{\alpha+m}(r) Z^m_\eta(\xi). \tag{2.2}$$

The series converges absolutely for each fixed value of $r \in \mathbb{R}$ and uniformly with respect to ξ, $\eta \in S^{d-1}$.

Our proof of this result is split into two parts. First we decompose the homogeneous polynomial $(x \mid \eta)^l$ with $x \in \mathbb{R}^d$ into harmonic components as in Theorem 1 and secondly we perform the summation of the exponential series in such a way as to single out the harmonic components of any given order. The first part is summed up by the following.

Lemma 2. *For any nonnegative integer l and $\alpha = (d-2)/2 \geq 1/2$, the function $x \mapsto (x \mid \eta)^l$ has the following expansion into spherical harmonics*

$$(x \mid \eta)^l = 2^{-l}\Gamma(\alpha)\Gamma(l+1)|x|^l \sum_{k=0}^{[l/2]} \frac{(\alpha+l-2k)}{k!\Gamma(\alpha+l-k+1)} C^\alpha_{l-2k}\big((\xi \mid \eta)\big), \qquad x = |x|\xi. \tag{2.3}$$

Inserting this formula into the exponential series we obtain

$$e^{ir(\xi|\eta)} = \sum_{l=0}^{\infty}\sum_{k=0}^{[l/2]} i^l \Big(\frac{r}{2}\Big)^l \frac{\Gamma(\alpha)(\alpha+l-2k)}{k!\Gamma(\alpha+l-k+1)} C_{l-2k}^{\alpha}\big((\xi \mid \eta)\big). \tag{2.4}$$

Now by changing the order of summation in the double sum on the right hand side of this equation (which is permitted since the series converges absolutely and uniformly for $\xi, \eta \in S^{d-1}$ and arbitrary $r \in \mathbb{R}$) we arrive at

$$\begin{aligned} e^{ir(\xi|\eta)} &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} i^{m+2k}\Big(\frac{r}{2}\Big)^{m+2k} \frac{\Gamma(\alpha)(\alpha+m)}{k!\Gamma(\alpha+m+k+1)} C_m^{\alpha}\big((\xi \mid \eta)\big) \\ &= \Gamma(\alpha)\Big(\frac{r}{2}\Big)^{-\alpha} \sum_{m=0}^{\infty} i^m (\alpha+m) J_{\alpha+m}(r) C_m^{\alpha}\big((\xi \mid \eta)\big). \end{aligned}$$

Thus the first part of the expansion (2.1) is proved. For the proof of the second formula in (2.1) we use the relations between small Bessel function (1.4) and the formula (1.7).

3 The Fourier transform of SO(d)-finite functions

We assume that the Fourier transform is defined by

$$\widehat{f}(x) = \mathcal{F}f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i(x|y)} f(y)\, dy, \qquad x \in \mathbb{R}^d,$$

for suitable, e.g. $L^1(\mathbb{R}^d)$, functions f. The formula may also be used for defining the Fourier transform of complex measures on $\mathbb{R}^d$ with support on the unit sphere.

The **SO**(d)-finite functions on the unit sphere are essentially the restrictions of polynomials to the unit sphere, hence to determine their Fourier transform it suffices to compute them for spherical measures of the form $P(\xi)d\sigma(\xi)$ with homogeneous P. In view of the Theorem 1.2 this can be reduced to the special case with harmonic P and for those the expansion (2.1) and the reproducing property of zonal polynomials provide natural tools. They give immediately the first of the expressions given in the Theorem 3 below, while the second of these, Eq. (3.2) below, was first given (with different proof) by F. J. Gonzalez Vieli in the paper [6]. The forthcoming paper [3] contains a detailed proof along these lines.

Theorem 3. *If $P \in \mathcal{P}^l$, then the Fourier transform of the measure $P(\xi)d\sigma(\xi)$ with support on the unit sphere S^{d-1} is given by the following equivalent formulae*

$$\begin{aligned} \int_{S^{d-1}} e^{i(x|\eta)} P(\eta)\, d\sigma(\eta) &= \Big(\frac{i}{2}\Big)^l \sum_{k=0}^{[l/2]} \frac{(-1)^k 2^{2k}\Gamma(\alpha+1)}{\Gamma(\alpha+l+1-2k)} j_{\alpha+l-2k}(|x|) h_{l-2k}(P)(x) \quad (3.1) \\ &= \Big(\frac{i}{2}\Big)^l \sum_{k=0}^{[l/2]} \frac{(-1)^k \Gamma(\alpha+1)}{k!\Gamma(\alpha+l+1-k)} j_{\alpha+l-k}(|x|)(\Delta^k P)(x) \quad (3.2) \end{aligned}$$

with $h_{l-2k}(P)$ denoting the harmonic components of P as in the Eq. (1.2).

3.1 Generalized Bochner identity

The celebrated Bochner identity, cf. for instance [1, 5], expresses the Fourier transform of a product of radial function with spherical harmonics in terms of the so-called Hankel transform, which is defined for functions defined on the positive real axis by

$$H_\nu(\phi)(t) = \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^\infty \phi(s) j_\nu(st) s^{2\nu+1} ds. \tag{3.3}$$

By use of the previous Theorem 3 we can rewrite in two equivalent forms the generalized Bochner identity obtained previously by one of the present authors in [11].

Corollary 1 (Generalized Bochner identity). *If $P \in \mathcal{P}^l$, then the Fourier transform of the function $f(|x|)P(x)$ is given by the following expressions:*

$$\widehat{fP}(y) = i^l \sum_{k=0}^{[l/2]} (-1)^k H_{\alpha+l-2k}(t^{2k} f)(|y|) h_{l-2k}(P)(y) \tag{3.4}$$

$$= i^l \sum_{k=0}^{[l/2]} \frac{(-1)^k}{2^k k!} H_{\alpha+l-k}(f)(|y|)) \Delta^k P(y) \tag{3.5}$$

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