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Canonical Analysis of Symmetry Enhancement with Gauged Grassmannian Model

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Abstract

We study the Hamiltonian structure of the gauge symmetry breaking and enhancement. After giving a general discussion of these phenomena in terms of the constrained phase space, we perform a canonical analysis of the Grassmannian nonlinear sigma model coupled with Chern-Simons term, which contains a free parameter governing explicit symmetry enhancement$\rightleftharpoons$ breaking according to $U(n+m) \rightleftharpoons U(n) \times U(m)$.

1 Introduction

The phenomena of gauge symmetry breaking or enhancement are very important subjects in physics. One of the most important applications is to the area of generating the masses of the gauge bosons [1]. Recently, the dynamical generation of gauge boson mass has been analyzed in the context of an enlarged $CP(N)$ model [2]. The model contains a free parameter r which could be understood as an explicit gauge symmetry breaking $\rightleftharpoons$ enhancement parameter; in the case of $r = 1$ the gauge symmetry of the model is $U(2)$, while it is $U(1) \times U(1)$ in the case of $r \neq 1$. It was shown in Ref. [2] that the dynamically generated gauge fields yield the $U(2)$ Yang-Mills theory in the limit of $r = 1$. Away from this limit, two of the gauge fields become massive with masses being induced radiatively through the loop corrections and the symmetry is broken to $U(1) \times U(1)$. That is, the gauge symmetry enhancement has occurred in the limit of $r = 1$.

In this work, we study extended version of this model in the Hamiltonian formulation. We first recall that the gauge symmetry is realized as the Gauss law constraints in the Hamiltonian method. In order to see the structure of gauge symmetry more explicitly, we couple the Grassmannian nonlinear sigma model [3] with some external gauge fields, which we choose to be described by the $U(n+m)$ Chern-Simons term. Then, we perform the Dirac analysis [4] of the resulting theory. The theory has both first and second class constraints, and it is found that for $r = 1$ the Gauss constraints satisfy $U(n + m)$ symmetry algebra, whereas for $r \neq 1$ only $U(n) \times U(m)$ algebra. What happens is that off-diagonal first class constraints generating the $U(n + m)$ gauge symmetry become second class constraints away from $r = 1$, reducing the resulting gauge symmetry.

This can be understood more geometrically as follows [5]. A phase space can be described by a manifold Γ with a non-degenerate closed 2-form. If a theory is constrained by the constraints, the space of physical interest will be the submanifold $\bar{\Gamma}$ consisting of all points of Γ satisfying the constraints. It is well-known [4] that one can decompose the constraints into two classes, first class constraints and the second class ones. The degeneracies of the closed two-form on the constrained submanifold are associated with the first class constraints, and the reduced phase space is identified with the quotient manifold of $\bar{\Gamma}$ where any two points of $\bar{\Gamma}$ are identified if they are related by a curve which lies along the degeneracy directions everywhere. The non-degenerate closed 2-form, or symplectic structure, on the reduced phase space defines the Dirac bracket. What happens in our model can be explained as follows. The Hamiltonian vector fields which are generated by the non-diagonal part of $U(n+m)$ constraints point in fixed directions in Γ. When $r \neq 1$, they are not tangent to $\bar{\Gamma}$. As the parameter r approaches one, the constraints change gradually and $\bar{\Gamma}$ becomes tangent to those vector fields at $r = 1$. Initially second class constraints become first class, the gauge symmetry being enlarged from $U(n) \times U(n)$ to $U(n+m)$.

It turns out that a smooth extrapolation from the $U(n) \times U(m)$ to $U(n+m)$ gauge symmetry algebra is not possible in the Dirac analysis. The reason is that in the Dirac method we have to compute the inverse of the Dirac matrix which is constructed with second class constraints only. This inverse matrix with parameter r becomes singular if we take the limit of $r \to 1$, because off-diagonal constraints change from second class into first class. When this happens, the Dirac matrix becomes degenerate and the inverse does not exist. From physical point of view, this singular behavior could be associated with the second order phase transition which one encounters in going to the limit $r = 1$ [2].

In the next section, we define the Grassmannian model coupled with Chern-Simons term and perform the canonical analysis. We also discuss the Dirac bracket structure in the case of $r = 1$ and $r \neq 1$ separately.

2 Chern-Simons gauged Grassmannian model

We start from the Lagrangian written in terms of the $N \times (n+m)$ matrix ψ such that

$$\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\left[(D_\mu\psi)^\dagger(D^\mu\psi) - \lambda(\psi^\dagger\psi - R)\right] + \mathcal{L}_{\mathrm{CS}}, \tag{2.1}$$

where the field, ψ, is made of $n+m$ complex N-vectors such that

$$\psi = \left[\psi_1, \cdots, \psi_n, \psi_{n+1}, \cdots, \psi_{n+m}\right]. \tag{2.2}$$

The hermitian $(n+m) \times (n+m)$ matrix λ is a Lagrange multiplier, and R is given by

$$R = \left[\begin{array}{cc} r_{n\times n} & 0 \\ 0 & {r^{-1}}_{m\times m} \end{array}\right], \tag{2.3}$$

where $r_{n\times n} = \mathrm{diag}\,\underbrace{(r, \cdots, r)}_{n \text{ times}}$ and ${r^{-1}}_{m\times m} = \mathrm{diag}\,\underbrace{(r^{-1}, \cdots, r^{-1})}_{m \text{ times}}$ with a real positive r. We will also use the notation $R_{ab} = r_a\delta_{ab}$ $(a, b = 1, \cdots, n+m)$ with $r_a = r(1 \leq a \leq n)$, and

$r_b = r^{-1}(n+1 \leq b \leq n+m)$. The covariant derivative is defined as $D_\mu\psi \equiv \partial_\mu\psi - \psi A_\mu$ with a $(n+m)\times(n+m)$ anti-hermitian matrix gauge potential A_μ associated with the local $U(n+m)$ gauge transformations. $\mathcal{L}_{\rm CS}$ is the non-Abelian Chern-Simons gauge action given by

$$\mathcal{L}_{\rm CS} = -\frac{\kappa}{2}\epsilon^{\mu\nu\rho}{\rm tr}\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right). \tag{2.4}$$

The kinetic term of the Lagrangian (2.1) is invariant under the local $U(n+m)$ transformation, while the matrix R with $r \neq 1$ explicitly breaks the $U(n+m)$ gauge symmetry down to $U(n)\times U(m)$. Thus, the symmetry of our model is $[SU(N)]_{\rm global}\times[U(n+m)]_{\rm local}$ for $r=1$, while $[SU(N)]_{\rm global}\times[U(n)\times U(m)]_{\rm local}$ for $r\neq 1$. Therefore, the parameter r could be regarded as an explicit symmetry breaking parameter.

Let us perform the canonical analysis using the Dirac method [4]. We first define the conjugate momenta of the ψ_a^α field by $\Pi_a^\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_a^\alpha}$, which gives

$$\Pi_a^\alpha = \frac{1}{g^2}(\dot{\psi}_a^{\alpha\dagger} + A_{0ab}\psi_b^{\alpha\dagger}). \tag{2.5}$$

The indices $a, b, \cdots$ represent the $U(n+m)$ indices, while Latin indices $\alpha, \beta, \cdots$ represent the global $SU(N)$ indices. We will occasionally omit the global $SU(N)$ indices, when the context is clear. Likewise, the conjugate momentum of the $\psi_a^{\alpha\dagger}$ field is given by

$$\Pi_a^{\alpha\dagger} = \frac{1}{g^2}(\dot{\psi}_a^\alpha - \psi_b^\alpha A_{0ba}). \tag{2.6}$$

The momentum for the Lagrangian multiplier field λ_{ab} is constrained to vanish,

$$\Pi_{ab}^\lambda = 0. \tag{2.7}$$

The conjugate momentum P_{ab}^μ for the gauge field $A_{\mu ab}$ is given by

$$P_{iab} = \kappa\epsilon_{ij}A_{jba}, \quad P_{0ab} = 0. \tag{2.8}$$

In the above, the indices $i, j, ..$ represent the spatial ones with 1 and 2. In the following analysis we will not treat the first equation as a constraint. Instead P_{iab} is removed from the beginning and replaced by $\kappa\epsilon_{ij}A_{jba}$. The second equation, together with (2.7), defines the primary constraint of the theory. The Poisson brackets are defined by

$$\begin{aligned}
&\{\psi_a^\alpha(x),\ \Pi_b^\beta(y)\} = \delta_{ab}\delta^{\alpha\beta}\delta(x-y),\\
&\{\lambda_{ab}(x), \Pi_{cd}^\lambda(y)\} = \delta_{ac}\delta_{bd}\delta(x-y),\\
&\{A_{0ab}(x), P_{0cd}(y)\} = \delta_{ac}\delta_{bd}\delta(x-y)\\
&\{A_{iab}(x), A_{jcd}(y)\} = -\tfrac{1}{\kappa}\epsilon_{ij}\delta_{ad}\delta_{bc}\delta(x-y).
\end{aligned} \tag{2.9}$$

After a straightforward Dirac analysis, we find that the system is described by the canonical Hamiltonian given by

$$\mathcal{H}_0 = g^2\Pi_a\Pi_a^\dagger + \frac{1}{g^2}(D_i\psi)_a^\dagger(D_i\psi)_a + \frac{1}{g^2}\lambda_{ab}(\psi_b^\dagger\psi_a - R_{ba}) + (\Pi_a\psi_b - \psi_a^\dagger\Pi_b^\dagger + \kappa F_{12ab})A_{0ba}, \tag{2.10}$$

where we denote $FG \equiv F^\alpha G^\alpha$ and F_{12ab} is the magnetic field given by

$$F_{12ab} = \partial_1 A_{2ab} - \partial_2 A_{1ab} + [A_1, A_2]_{ab}. \tag{2.11}$$

Including all secondary constraints, we find that the dynamics is governed by the following constraints;

$$\begin{aligned} C^{(0)}_{ab} &= \Pi^\lambda_{ab} \approx 0, \\ C^{(1)}_{ab} &= P^0_{ab} \approx 0, \\ C^{(2)}_{ab} &= \psi^\dagger_a \psi_b - R_{ab} \approx 0, \\ C^{(3)}_{ab} &= \Pi_a \psi_b - \psi^\dagger_a \Pi^\dagger_b + \kappa F_{12ab} \approx 0, \\ C^{(4)}_{ab} &= \Pi_a \psi_b + \psi^\dagger_a \Pi^\dagger_b - \tfrac{1}{g^2}[A_0, R]_{ab} \approx 0. \end{aligned} \tag{2.12}$$

One can check that the time evolution of the above constraints are closed with a total Hamiltonian $\mathcal{H}_T = \mathcal{H}_0 + \sum_{u=0}^4 \Lambda^{(u)}_{ab} C^{(u)}_{ab}$ using the relations (2.9).

To separate the constraints into first and second-class, we first calculate the commutation relations of (2.12) to yield the nonvanishing Poisson brackets

$$\{C^{(1)}_{ab}(x), C^{(4)}_{cd}(y)\} = \frac{1}{g^2}(r_c - r_d)\delta_{ad}\delta_{bc}\delta(x-y), \tag{2.13}$$

$$\{C^{(2)}_{ab}(x), C^{(3)}_{cd}(y)\} = (r_c - r_d)\delta_{ad}\delta_{bc}\delta(x-y), \tag{2.14}$$

$$\{C^{(2)}_{ab}(x), C^{(4)}_{cd}(y)\} = (r_a + r_b)\delta_{ad}\delta_{bc}\delta(x-y), \tag{2.15}$$

$$\{C^{(3)}_{ab}(x), C^{(3)}_{cd}(y)\} = \left(\delta_{bc} C^{(3)}_{ad} - \delta_{ad} C^{(3)}_{cb}\right)\delta(x-y), \tag{2.16}$$

$$\{C^{(3)}_{ab}(x), C^{(4)}_{cd}(y)\} = \frac{1}{g^2}([A_0, R]_{ad}\delta_{bc} - [A_0, R]_{bc}\delta_{ad})\delta(x-y), \tag{2.17}$$

$$\{C^{(4)}_{ab}(x), C^{(4)}_{cd}(y)\} = \kappa(F_{12cb}\delta_{ad} - F_{12ad}\delta_{bc})\delta(x-y). \tag{2.18}$$

Note that (2.16) satisfies $U(n+m)$ Gauss law algebra. Nevertheless, $C^{(3)}_{ab}$ with $a \neq b$ become second class constraints for $r \neq 1$, because in this case the right hand sides of (2.14) and (2.17) are nonvanishing for $c \neq d$.

Let us discuss the Dirac brackets [4] of our model. It turns out that transition from $r \neq 1$ to $r = 1$ is singular and we have to carry out the cases of $r = 1$ and $r \neq 1$ separately.

$r = 1$ case. For the case of $r = 1$, we have $R_{ab} = \delta_{ab}$, and it is easy to infer from the constraints algebra (2.13)-(2.18), that only $C^{(2)}_{ab}$ and $C^{(4)}_{ab}$ are second class constraints. All of $C^{(3)}_{ab}$'s are the first class constraints whose Gauss law satisfies the $U(n+m)$ algebra (2.16). $C^{(0)}_{ab}$ and $C^{(1)}_{ab}$ completely decouple from the theory and can be put equal to zero.

Let us define the second class constraints as, $C_K \equiv (C^{(2)}_A, C^{(4)}_B)$ $(K = 1, 2, ..., 2(n+m)^2)$ where $C^{(2)}_A$ stands for $C^{(2)}_{ab}$ collectively and $C^{(4)}_A$ for $C^{(4)}_{ab}$. One obtains the following Poisson bracket relations $\Theta_{KL} = \{C_K, C_L\}$,

$$\Theta = \left[\begin{array}{cc} O & M \\ -M^T & N \end{array} \right], \tag{2.19}$$

where

$$M_{AB} \equiv M_{ab;cd} = 2\delta_{ad}\delta_{bc} = 2(I \otimes I)_{ab;dc}, \tag{2.20}$$

$$N_{AB} \equiv N_{ab;cd} = \kappa\left[-F_{12} \otimes I + I \otimes {F_{12}}^T\right]_{ab;dc}. \tag{2.21}$$

The inverse matrix of Θ is given by

$$\Theta^{-1} = \begin{bmatrix} M^{T-1}NM^{-1} & -M^{T-1} \\ M^{-1} & O \end{bmatrix}, \tag{2.22}$$

with

$$M^{-1}{}_{AB} = M^{-1}{}_{ab;cd} = \frac{1}{2}(I \otimes I)_{ab;dc}. \tag{2.23}$$

$\underline{r \neq 1 \text{ case.}}$ Let us divide the index by $\bar{a}$ for $1 \leq \bar{a} \leq n$, and $\tilde{a}$ for $n+1 \leq \tilde{a} \leq n+m$. In this case, we first note that off-diagonal constraints $C^{(3)}_{\bar{a}\tilde{b}}$ and $C^{(3)}_{\tilde{a}\bar{b}}$ which were first-class in the case of $r = 1$ become second-class, because the gauge symmetry is reduced to $U(m) \times U(n)$. This is evident from (2.13), whose right hand side is nonvanishing for $r_{\bar{c}} = r \neq r_{\tilde{d}} = r^{-1}$. We use the Weyl gauge with $A_0 = 0$ which makes the constraints $C^{(3)}_{ab}$ commute with $C^{(4)}_{ab}$, this makes the computation considerably easier. Therefore, all the second class constraints are given by

$$C_K = (C^{(2)}_{\bar{a}\bar{b}}, C^{(2)}_{\bar{a}\tilde{b}}, C^{(2)}_{\tilde{a}\bar{b}}, C^{(2)}_{\tilde{a}\tilde{b}}, C^{(4)}_{\bar{a}\bar{b}}, C^{(4)}_{\tilde{a}\tilde{b}}, C^{(3)}_{\bar{a}\tilde{b}}, C^{(3)}_{\tilde{a}\bar{b}}). \tag{2.24}$$

In order to compute the Dirac matrix $\Theta_{KL} = \{C_K, C_L\}$, we use the following non-vanishing components;

$$\begin{array}{l}
\{C^{(2)}{}_{\bar{a}\tilde{b}}(x), C^{(3)}{}_{\tilde{c}\bar{d}}(y)\} = (r_{\tilde{c}} - r_{\bar{d}})\delta_{\bar{a}\bar{d}}\delta_{\tilde{b}\tilde{c}}\delta(x-y) \\
\{C^{(2)}{}_{\tilde{a}\bar{b}}(x), C^{(3)}{}_{\bar{c}\tilde{d}}(y)\} = (r_{\bar{c}} - r_{\tilde{d}})\delta_{\tilde{a}\tilde{d}}\delta_{\bar{b}\bar{c}}\delta(x-y) \\
\{C^{(2)}{}_{\tilde{a}\tilde{b}}(x), C^{(4)}{}_{\tilde{c}\tilde{d}}(y)\} = (r_{\tilde{c}} + r_{\tilde{d}})\delta_{\tilde{a}\tilde{d}}\delta_{\tilde{b}\tilde{c}}\delta(x-y) \\
\{C^{(2)}{}_{\bar{a}\bar{b}}(x), C^{(4)}{}_{\bar{c}\bar{d}}(y)\} = (r_{\bar{c}} + r_{\bar{d}})\delta_{\bar{a}\bar{d}}\delta_{\bar{b}\bar{c}}\delta(x-y) \\
\{C^{(4)}{}_{\bar{a}\bar{b}}(x), C^{(4)}{}_{\bar{c}\bar{d}}(y)\} = \kappa(F_{12\,\bar{c}\bar{b}}\delta_{\bar{a}\bar{d}} - F_{12\,\bar{a}\bar{d}}\delta_{\bar{b}\bar{c}})\delta(x-y) \\
\{C^{(4)}{}_{\tilde{a}\tilde{b}}(x), C^{(4)}{}_{\tilde{c}\tilde{d}}(y)\} = \kappa(F_{12\,\tilde{c}\tilde{b}}\delta_{\tilde{a}\tilde{d}} - F_{12\,\tilde{a}\tilde{d}}\delta_{\tilde{b}\tilde{c}})\delta(x-y).
\end{array} \tag{2.25}$$

Now we find the Dirac matrix of the form

$$\Theta = \begin{bmatrix} O & M \\ -M^T & N \end{bmatrix}, \tag{2.26}$$

where

$$M = \begin{bmatrix} l & 0 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & 0 & p \end{bmatrix}, \; N = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r \end{bmatrix}, \tag{2.27}$$

with

$$M_{KL} = \left\{ \begin{array}{l} l_{\bar{a}\bar{b};\bar{c}\bar{d}} = 2r\delta_{\bar{a}\bar{d}}\delta_{\bar{b}\bar{c}} \\ m_{\bar{a}\tilde{b};\tilde{c}\bar{d}} = (r^{-1} - r)\delta_{\bar{a}\bar{d}}\delta_{\tilde{b}\tilde{c}} \\ n_{\tilde{a}\bar{b};\bar{c}\tilde{d}} = (r - r^{-1})\delta_{\tilde{a}\tilde{d}}\delta_{\bar{b}\bar{c}} \\ p_{\tilde{a}\tilde{b};\tilde{c}\tilde{d}} = 2r^{-1}\delta_{\tilde{a}\tilde{d}}\delta_{\tilde{b}\tilde{c}} \end{array} \right\}, \; \left\{ \begin{array}{l} q_{\bar{a}\bar{b};\bar{c}\bar{d}} = \kappa(F_{12\,\bar{c}\bar{b}}\delta_{\bar{a}\bar{d}} - F_{12\,\bar{a}\bar{d}}\delta_{\bar{b}\bar{c}}) \\ r_{\tilde{a}\tilde{b};\tilde{c}\tilde{d}} = \kappa(F_{12\,\tilde{a}\tilde{b}}\delta_{\tilde{a}\tilde{d}} - F_{12\,\tilde{a}\tilde{d}}\delta_{\tilde{b}\tilde{c}}) \end{array} \right\}. \tag{2.28}$$

The inverse matrix of Θ is given by

$$\Theta^{-1} = \begin{bmatrix} M^{T^{-1}} N M^{-1} & -M^{T^{-1}} \\ M^{-1} & 0 \end{bmatrix}, \; M^{-1} = \begin{bmatrix} l^{-1} & 0 & 0 & 0 \\ 0 & 0 & m^{-1} & 0 \\ 0 & n^{-1} & 0 & 0 \\ 0 & 0 & 0 & p^{-1} \end{bmatrix}, \tag{2.29}$$

with $(r - r^{-1} \equiv r_s)$

$$M^{-1}_{KL} = \left\{ (l^{-1}_{\bar{a}\bar{b};\bar{c}\bar{d}},\ m^{-1}_{\bar{a}\tilde{b};\bar{c}\tilde{d}},\ n^{-1}_{\tilde{a}\bar{b};\tilde{c}\bar{d}},\ p^{-1}_{\tilde{a}\tilde{b};\tilde{c}\tilde{d}}) = (1/2r,\ \ -r_s^{-1},\ r_s^{-1},\ r/2)\delta_{\tilde{a}\tilde{d}}\delta_{\tilde{b}\tilde{c}} \right\}. \tag{2.30}$$

Note that the inverse matrix M^{-1} of (2.29) becomes singular in the limit of $r \to 1$. This is because part of the constraints change from second class into first class in the limit $r = 1$, and determinant of the Dirac matrix becomes zero. The resulting Dirac brackets for both cases of $r = 1$ and $r \neq 1$ are very involved, and will not be presented here. The case of $n = m = 1$ can be found in Ref. [5].

3 Conclusion

We performed canonical analysis of the gauge symmetry enhancement in the Grassmannian model coupled with $U(n+m)$ Chern-Simons term. We discussed that the conventional Dirac method does not allow a smooth extrapolation of the symmetry enhanced and broken phases. This was essentially due to the fact that Dirac procedure requires an inverse of the Dirac matrix which is constructed with second class constraints only, and becomes singular when some of the second class constraints become first class. Physically, second order phase transition occurring as the symmetry breaking parameter r approaches the critical value 1 could be responsible for the non-smooth transition. It would be interesting to extend our analysis to the supersymmetric case and perform other quantization methods of our model.

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