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On One Modern Approach in Economic-Mathematical Modeling

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Abstract— **The problem of economic growth has always been a major issue in literature on micro-economics traditionally based on advanced mathematical theories. Aggregation method ("consolidation") in economic-mathematical simulation of has been successfully used by many authors to build dynamic models in macroeconomics, in particular, to model dynamics of net gross product (NGP). In constructing mathematical models, the authors, in addition to the classical Keynes multiplier (according to R. Allen) use the concept of accelerator. In the economic models of NGP dynamics traditionally this relationship is a linear function, and in more complex mathematical models it is a nonlinear function. R. Allen also pointed to the need to consider the delay ("lag"). Different types of delay are often based on mathematical differential equations with lag (differential equations with deviating argument). In the works of P.M. Simonov, it was first proposed to simulate the lag in the form . In this paper a nonlinear accelerator is considered rather extensively and mathematically justified. It is applied to the modified model of NGP by Phillips-Goodwin allowing for lag of the input of induced investments.**

Keywords— Aggregation method, economic-mathematic modeling, differential lag equations.

1. Introduction

Modern economists pay much attention to the problem of economic growth [15, 18-24] with mathematical methods being the tools for economic-mathematical modeling [2, 3, 7-11, 14]. Aggregation method of has been successfully used by authors to model dynamics of net gross product (NGP) [1, 12]. In constructing mathematical models, the authors, in addition to the classical Keynes multiplier (according to R. Allen [1] use the concept of accelerator. In the economic models of NGP dynamics traditionally this relationship is a linear function, and in more complex mathematical models it is a nonlinear function. R. Allen also pointed to the need to consider the delay ("lag"). Different types of delay are often based on mathematical differential equations with lag (differential equations with deviating argument). In the works of P. M. Simonov [16], it was first proposed to simulate the lag in the form $h_0(t)=\left[\frac{t}{\tau}\right]\tau$. In this paper a nonlinear accelerator is considered rather extensively and mathematically justified. It is applied to the modified model of NGP by Phillips-Goodwin allowing for lag of the input of induced investments.

Let us introduce the following notation: $Y(t)$ - intensity of NGP reproduction at the time t; $C(t)$ - intensity of the final non-production consumption at the time t; τ - lag of the input of real induced investments; I(t) - intensity of the input of real induced investments at the time t. Basing on the macroeconomic identity $Y(t)=I(t)+C(t)$, we assume that there is a relationship between the amount of investment and the future GNP growth. Substitute $I(t)$ into the identity:

$$Y(t)=B_1\left(\tau Y''(t)+Y'\left(\left[\tfrac{t}{\tau}\right]\tau\right)\right)+C(t).$$

In the case of a strictly monotone function B_1 there is an inverse function B_1^{-1} and then $\tau Y''(t)+Y'\left(\left[\frac{t}{\tau}\right]\tau\right)=B_1^{-1}\left(Y(t)-C(t)\right)$. Dividing both parts into $\tau>0$ and assuming $h_0(t)=\left[\frac{t}{\tau}\right]\tau$, we obtain the basic equation

$$(\Lambda Y)(t)=Y''(t)+\tfrac{1}{\tau}Y'(h_0(t))=\tfrac{1}{\tau}B_1^{-1}\left(Y(t)-C(t)\right) \quad (1)$$

Assuming that the Nemytsky operator from the right-hand side of equation (1) has the form $(Ny)(t)=B_1^{-1}\left(Y(t)-C(t)\right)=f(t,y(t))$, we introduce the functionals $l_1(Y)=Y(a_3),\quad l_2(Y)=Y(b_3)$. Next, we consider equation (1) in the operator notation $(\Lambda Y)(t)=(NY)(t)$ together with the boundary conditions

$$l_1(Y)=\alpha_3,\, l_2(Y)=\beta_3 \quad (2)$$

We use the notation: R^n - space of n - dimensional vector columns with the norm $\|\cdot\|$; L_2 - space of summable functions to the 2nd power on the interval $[a,b]$ $x:[a,b]\to R$ with the norm $\|x\|_{L_2}=\sqrt{\int_a^b x^2(t)dt}$; C - space of continuous $[a,b]$ functions with the norm $\|x\|_C=\max_{a\le x\le b}\|x(t)\|$; X, Y - Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ respectively;

$< u,v >_{L_2} = \int_a^b < u(t),v(t) > dt$ - the scalar product in L_2, defined by <y, x> - bilinear form given on $X \times X^*$, $x \in X$, $y \in X^*$; A^* - operator conjugate to A.

For a nonlinear operator M1, operating from X to X^*, we say that it satisfies the conditions (3A, θ) и (3B, η) [13], if there exist two nonnegative constants θ and η, that the inequalities are realized

$$[M_1(z_1) - M_1(z_2)](z_1 - z_2) \geq \theta(z_1 - z_2)^2, \qquad (3A,\theta)$$

$$|M_1(z_1) - M_1(z_2)| \leq \eta |z_1 - z_2|. \qquad (3B,\eta)$$

Hence, $\theta \leq \eta$.

Definition. Function f(t,u) satisfies the condition $L_1(p_1, 3A\theta, 3Б\eta)$ if there exists a summable function on the interval [a,b] and an operator $M_1 : L_2 \to L_2$, that $f(t,u(t)) = p_1(t)u(t) + (M_1u)(t)$, for M1 the conditions $(3A\theta, 3Б\eta)$ are realized.

Let $A : X \to X$ - linear invertible operator and $\alpha > 0$.

Definition. Operator $F : X \to X^*$ is called (A, α)-strongly monotone if $\forall u,v \in X$ the inequality is realized $< Fu - Fv, A(u-v) >_X \geq \alpha \|u-v\|_X^2$.

Lemma 1. Suppose that there exists such a linear invertible operator $A : X \to X$ and $\alpha > 0$, that a continuous operator $F : X \to X^*$ is a (A, α)-strongly monotone. Then the equation $Fx = w$ has the only solution for any $w \in X^*$.

Together with boundary value problem (1) - (2) we consider the auxiliary boundary value problem

$$\begin{cases} (\Lambda_0 y)(t) = z(t) & (4) \\ l_1 y = 0, \quad l_2 y = 0 & (5)-(6) \end{cases}$$

and denote it by Green's function $G(t,s)$. The solution of the boundary value problem (4) - (6) has the form $y_0(t) = \int_a^b G(t,s)z(s)ds$. We seek the solution of problem (1) - (2) in the form $Y(t) = \int_a^b G(t,s)z(s)ds + u(t)$, where $u(t)$ - the solution of another auxiliary problem

$$\begin{cases} (\Lambda_0 u)(t) = u''(t) = 0 & (7) \\ (l_1 u) = \alpha_3, \quad (l_2 u) = \beta_3 & (8)-(9) \end{cases}$$

Consider the family of operators $A_\alpha = \{I - \alpha G : L_2 \to L_2\}$, where $G : L_2^1 \to D_2^1$ - The Green's operator of the boundary value problem (4-6) and I – identical operator, $\alpha \in R^1$ – parameter and with $0 \leq \alpha < +\infty$ such operators are invertible.

Suggestion: there is such a deviation of the argument h(t) and such a nonnegative function $V : R^1 \times R^1 \to R^1$, that for all the problem solutions (1) - (2) there is inequality $\|Y'(h_0(t))\|_{L_2} \leq V(n,\tau)\|Y(t) - Y(h(t)\|_{L_2}$.

$V(n,\tau)$ will be denoted by V and called a constant. We denote $\alpha_1 = 2\sqrt{\int_{a_3}^{b_3} |h(t)-t|^2 dt} \geq 0$. If the deviation of the argument h(t) coincides with $h_0(t) = \left[\frac{t}{\tau}\right]\tau$, then we can obtain an effective bound for the constant V: $\alpha_1 \leq 2\sqrt{(b-a)\tau^2} = 2\tau\sqrt{b-a}$.

Therefore, the inequality: $< (S_h G - G)z, z >_{L_2} \geq -\alpha_1 \|z\|_{L_2}^2$ is correct.

We assume that the Nemytskii operator N has the condition $L_1(p_1, 3A\theta, 3Б\eta)$, that is, it is representable in the form

$$(NY)(t) = p_1(t)Y(t) + (M_1Y)(t) + b_4(t),$$ then equation (1) can be written in the form

$$Y''(t) + \tfrac{1}{\tau}(S_h Y')(t) = \tfrac{1}{\tau}\left(p_1(t)y(t) + (M_1 y)(t) + b_4(t)\right) \quad (10)$$

In addition, let the operator M_1 have the condition $L_1(p_1, 3A\theta, 3Б\eta)$ under assumptions $p_1(t) \equiv p_1$, $\frac{p_1 - 1}{\tau} > 0$.

We denote by $U_1 z = z + \frac{1}{\tau}S_h\left(\frac{d}{dt}Gz\right) + \frac{p_1-1}{\tau}Gz$, $F_1 z = U_1 z - \frac{1}{\tau}M_2 Gz$ and consider equation

$$F_1 z = \tfrac{b_5}{\tau} \qquad (11)$$

The following assertion enables us to replace the study of the boundary value problem (1) - (2) by studying equation (11).

Lemma 2. [2] $z \in L_2^1$ is a solution of equation (11) if and only if, когда $Y(t) = \int_a^b G(t,s)ds + u(t)$ is a solution of the problem (1)-(2).

We use the following notation for constants $a_\tau = \left(1 - \frac{\alpha_1 V}{\tau}\right)$, $a_2 = \left(\frac{\alpha_1 V}{\tau}\right)^2$, $P_1 = \frac{p_1 - 1}{\tau}$, $P_2 = \frac{\pi^2 \alpha_1}{(b-a)^2} + P_1$, $b_2 = \left(\frac{2\alpha_1 V\eta}{\tau^2} - 4a_\tau P_2\right)$, $b_V(\alpha) = -\frac{\alpha\alpha_1 V + \eta}{\tau}$, $c_1 = \left(\frac{\eta}{\tau}\right)^2 - 4a_\tau\left(P_2 \frac{\pi^2\alpha_1}{(b-a)^2} + \theta\right)$, $c(\alpha) = P_2\alpha + P_1\frac{\pi^2\alpha_1}{(b-a)^2} + \theta$, $D_1 = b_2^{\ 2} - 4a_2 c_1$, $r_2 = \dfrac{a_\tau}{c_\alpha} - \dfrac{b_V^{\ 2}}{4c_\alpha}$.

Theorem 3. Suppose that the assumptions are fulfilled:

a) function $f(t,u)$ satisfies the condition $L_1(p_1, 3A\theta, 3Б\eta)$ and $p_1(t) \equiv p_1$, $p_1 > 0\ (\alpha > 0)$;

b) $D_1 \geq 0$ and at least one of the inequalities is fulfilled: $c_1 < 0$ or $b_2 < 0$.

Then the boundary value problem (1) - (2) has the only solution $Y(t)$, which satisfies the estimate

$$\|Y\|_C \leq Q(\alpha), \tag{12}$$

where $Q(\alpha) = \frac{(b-a)^2}{4}\frac{\|A_\alpha^* b_1\|_{L_2(a,b)}}{r_2(\alpha)} + \|u\|_{C[a,b]}$.

Proof. We shall prove existence of a solution of the auxiliary boundary value problem

$$\begin{cases}(\Lambda Y)(t) = \tilde{f}(t, Y(t)) \\ Y(a_3) = \alpha_3, \quad Y(b_3) = \beta_3, \quad t \in [a_3, b_3],\end{cases} \tag{13}$$

satisfying the estimate (12), where the function $\tilde{f}(t, Y(t))$ coincides with $f(t, Y(t))$ in the set $(Y,t) \in [-Q(\alpha_3), Q(\alpha_3)] \times [a_3, b_3]$. Consider the operator $\tilde{F}: L_2 \to L_2$, defined by

$$\tilde{F}z = U_1 z - \tilde{f}(t, (Gz)(t)) \Rightarrow$$
$$\tilde{F}z = z + \tfrac{1}{\tau}(S_h G)z - \tilde{f}(t, (Gz)(t)).$$

The conditions of the theorem imply A_α - strong monotonicity of the operator $\tilde{F}: L_2 \to L_2$, the operator $A_\alpha^*: L_2 \to L_2$ is invertible and $r_2(\alpha) > 0$. Thus, all the conditions of Lemma 1 are satisfied; therefore, equation $F_1 z = \frac{b_1}{\tau}$ has a unique solution $z \in L_2$, and the boundary value problem (2) has a unique solution $Y(t)$, satisfying the estimate (12). Then this $Y(t)$ is the unique solution of problem (1) - (2). The proof is complete.

Further: $\rho = \frac{1}{B}$ - Technological growth index (growth rate) of the GNP in the case of a linear accelerator; B - capacity, accelerator ratio, capital intensity of GNP [12]; $[a,b] = [0, nT]$, $B(u,t) \equiv Bu = \rho^{-1}u$. For this linear case $(Ny)(t) = \rho Y(t) - \rho c(t)$ and $M(y(t)) = My(t)$.

Then equation (1) can be written in the form

$$Y''(t) + \tfrac{1}{\tau}Y'(h(t)) = \tfrac{p_1}{\tau}Y(t) + \tfrac{M(y(t))}{\tau} + \tfrac{b_1(t)}{\tau}.$$

For such a linear case, we can have $\eta = \theta = 0$, $\rho = p_1$ and

$$\alpha_1 = 2\sqrt{\int_a^b |h(t) - t|^2\,dt} = 2\sqrt{\int_0^{n\tau} |h(t) - t|^2\,dt} \leq 2\sqrt{\int_0^{n\tau} \tau^2 dt} = 2\sqrt{\tau^2 n\tau} = 2\tau\sqrt{n\tau}.$$

Hence, as a corollary of Theorem 3, we obtain the assertion about the solvability of the linear boundary-value problem

$$Y''(t) + \tfrac{1}{\tau}Y'(h(t)) = \tfrac{p_1}{\tau}Y(t) + \tfrac{b_1(t)}{\tau},$$
$$Y(a_3) = \alpha_3, \quad Y(b_3) = \beta_3, \quad t \in [a_3, b_3]. \tag{13}$$

Theorem 4. Let at least one of the three conditions hold:

$D_1 = b_2^{\ 2} - 4a_2 c_1$, $D_1 \geq 0$ and $\rho > 1 - \dfrac{2\pi^2}{n\sqrt{n}}\sqrt{\tau}\tau$;

B1) $\sqrt{n\tau} > \frac{1}{2}$, $\rho > \frac{2}{n\sqrt{n\tau}} + 1$; or B2) $\sqrt{n\tau} < \frac{1}{2}$, $\rho < \frac{2}{n\sqrt{n\tau}} + 1$

Then the boundary value problem (13) is solvable.

In the case of a linear accelerator, let us consider the simplest linear model of the GNP dynamics, taking into account the lag in introduction of induced investments:

$$\tau Y''(t) + Y'\left(\left[\tfrac{t}{\tau}\right]\tau\right) - \rho Y(t) = -\rho C(t),\ t \geq 0.$$

We introduce the notation: $\frac{1}{\tau} = p$, $-\frac{\rho}{\tau} = q$, $-\frac{\rho}{\tau}C(t) = f(t)$. Then the equation takes the form:

$$Y''(t) + pY'\left(\left[\tfrac{t}{\tau}\right]\tau\right) + qY(t) = f(t) \quad t \in [0, n\tau] \tag{14}$$

Consider the boundary-value problem for such a linear differential equation in the form

$$\int_0^{n\tau} Y(s)ds = \beta, \quad \omega' Y(0) = Y'(n\tau). \tag{15)-(16}$$

It is shown that such a problem «about ω - multiple change in the GNP growth in the base period» can be reduced to the problem considered earlier (13) as follows: we prove that the GNP increment: $\beta_3 - \alpha_3$ (in the notation of problem (1) - (2)) is uniquely determined for the base period. In this case we assume that $\alpha_3 = Y(0)$ (GNP at time zero) is known (from statistics) and therefore we obtain conditions for the unique solvability of such a problem from Theorem 4.

We find from (14), (15) and (16) the difference of two numbers $Y(0) = \alpha_3$ and $Y(n\tau) = \beta_3$. We denote: $d_5 = d_5(p_3, q_3, \omega) = \left(-p_3 - \frac{q_3}{p_3}\right)(\omega - 1)$;

$$d_4 = d_4(p_3, q_3, \omega, \beta, f_3(t)) = \left[f_3(n\tau) - f_3(0)\right] - \tfrac{q_3}{p_3}\left[q_3\beta - \int_0^{n\tau} f_3(s)ds\right];$$

$$\Delta_{30} = \begin{vmatrix} 1 & -\frac{\omega-1}{p_3} \\ q_3 & d_5 + p_3(\omega-1) \end{vmatrix};$$

$$\Delta_{31} = \begin{vmatrix} \frac{1}{p_3}\left[q_3\beta - \int_0^{n\tau} f_3(s)ds \right] & -\frac{\omega-1}{p_3} \\ f_3(n\tau) - f_3(0) - d_4 & d_5 + p_3(\omega-1) \end{vmatrix};$$

$$\Delta_{33} = \begin{vmatrix} 1 & \frac{1}{p_3}\left[q_3\beta - \int_0^{n\tau} f_3(s)ds \right] \\ q_3 & f_3(n\tau) - f_3(0) - d_4 \end{vmatrix}.$$

Lemma 5. Let $\Delta_{30} \neq 0$. Then, if the equalities (14), (15), (16) hold, the following assertions hold:

a) The GNP increase in the period from $t = 0$ to $t = n\tau$ is calculated by the formula $Y(n\tau) - Y(0) = \frac{\Delta_{31}}{\Delta_{30}}$;

b) The growth rate of GNP at the initial moment of time is calculated by the formula $Y'(0) = \frac{\Delta_{32}}{\Delta_{30}}$;

c) increase in the GNP acceleration in the base period from $t = 0$ to $t = n\tau$ is calculated by the formula $Y''(n\tau) - Y''(0) = d_5 \frac{\Delta_{32}}{\Delta_{30}} + d_4$.

Conclusion

Let us further consider the problem of identifying the parameters of the linear problem «about ω - multiple change in the GNP growth in the base period». A numerical experiment was conducted, with the base period of 2007-2016. The data of GNP statistics $Y^*(t)$ and non-productive consumption $C(t)$ are approximated in the form of curves of exponential type. With input influences given by the years of the base period, the deviation measure of the calculated values from the statistical values was computed in two ways in two different approximate calculation methods. In the first metod, described in detail in [17] and essentially based on the W-method by N. V. Azbelev [4, 5, 6], the deviation modulus normalized with respect to statistical data is applied as a measure. In the second method, based on the classical Ritz method, the total normalized quadratic deviation from the statistically derived GNP curves and non-production consumption is taken as a measure of deviation. Several algorithms for identifying parameters that have been tested in 10 different world economies are presented. The result of a numerical experiment confirms adequacy of a linear model to real processes, and for a nonlinear model, it indirectly confirms correctness of its structure.

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