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Analysis of the Dynamics of Market Graph Characteristics

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***Abstract*—In recent years the network models have been successfully employed for the analysis of the stock market. The network model of the stock market is defined as a full weighted graph, which vertices correspond to the returns of market assets, and the weights of the edges are determined by the measure of its interdependencies. To obtain important information from the network model, many researches extract subgraphs, which are called network structures. One of the most popular network structures is the so called market graph. In this paper the market graph is constructed as follows: each company is a node and the value of sign correlation between assets of the two stocks establishes a link between them. Network analysis is carried out for the companies whose shares are traded on the NYSE and NASDAQ for the period from November 22, 2013 to November 10, 2017. It was shown that distribution of degrees and clustering coefficient for our network follows the power law.**

***Index Terms*—network analysis, market graph, degree distribution, maximum clique**

I. Introduction

One of the most key tasks in economics and finance is the exploration for effective practices for generalization and visualization of the stock market data. In many cases it can bring researchers and practitioners with helpful information about the behavior of the finance market. Presently, a huge amount of financial assets are traded on the stock markets and their amount is firmly increasing. The enormous amount of data is being created by the financial market every day. This data can be presented and visualized by tables or charts matching the price of every asset in a given point of time. The exploration and examination of such data is becoming progressively difficult with the increase of the number of financial assets on the market.

One of the important aspects of modern financial markets is that they operate as complex systems with a huge amount of interdependent components and connections between them. The last two decade have drawn attention to the analysis of the financial market as a network. It should be noted that the concept of a market graph was introduced and examined in the paper [1]. The market network was described as a full weighted graph in which nodes represent the assets and the weights of the links between any two assets represent the similarity in the behavior of the two assets. In the paper [1] the correlation coefficient was used to quantify the similarity measure between two assets and the link between two assets (graph nodes) is tucked in into the market graph if the value of the corresponding Pearson correlation is above a fixed threshold.

The market graph approach has been proved to be successful in applications and examination of market data. Recently, many researchers have presented empirical results using real market data in which there have been examined the important structural characteristics and features of the market network. For example, papers [2]–[5] examines maximum clicks, maximum independent sets, the distribution of powers. Clustering of the Pearson correlation is studied in [6], the examination of the dynamics of the market graphs of the US market can be found in the work [7]. The paper [8] is devoted to investigation of the complexity of the market graph. The papers [3], [9]–[12] examine the distinctive features of individual financial markets. The market graphs constructed with use of similarity measures that differ from the Pearson correlation are explored in [9], [13]–[17].

The market graph analysis is similar to the social network analysis (SNA). SNA was developed to analyze the structure of relations in an organization and social networks [18], [19]). The work [20] acknowledges SNA as a method for analysis of connections among social entities. The basic concepts of SNA are node and link. A node represents the unit (individual, object, item) and a link reflects the relationship between nodes.

Historical data of financial markets can be conveniently converted into a network representation. A market network is defined as a set of assets which have links in pair to reflect their connections. Two assets are viewed to be connected if there is positive significant correlation between them. Thus, a company is regarded as “node” in the market graph or “vertex” and the connection between nodes is presented by an “edge” or “link”. Market networks is modeled by undirected unweighted graphs. Various type of SNA metrics may be employed to analyze market graphs, including edge density, degree distribution, maximum clique and maximum independent set.

The main goal of this paper is to describe the evolution in time of the structural properties of the market graph constructed with use of the sign correlation. The research constructs market graphs based on stock prices data for different periods of time during 2013–2017 to study the dynamics of some features of these market networks. In our research we would like to find the type of the degree distribution, the type of the clustering-degree distribution exhibited by the market network. Moreover we would like to estimate the size of the maximum clique in the market graph.

In our paper we would like to find answers to the following questions.

1) What type of the degree distribution exhibits the market network constructed with use of the sign correlation? What type of the functional form has the clustering-degree relation for the network? How does it evolve over time?
   Recent results in the area of degree distribution analysis of complex networks arisen in different areas, such as sociology, physics, and biology, have shown that many networks exhibit comparable degree distributions [21]–[26]. It turned out that most of real networks have degree distributions that are scale-free [21]. In other words, their degree distributions are power-law.
2) What are the size of the maximum clique in the market graph constructed with use of the sign correlation? How does the size of the maximum clique evolve over time?

II. Data

The database for constructing and analyzing the market graph was taken from Thomson Reuters database. The Thomson Reuters database was employed to acquire historical prices of the stocks traded in the NYSE and NASDAQ for the period from November 22, 2013 to November 10, 2017 (i. e. 1000 trading days). We use the daily closing prices which have been adjusted for dividends and splits. We include in the analysis only stocks which had been traded without gaps and omissions during the period. After elimination, there are remained 3736 different stocks, and only 15 stocks from S&P500 were ignored).

In this study we examine the dynamics of the market graph. We divide the 1000-day trading days interval into 10 consecutive periods, each of which is 500-day long. Every period except the first one is picked up by shifting the previous one by 50 days. Thus, two adjoining periods have 450 common days. The dates of each period can be seen in Table I.

Table I
TIME INTERVALS

Period	Start	End	The number of stocks
1	22.11.2013	13.11.2015	3736
2	04.02.2014	26.01.2016	3736
3	17.04.2014	07.04.2016	3736
4	30.06.2014	20.06.2016	3736
5	10.09.2014	31.08.2016	3736
6	21.11.2014	11.11.2016	3736
7	03.02.2015	24.01.2017	3736
8	16.04.2015	06.04.2017	3736
9	29.06.2015	19.06.2017	3736
10	09.09.2015	30.08.2017	3736
11	20.11.2015	10.11.2017	3736

A. Market Network Construction

We construct the market network using the sign correlation; it means that two companies is connected if the value of the sign correlation between the two assets are above a given threshold (in this period of time).

More precisely, the market graph is constructed as follows. We denote by $P_i(t)$ the price of the asset i in day t. Then

$$R_i(t) = \ln \frac{P_i(t)}{P_i(t-1)} \tag{1}$$

is the logarithm of the ratio of the price of the asset i in day t to the price in the previous day $t-1$.

Let n be the number of assets. We will suppose that random variable $R_i(t)$, $t = 1, 2, \ldots, N$, has a corresponding distribution R_i, $i = 1, 2, \ldots, N$, and the joint distribution of random $R_1, R_2, \ldots, R_N$ is not known.

The Pearson correlation coefficient between random variables R_i and R_j is defined by

$$r_{ij} = \frac{\sum(R_i(t) - \overline{R_i})(R_j(t) - \overline{R_j})}{\sqrt{\sum(R_i(t) - \overline{R_i})^2}\sqrt{(R_j(t) - \overline{R_j})^2}},$$

where $\overline{R_i} = \frac{1}{T}\sum_{i=1}^{T} R_i(t)$ denotes the mean value of R_i.

The Pearson correlation is the most popular measure exercised in the examination of the finance market. The main shortcoming of the Pearson correlation is weak robustness to deviations from the assumptions on identity distribution of the random variables in question.

In this paper we will use the pairwise similarity measure for stocks i and j using the sign correlation which is based on the probability of their returns to have the same sign with respect to $\mu_i = E(R_i)$, i.e. the expected value of R_i, $i = 1, 2, \ldots, N$. The sign correlation was introduced by Fechner and was examined in the book [27]. The sign correlation is defined by

$$s_{ij} = \frac{1}{n}\sum_{i=1}^{n} \text{sgn}(R_i(t) - \mu_i)\text{sgn}(R_j(t) - \mu_j),$$

where $\text{sgn}(a)$ denotes the sign of a.

The edge between the vertices i and j is added to the graph if $s_{ij} \geq \theta$, which means that the prices for these two assets behave identically over time, and the degree of this similarity is determined by the corresponding value of the Pearson correlation coefficient.

The sign correlation can be easily interpreted when we analyze the dependence between two stock. If $T = 1000$ and $s_{ij} = 0.25$ then we have 625 days of 1000 with the log returns of stocks changing in one direction, and we have 375 days of 1000 when the prices have changed in different directions. In this example the stocks are highly correlated. If s_{ij} is close to 0 then the log returns of stocks i and j are more likely independent. In the case $s_{ij} < 0$ we will see more days on which the log returns of stocks i and j are changing in contrasting directions, and, consequently, such stocks are anti-correlated.

The market graph constructed with use of the measure linearly dependant of the sign correlation was studied in [9]. The paper showed that the measure is capable for the analysis of the market graphs. As it pointed out in [9], the sign correlation has important differences from the Pearson correlation, which make it more applicable to our analysis than the classical correlation:

- In the case when we do not know the joint distribution of random variables $R_1, R_2, \ldots, R_n$, the classic correlation coefficient cannot be interpreted in such a simple way, except the case where it is equal to 1.
- It should be noted that the sign correlation as a measure of similarity has transitivity property and it is crucial aspect in the case of the market graph analysis.
- Another key property of sign correlation is that it is nonparametric. Therefore, sign correlation is robust to deviations from the normal distribution. To calculate it one needs the frequency analysis only and does not need to compute the detailed joint distribution.

III. Network Analysis

A. Edge Density

The edge density of a simple undirected graph G is defined as the ratio of the number of edges of a graph to the maximum possible number of edges in it [28]:

$$D = \frac{2|E|}{|V|(|V|-1)}, \qquad (2)$$

where V is the number of vertices of the graph and E is the number of edges of a graph.

The edge density is one of the key characteristics of the market graph. The expansion of the edge density shows a certain "globalization" of the finance market, i. e. that more and more stocks essentially influence each other and the change in prices of one stock causes a change in the prices of connected finance assets.

B. Degree Distribution

The graph $G = (V, E)$ is connected if there is a path from any vertex to any vertex in the set V. If the graph is disconnected, it can be decomposed into several connected subgraphs, which are referred to as the connected components of G.

The degree of a vertex is the number of edges emanating from it. For every integer number k one can calculate the number of vertices $n(k)$ with the degree equal to k, and then get the probability that a vertex has the degree k as $P(k) = n(k)/n$, where n is the total number of vertices. The function $P(k)$ is referred to as the degree distribution of the graph. The degree distribution is an important characteristic of a graph representing a dataset.

It should be noted that real graphs that arise in different fields (economics, Internet, telecommunications, finance, medicine, biology, sociology) exhibit the degree distribution that follows the power-law model [21]–[26]. According to this model, the probability that a vertex has degree k (that is, there exist k edges originating from it) asymptotically follows

$$P(k) \propto k^{-\gamma} \text{ or } \log P(k) \propto -\gamma \log k,$$

which shows that this function has a linear dependence in the logarithmic scale.

An important characteristic of this model is its scale-free property. It implies that the fractal structure of a network remains constant despite its development and growth over time [29].

C. Clustering Analysis

The local clustering coefficient for node i is defined by

$$C_i = \frac{E_i}{k_i(k_i-1)},$$

where E_i is the number of links connecting the immediate neighbors of node i, and k_i is the degree of node i. The average value of clustering coefficients of all nodes in a network is called the average clustering coefficient. The value of the average clustering coefficient quantifies the strength of connectivity within the network. The paper [30] examines protein-protein interaction networks and metabolic networks which have to demonstrate large average clustering coefficients. The analogues result has been established for collaboration networks in academia and the entertainment industry in papers [31], [32]. Let $C(k)$ denote the average clustering coefficient of nodes with degree k. It has been found that for most of real networks $C(k)$ follows

$$C(k) \sim \frac{B}{k^\beta},$$

where the exponent β usually lies between 1 and 2 [33]–[35].

D. Maximum Cliques

Given a subset $S \subseteq V$, by $G(S)$ we denote the subgraph induced by S. A subset $C \subseteq V$ is a clique if $G(C)$ is a complete graph, i. e. it has all possible edges. The maximum clique problem is to find the largest clique in a graph.

The clique is a set of vertices which are fully interconnected. That is why any financial asset which belongs to the click is strongly correlated with all other financial assets in this click. Because of this fact the asset is bound to a specific click only in case when its behavior is similar to all other assets in this group. It is clear that one of the main characteristics of stock market is the maximum size of clique, because it shows the largest possible group of similar objects (financial assets which are cross correlated to each other).

The maximum clique problem (as well as the maximum independent set problem) is known to be NP-hard [36]. Moreover, it turns out that these problems are difficult to approximate [37], [38]. This makes these problems especially challenging in large graphs. However, as we will see later, a special structure of the co-mention graph allows us to get the exact solution of the maximum clique problem.

The variant of Bron–Kerbosch algorithm is used in order to calculate an accurate maximum click. Bron–Kerbosch algorithm is the algorithm which allows to find maximal cliques in the undirected graph [39]. Dutch scientists Bron Conradomi and Jupe Kerbosh developed this algorithm and published it in 1973. There some other algorithm which can solve the problem of maximum clique and works better in some graphs with a little quantity of vertexes. Actually Bron–Kerbosch algorithm and its improvements work effectively.

The main form of Bron–Kerbosch algorithm is recursive search algorithm with return. It founds all maximum cliques in the graph G. The algorithm is linear relative to the number of cliques in the graph. The working time of this algorithm with some extra tests $O(n^{n/3})$. But the algorithm is more effective for random graphs.

IV. Evolution of Market Network

A. Market graphs constructed with use of Pearson correlation

The density of distribution of correlation coefficients for the US stock market is almost symmetrical and has a form which is similar to the normal with the mean around 0.2 (Fig. 1, 2). The comparison of densities for different periods of time shows that distributions are similar to each other. The proportion of present edges to all possible edges in the network are shown in Table II. The density of edges increases over the time, its peak is reached during 5 and 6 periods and after that it goes down. (Table II). A positive mean implies that financial assets of the USA market are related to each other on average. The correlation in case of negative mean is rather rare. Because of that, it is more difficult to form a diversified portfolio of shares whose yields move in different directions. The hypothesis on power-low degree distribution of vertexes' degree is confirmed. It means that degree distribution of vertexes is approximated by a power-law model. At the same time the power coefficient γ is less than 1 for all periods of time. For the given networks, the clustering-degree distribution relation also follows the power law (Fig. 3). The resulting models is statistically significant at any significance level. Herewith, the exponent β turns out less 1 for all the subgraphs under consideration (Table II). The plot of the clustering-degree relation, i. e. $C(k)$ as a function of node degree k, is shown in Fig. 4.

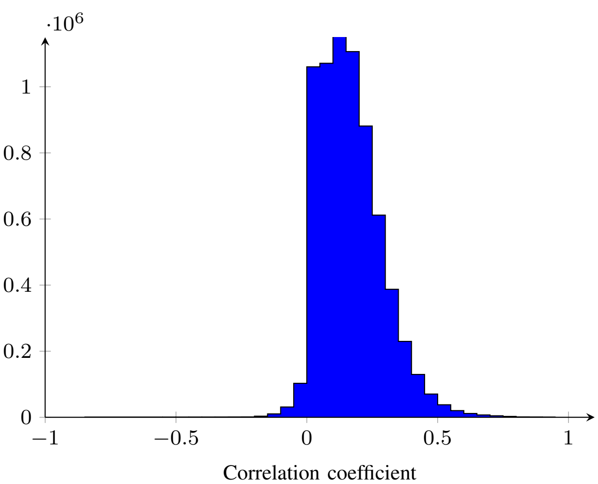

Figure 1. Distribution of Pearson correlation coefficients (1st period)

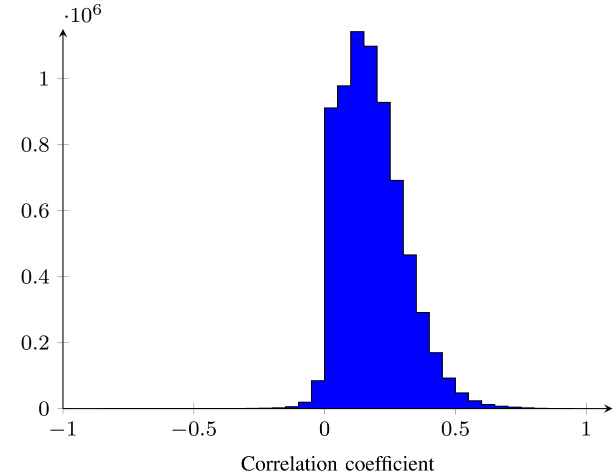

Figure 2. Distribution of Pearson correlation coefficients (11th period)

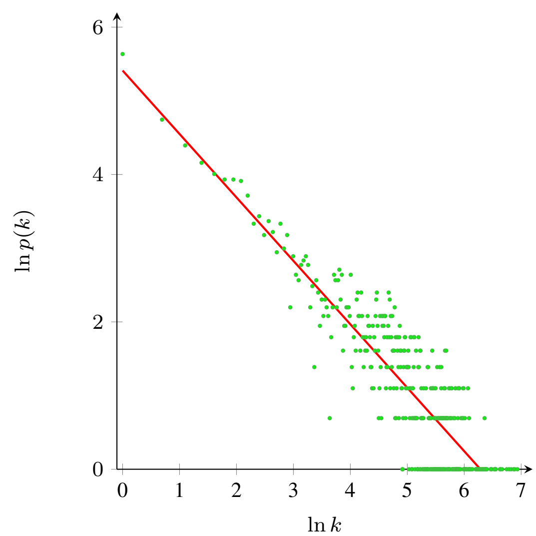

Figure 3. The degree distribution of the market network constructed with use of Pearson correlation

The sizes of the maximum cliques of the market of the USA big enough (Table II), and the peak is account form September 2014 till August 2016. A clique is a set of fully interconnected vertices. That is why any asset owned by the clique is strongly associated with all other assets in this clique. In this way, an increase of the maximum clique may mean an intensity increase of markets' globalization in this period of time.

B. Market graphs constructed with use of the sign correlation

The density of distribution of sign correlation coefficients are presented in Fig. 5, 6. The hypothesis on power-low degree distribution of nodes' degree for the market graphs constructed with use of the sign correlation is fulfilled, i.e. degree distribution of nodes is approximated by a power-law model. At the same time the power coefficient γ is more than 1 for all periods of time. As it is for market graphs constructed with use of Pearson correlation, the clustering-degree distribution

Table II
CHARACTERISTICS OF GRAPHS CONSTRUCTED WITH USE OF PEARSON CORRELATION

Period	1	2	3	4	5	6	7	8	9	10	11
Density	0.014	0.018	0.019	0.02	0.023	0.023	0.02	0.02	0.019	0.014	0.012
Coefficient γ	0.86	0.84	0.82	0.82	0.79	0.81	0.84	0.84	0.85	0.92	0.82
Coefficient β	0.25	0.25	0.24	0.24	0.22	0.23	0.23	0.23	0.23	0.23	0.22
Clique size	116	127	139	147	166	159	149	149	149	141	137

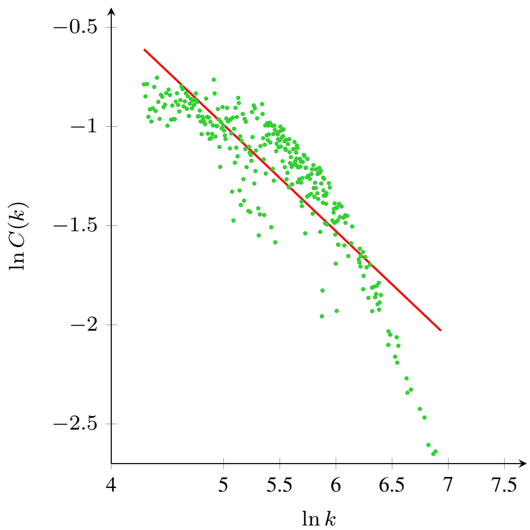

Figure 4. The clustering-degree relation of the market network constructed with use of Pearson correlation

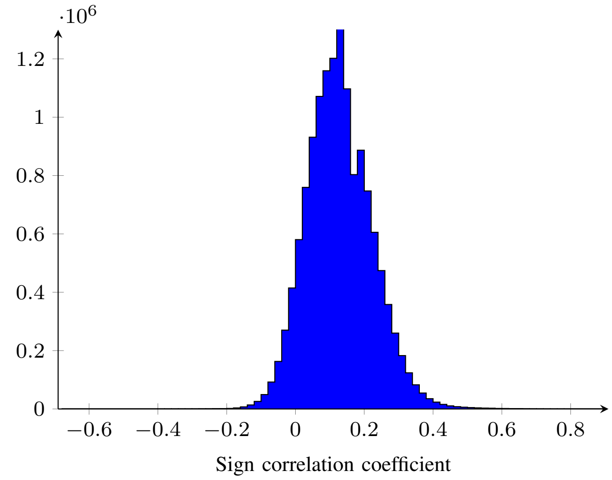

Figure 6. Distribution of sign correlation coefficients (11th period)

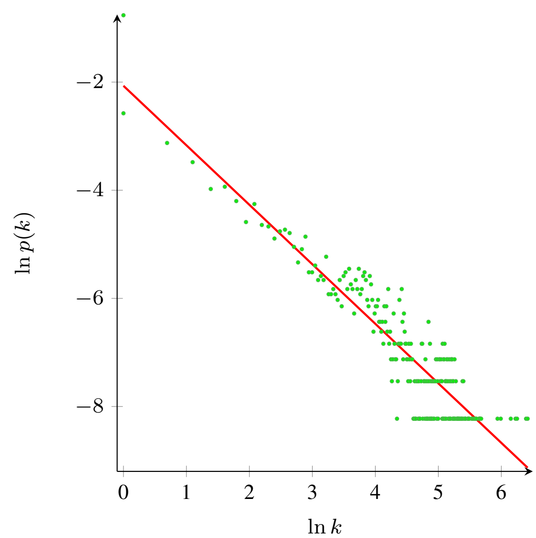

Figure 7. The degree distribution of the market network constructed with use of the sign correlation

relation for the market graphs constructed with use of the sign correlation follows the power law (Fig. 7). The resulting models is statistically significant at any significance level. The exponent β of the clustering-degree relation turns out less 1 for all the subgraphs (Table III). The plot of the clustering-degree relation is presented in Fig. 8.

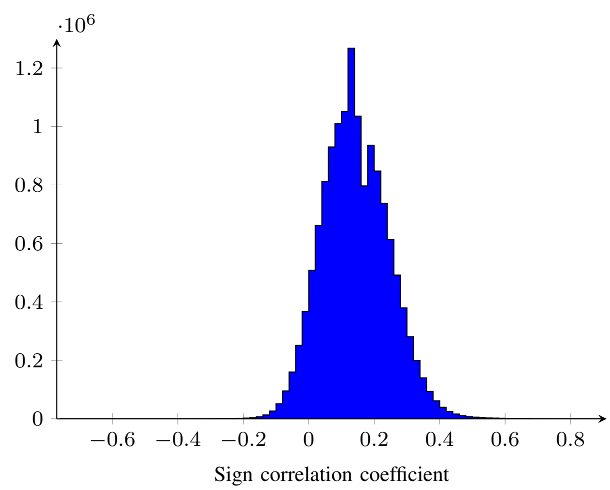

Figure 5. Distribution of sign correlation coefficients (1st period)

V. CONCLUSION

The network model of the stock market is defined as a full weighted graph, which vertices correspond to the returns of market assets, and the weights of the edges are determined by the measure of its interdependencies. To obtain important information from the network model, many researches extract subgraphs, which are called network structures. One of the

Table III
CHARACTERISTICS OF GRAPHS CONSTRUCTED WITH USE OF THE SIGN CORRELATION

Period	1	2	3	4	5	6	7	8	9	10	11
Density	0.005	0.007	0.007	0.009	0.009	0.009	0.007	0.007	0.007	0.005	0.004
Coefficient γ	1.10	1.04	1.06	1.01	1.00	1.04	1.08	1.07	1.09	1.12	1.13
Coefficient β	0.36	0.34	0.34	0.32	0.33	0.32	0.31	0.28	0.28	0.21	0.16

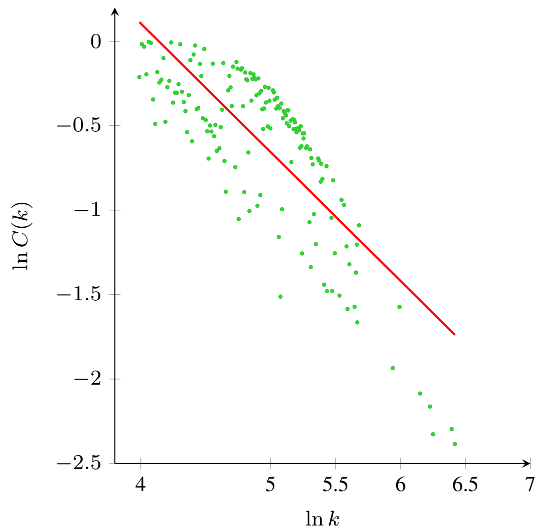

Figure 8. The clustering-degree relation of the market network constructed with use of the sign correlation

most popular network structures is the so called market graph. In this paper the market graph is constructed as follows: each company is a node and the value of sign correlation between assets of the two stocks establishes a link between them. Network analysis is carried out for the companies whose shares are traded on the NYSE and NASDAQ for the period from November 22, 2013 to November 10, 2017.

In this paper we transform financial data into the market graph. The examination of graph properties gives new understanding of the financial internal structure. We investigated the dynamics and changes of the market graph structural properties over time. It was shown that the power-law structure of the market graph is fairly stable over time. Unlike real social graphs, the market graph displays power-low distribution of degrees with non-typical indicators of degree exponent. The same results were obtained for the market graphs constructed with use of Pearson correlation. It would be interesting to examine the dependence between market graph chrasteristics and stock market equilibria [40], [41].

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