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Numerical and Theoretical Solutions for the Free Vibration Analysis of Moderately Thick Plates

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***Abstract*—The vibrations of plates are a topic of undoubted interest in the field of civil engineering and aeronautics. Today, we find many examples where this type of phenomena occur, and there are structural elements that we can study as plates or shells. The analytical solutions for the governing equations for the dynamics of higher order plates are very difficult or impossible to obtain so we have to resort to numerical methods. The Finite Element Method (FEM) has been and is a very powerful tool to solve differential or integral equations. The method of Galerkin is another interesting possibility that can be applied without difficulty and that has not been given as much interest as the FEM. As with the FEM, there are different techniques to relieve numerical instabilities (shear locking of the stiffness matrix) for the solutions of thin plates. In this communication, we studied this phenomenon by making use of a higher order shear deformation plate theory deduced by the author. We made use of D'Alembert's principle and used a modified method of Galerkin. The results obtained are in very good agreement with those present in the literature.**

Keywords*— *plates; vibrations; structures; galerkin method

I. INTRODUCTION

The first studies on free vibration in structural elements date back to at least 1800s. Kirchhoff introduced the famous biharmonic equation that relates the vertical displacements of the plate with the transverse loads applied. However, this theory is only valid for thin plates. It was Reissner [1] who introduced the shear deformation in plates and who even proposed including the dynamic formulation of the problem. From there, numerous higher order plate theories have emerged with infinity of variants that analysed both the dynamic and the static problem [2]. The analytical solutions for these equations are very difficult or impossible to obtain so we have to resort to numerical methods. The Finite Element Method (FEM) has been and is a very powerful tool to solve differential or integral equations [3].

The method of Galerkin is another interesting possibility that can be applied without difficulty and that has not been given as much interest as the MEF. One of the advantages of this method with regard to the FEM is that it does not need to operate with any functional and that the discretization is performed over the whole domain of the problem. As with the FEM, there are different techniques to relieve shear locking [4].

In this communication, we studied this phenomenon by making use of a higher order shear deformation plate theory [2].

II. DEFINITION OF THE STUDY PROBLEM OF TRANSVERSAL OSCILLATIONS OF MODERATELY THICK PLATES

The equations of calculation of the classical theory of plates including shear deformation in dynamic regime without applied external forces are, [5],

$$-\frac{\partial M_x}{\partial M} - -\frac{\partial M_{xy}}{\partial y} + Q_x - \gamma I \frac{\partial^2 \theta_x}{\partial t^2} = 0$$
$$-\frac{\partial M_{xy}}{\partial x} - -\frac{\partial M_y}{\partial y} + Q_y - \gamma I \frac{\partial^2 \theta_y}{\partial t^2} = 0$$
$$\frac{\partial Q_x}{\partial x} - -\frac{\partial Q_y}{\partial y} - \gamma h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1)$$

Where M_x, M_{xy} and M_y are the generalized moments, Q_x and Q_y are the generalized shear forces, γ is the density, I is the moment of inertia, h is the thickness of the plate, θ_x the angle rotated by the rectilinear segment normal to middle surface around the Ox-axis, θ_y the angle rotated around the Oy-axis and w is the vertical displacement. We have to notice that in the classic theories of plates with shear deformation and in the higuer order shear deformation plate theories, the rotations are decoupled from the shifts, resulting in a system of 3 partial differential equations that, in general, do not have analytical solutions. The first two equations take into account the rotational inertia or, what is equivalent, the influence of shear deformation on the phenomenon of vibration.

We also note that once the equilibrium equations of a certain plate theory are known, we can approach their dynamic study in order to apply the D'Alembert principle. Therefore, for the dynamic study of moderately thick plates we can obtain a system of differential equations equivalent to the previous equations making use of the equilibrium equations written in terms of the displacements and rotations either from Bolle-Reissner, either from Mindlin, from Vlasov etc. all of them described in Panc's book [6].

We can express these equations in terms of the displacements, rotations and the geometric characteristics of the plate as well as its material properties,

$$\Delta\theta_x - \frac{5(1-\mu)}{h^2}(\theta_x - \frac{\partial \hat{w}}{\partial y}) - \gamma I \frac{\partial^2 \theta_x}{\partial t^2} = 0 \qquad (2)$$

These equations are known as Bolle Reissner equations and can be expressed in a more compact form as a function of the moment sum, the displacements and the applied loads.

We also take into account another important condition; that is, the rotation w_{xy} of a differential element around the z-axis is null for all points of the plate, [2],

$$w_{xy} = \frac{1}{2}(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}) = 0 \qquad (3)$$

which is equivalent to,

$$\frac{\partial \theta y}{\partial y} + \frac{\partial \theta x}{\partial x} = 0 \qquad (4)$$

Doing so, the transformed Bolle Reissner equations are,

$$\frac{\partial \vartheta y}{\partial x} - \frac{\partial \vartheta x}{\partial y} + \Delta w - \gamma h \frac{\partial^2 w}{\partial t^2} = -\frac{12(1+\mu)}{5Eh}P$$

$$\Delta \vartheta y - \gamma I \frac{\partial^2 \theta_y}{\partial t^2} = \frac{5(1-\mu)}{h^2}(\vartheta y + \frac{\partial \hat{w}}{\partial x})$$

$$\Delta \vartheta x - \gamma I \frac{\partial^2 \theta_x}{\partial t^2} = \frac{5(1-\mu)}{h^2}(\vartheta x - \frac{\partial \hat{w}}{\partial y}) \qquad (5)$$

Or also expressed in a more compact form,

$$\Delta M - \gamma h \ddot{w} = -P$$

$$\Delta \ddot{w} = -\frac{M}{D} - \frac{6P}{5GH} \qquad (6)$$

where M, Marcus Moment, [5], is

$$\frac{M_x + M_y}{(1+\mu)} = M \qquad (7)$$

III. Analytical Solution for Simply Supported Rectangular Plates

Our first purpose is to obtain analytical solutions for the vibrations of rectangular plates with influence of the shear deformation. One of the great advantages of our system of equations that we present in equations (8) and (9) is that the only variable to study is the vertical displacement w, which is why this variable has to be developed in Fourier series.

Since our study is for simply supported plates, (see Figure I) the conditions of support in the contours (w = M = 0) are met if the solution is expressed through

$$w = (c_1 \cos(ft) + c_2 sen(ft)) \cdot \sum A_{mn} sen\frac{m\pi x}{a} sen\frac{n\pi y}{b}$$

$$M = (c_1 \cos(ft) + c_2 sen(ft)) \cdot \sum B_{mn} sen\frac{m\pi x}{a} sen\frac{n\pi y}{b} \qquad (8)$$

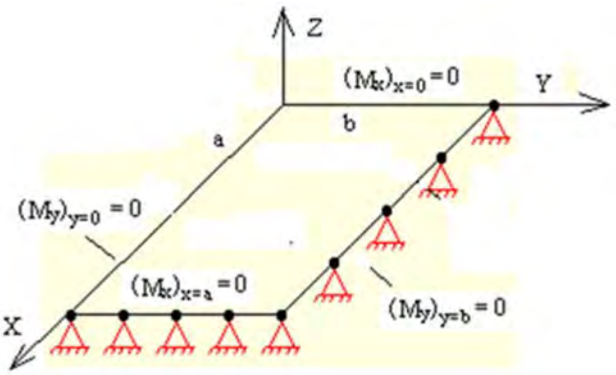

FIGURE I. SIMPLY SUPPORTED RECTANGULAR PLATE BOUNDARY CONDITIONS

Where w is the natural frequency and τ is the variable time. From them we deduce,

$$\ddot{w} = -f^2(c_1 \cos(ft) + c_2 sen(ft)) \cdot \sum A_{mn} sen\frac{m\pi x}{a} sen\frac{n\pi y}{b}$$

$$\Delta w = -(c_1 \cos(ft) + c_2 sen(ft)) \cdot \sum[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2] A_{mn} sen\frac{m\pi x}{a} sen\frac{n\pi y}{b}$$

$$\Delta M = -(c_1 \cos(ft) + c_2 sen(ft)) \cdot \sum[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2] B_{mn} sen\frac{m\pi x}{a} sen\frac{n\pi y}{b} \qquad (9)$$

Substituting in equation 8

$$B_{mn} = f^2 A_{mn}\left[\frac{\gamma h}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} + \frac{\mu\gamma h^3}{10(1+\mu)}\right]. \qquad (10)$$

and making use of equation 9,

$$f_{mn} = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \cdot \sqrt{\frac{D}{h\gamma}} \frac{1}{\sqrt{1 + h^2\left(\frac{\mu(1-\mu)+2}{10(1-\mu^2)}\right)\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}} \qquad (11)$$

which tells us that the fundamental frequency, for m = n = 1, is

$$f_{11} = \left(\frac{\pi}{b}\right)^2\left[1 + \left(\frac{b}{a}\right)^2\right] \cdot \sqrt{\frac{D}{h\gamma}} \frac{1}{\sqrt{1 + \left(\frac{\pi}{b}\right)^2 h^2 \left(\frac{\mu(1-\mu)+2}{10(1-\mu^2)}\right)\left[1 + \left(\frac{b}{a}\right)^2\right]}} \qquad (12)$$

Which leads to the value of the fundamental frequency corresponding to thin plates when the thickness is small, Leissa [7]

$$f_{11} = \left(\frac{\pi}{b}\right)^2 \left[1 + \left(\frac{b}{a}\right)^2\right] \cdot \sqrt{\frac{D}{h\gamma}} \quad (13)$$

IV. Numerical Solution of the Problem by the Galerkin Method

A. *Approach to the Weak Integral Formulation by the Galerkin Method*

The system formed by equations (6) and (7) is a variant of the typical Dirichlet problem (resolution of the Poisson equation with Dirichlet-type boundary conditions), in which the uniqueness of the solution is ensured and is known as Green function, [8]. It admits its resolution by numerical methods like finite elements and it is strongly convergent even for not very dense meshes, approaching the weak integral formulation by the Galerkin method.

Taking V=V(x,y), with V=0 at the boundary, we form the integral equation:

$$\iint_\Omega \left(\Delta M - \gamma h \ddot{w} + \frac{\mu \gamma h^3}{10(1+\mu)} \Delta \ddot{w}\right) V dA = 0 \quad (14)$$

Transforming the first and the second summing term of this equation, according to the 2nd and 1st Green´s Identity,

$$\iint_\Omega \Delta M V dA = \int_\Gamma \frac{\partial M}{\partial n} V ds - \iint_\Omega \nabla M \cdot \nabla V dA = -\iint_\Omega \nabla M \cdot \nabla V dA \quad (15)$$

or

$$\iint_\Omega \Delta \ddot{w} V dA = \int_\Gamma \frac{\partial \ddot{w}}{\partial n} V ds - \iint_\Omega \nabla \ddot{w} \cdot \nabla V dA = -\iint_\Omega \nabla \ddot{w} \cdot \nabla V dA \quad (16)$$

for choosing V = 0 at the border.

So, we obtain,

$$-\iint_\Omega \nabla M \cdot \nabla V dA = \iint_\Omega r h \ddot{w} V dA + \frac{\mu r h^2}{10(1+\mu)} \iint_\Omega \nabla \ddot{w} \cdot \nabla V dA \quad (17)$$

Following the Ritz-Galerkin method we adopted,

$$M = \sum_\Gamma \alpha_f \phi_f(x,y) + \sum_\Omega \alpha_i \phi_i(x,y) \quad (18)$$

being α_f constants and $\phi_f(x,y)$ shape functions or interpolation functions, for example piece-wise continuous functions, with unit value in the node in which they are defined and null in the rest of the nodes of the domain, even in the border (Kronecker delta property: the shape function at any node has value of 1 at that node and a value of zero at all other nodes)). With the subscript "f" we denote the possible nodes that we could place at the edges of the domain. In the case of simply supported plates, where M is null in the border (M = 0), we would have for the previous expression,

$$M = \sum_\Omega \alpha_i \phi_i(x,y) \quad (19)$$

Or also,

$$M = \sum_\Omega M_i \phi_i(x,y) \quad (20)$$

Being M_i the value of M in node i.

In the same way we adopt for the displacements w,

$$w = \sum_\Omega w_i \phi_i(x,y) \quad (21)$$

We can express the displacements w_i as,

$$\widehat{w}_i = \sum_i w_i \phi_i(x,y) sen\,(ft + \beta) \quad (22)$$

Operating with the previous equations, (9),(10) and (11), we have,

$$-\sum M_i^0 \iint_\Omega \nabla\phi_i \cdot \nabla V dA = -\left[\sum \gamma h f^2 w_i^0 \iint_\Omega \phi_i(x,y) V dA - \sum \frac{\mu\gamma h^2}{10(1+\mu)} f^2 w_i^0 \iint_\Omega \nabla\phi_i \cdot \nabla V dA\right] sen(ft+\beta) \quad (23)$$

And also,

$$\left[-\sum w_i^0 \iint_\Omega \nabla\phi_i \nabla\phi_j dA\right] sen(ft+\beta) = -\sum \frac{1}{D} M_i^0 \iint_\Omega \phi_i \cdot \phi_j dA - \frac{6\gamma}{5G(1+\mu)} f^2 \left[\sum w_i^0 \iint_\Omega \phi_i \phi_j dA\right] sen(ft+\beta) \quad (24)$$

In the same way for the 2nd of the elliptic equations 1 we form the integral equation

$$\iint_\Omega \Delta w V dA = -\frac{1}{D} \iint_\Omega M V dA + \frac{6\gamma}{5G(1+\mu)} \iint_\Omega \ddot{w} \cdot V dA \quad (25)$$

That we transform in the same way and obtain

$$\left[-\sum w_i^0 \iint_\Omega \nabla\phi_i \nabla\phi_j dA\right] sen(ft+\beta) = -\sum \frac{1}{D} M_i^0 \iint_\Omega \phi_i \cdot \phi_j dA - \frac{6\gamma}{5G(1+\mu)} f^2 \left[\sum w_i^0 \iint_\Omega \phi_i \phi_j dA\right] sen(ft+\beta) \quad (26)$$

That we also write in matrix form as

$$\psi\{w_i^0\} sen(ft+\beta) = \frac{1}{D} \Phi M_i^0 + \frac{6\gamma}{5G(1+\mu)} f^2 \Phi \{w_i^0\} sen(ft+\beta) \quad (27)$$

Rearranging terms and if we take $\lambda = \frac{1}{f^2}$, we obtain a typical eigenvalue problem in the form,

$$\lambda I\{w_i^0\} = \left[\frac{\gamma h}{D} \psi^{-1} \Phi \psi^{-1} \Phi + \frac{(12+6\mu-6\mu^2)}{5E} \psi^{-1} \Phi\right] \{w_i^0\} \quad (28)$$

Where $\psi = \iint_\Omega \nabla\phi_i \nabla\phi_j dA$ and $\Phi = \iint_\Omega \phi_i \phi_j dA$.

We must bear in mind that the system of equations to be solved is very similar to the one which is formulated by the finite element method (MEF) by means of the stiffness matrix K and the mass matrix M.

V. RESULTS

In order to compare the results, we consider a simple supported isotropic rectangular plate with $\frac{a}{b} = 0.4$, [9], in which the pb-2 Rayleigh-Ritz method was adopted. These results are referred to the nondimensional frequency parameter ϖ given by,

$$\varpi = \frac{f^2 a^2}{\pi^2}\sqrt{\frac{\rho h}{D}} \tag{29}$$

Results are shown for different ratios thickness/length (0.01, 0.1 and 0.2) in the following Table I.

TABLE I. NATURAL FREQUENCIES FOR SIMPLY SUPPORTED PLATES

Mode	h/a=0.01	h/a=0.1	h/a=0.2
1	7.250	6.4769	5.1831
4	22.231	16.847	11.489
8	33.999	23.400	15.036
Sol.1(Liew)	7.250	6.4773	5.1831
Sol.4(Liew)	22.233	16.845	11.487
Sol.8(Liew)	33.998	23.399	15.034

For the present results we have chosen linear interpolation functions (shape functions) corresponding to a linear triangle. The definition of the interpolation functions and the vector $\nabla\phi_i$ on each triangle is the one corresponding to a flat triangle.

VI. CONCLUSIONS

In the present work we have shown a methodology for the numerical and theoretical obtaining of the natural frequencies and modes of vibration of the transverse oscillations of moderately thick plates. We have based on a refined shear deformation plate theory.

The comparison with other competitive numerical methods has allowed us to establish the goodness of the obtained solutions.

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