Singular Strain Field for Composite Materials

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***Abstract*—Basic equations of composite mechanics have been established according to the orthotropic material properties. The analytic function of general complex variable is analyzed fully. The stress boundary problem of composite plate with Mode I crack is solved by using the complex variable method. The stress and strain fields are derived.**

Keywords—composite mechanics; orthotropic material; analytic function; stress and strain fields

I. INTRODUCTION

Fracture mechanics of non-homogeneous materials or anisotropic materials has more applications in new engineering structure materials. It is well known that the prediction of the crack initiation or propagation must be based on the fracture mechanics. Fiber-reinforced polymer matrix materials are the most typical composites and usually modeled as anisotropic materials at the macroscopic level [1~4]. The orthotropic plate may have been the base of composites in common use. The methods of elasticity are used to obtain stress and strain fields in cracked bodies [5~8]. The only viable method to solve the stress boundary problems in anisotropic composites may be to use the complex analytic function theory, and the results have been reported [9~12]. But the general solution may have not been solved completely. So this paper focuses on the fracture of orthotropic materials. Mode I stress fields and strain fields in the cracked plate are mainly analyzed.

II. STRESS FUNCTION

A. Mode I Crack

The plane stress state of composite plates may be common and very importance for the application. It is the key point to solve stress-field problems in orthotropic materials. For the crack problem, the general rectangular and polar coordinates are shown in Figure I. The polar coordinate system centered at the crack tip must be adequate for local stress analysis. In the consideration of cracked orthotropic materials, and also in the construction of suitable stress functions, it is very advantageous to use complex variables. Usually, two real numbers *x* and *y* form the complex number z, that is: $z = x + iy$ ($i = \sqrt{-1}$). And the conjugate complex number $\bar{z}$ is as: $\bar{z} = x - iy$. For the convenience of general investigation, now we introduce another complex variable (w), that is: $w = x + ihy$. The constant h is real. Definitely, we suppose the constant *h* to be positive ($h > 0$), also it can be called tensile or compressive ratio for the coordinate system. In terms of two reference coordinates, the complex variables can be written as:

$$\begin{cases} w = x + ihy = a + r\cos\theta + ihr\sin\theta \\ \overline{w} = x - ihy = a + r\cos\theta - ihr\sin\theta \end{cases} \tag{1}$$

Evidently, the derivative relation must be given by:

$$\frac{\partial w}{\partial x} = \frac{\partial \overline{w}}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = -\frac{\partial \overline{w}}{\partial y} = ih$$

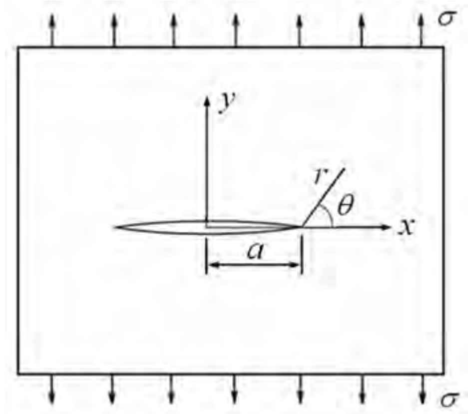

FIGURE I. OPENING MODE CRACK

For opening crack problem (Mode I), the plate with a line crack is subjected to symmetric loading σ at far edge and along y direction shown in Figure I. Therefore, the stress boundary conditions for this problem are:

$$\sigma_y = \tau_{xy} = 0 \qquad \text{for } |x| < a \text{ and } y = 0$$

$$\sigma_y = \sigma, \ \sigma_x = \tau_{xy} = 0 \qquad \text{for } x \to \infty \text{ or } y \to \infty$$

B. Analytic Function and Stresses

In order to solve the typical problem for Mode I crack at symmetrical loading, we can consider the complex function:

$$\Phi = \frac{1}{2}\sqrt{w^2(w^2 - a^2)} - \frac{a^2}{2}\ln(\sqrt{w^2} + \sqrt{w^2 - a^2})$$

Note that $\Phi(w)$ is an analytic function. The derivatives for each order can be obtained as follows:

$$\begin{cases} \Phi' = \dfrac{d\Phi}{dw} = w \cdot \sqrt{\dfrac{w^2 - a^2}{w^2}} \\ \Phi'' = \dfrac{d^2\Phi}{dw^2} = \sqrt{\dfrac{w^2}{w^2 - a^2}} \\ \Phi''' = \dfrac{d^3\Phi}{dw^3} = \dfrac{-a^2 w}{\sqrt{w^2(w^2 - a^2)^3}} \end{cases} \quad (2)$$

Furthermore, we can derive the partial derivatives of $\Phi(w)$ with respect to *x* and *y*, namely in the following:

$$\frac{\partial\Phi}{\partial x} = \frac{d\Phi}{dw}\frac{\partial w}{\partial x} = \Phi' = \operatorname{Re}\Phi' + i\operatorname{Im}\Phi' = \frac{\partial \operatorname{Re}\Phi}{\partial x} + i\frac{\partial \operatorname{Im}\Phi}{\partial x}$$

$$\frac{\partial\Phi}{\partial y} = \frac{d\Phi}{dw}\frac{\partial w}{\partial y} = ih\Phi' = ih\operatorname{Re}\Phi' - h\operatorname{Im}\Phi' = \frac{\partial \operatorname{Re}\Phi}{\partial y} + i\frac{\partial \operatorname{Im}\Phi}{\partial y}$$

Obviously, it is easy to determine the following differential equations:

$$\begin{cases} \operatorname{Re}\Phi' = \dfrac{\partial \operatorname{Re}\Phi}{\partial x} = \dfrac{1}{h}\dfrac{\partial \operatorname{Im}\Phi}{\partial y} \\ \operatorname{Im}\Phi' = \dfrac{\partial \operatorname{Im}\Phi}{\partial x} = -\dfrac{1}{h}\dfrac{\partial \operatorname{Re}\Phi}{\partial y} \end{cases} \quad (3)$$

Similarly, the partial derivatives of $\Phi'(w)$ can be given by:

$$\begin{cases} \operatorname{Re}\Phi'' = \dfrac{\partial \operatorname{Re}\Phi'}{\partial x} = \dfrac{1}{h}\dfrac{\partial \operatorname{Im}\Phi'}{\partial y} \\ \operatorname{Im}\Phi'' = \dfrac{\partial \operatorname{Im}\Phi'}{\partial x} = -\dfrac{1}{h}\dfrac{\partial \operatorname{Re}\Phi'}{\partial y} \end{cases} \quad (4)$$

For typical plane stress problems, it is well known that the stress components can be expressed by a real stress function *F*=*F*(*x, y*), which is defined as follows:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \quad (5)$$

The three stress components must be suitable for equilibrium equations in x-y plane (no body forces). According to the experience at solving the opening crack problem, the stress function can be determined by the below form:

$$F = C_1 \operatorname{Re}\Phi + C_2 y \operatorname{Im}\Phi' + \frac{T}{2}y^2 \quad (6)$$

By substituting this function into the stress expression (5), the stress components can be obtained as follows:

$$\begin{cases} \sigma_x = (2C_2 - hC_1)h\operatorname{Re}\Phi'' \\ \qquad - C_2 h^2 y \operatorname{Im}\Phi''' + T \\ \sigma_y = C_1 \operatorname{Re}\Phi'' + C_2 y \operatorname{Im}\Phi''' \\ \tau_{xy} = (hC_1 - C_2)\operatorname{Im}\Phi'' \\ \qquad - C_2 hy \operatorname{Re}\Phi''' \end{cases} \quad (7)$$

And then by using the partial derivatives in the expression (2), the stress equation (7) can be written as

$$\begin{cases} \sigma_x = (2C_2 - hC_1)h\operatorname{Re}\sqrt{\dfrac{w^2}{w^2 - a^2}} \\ \qquad + C_2 h^2 y \operatorname{Im}\dfrac{a^2 w}{\sqrt{w^2(w^2 - a^2)^3}} + T \\ \sigma_y = C_1 \operatorname{Re}\sqrt{\dfrac{w^2}{w^2 - a^2}} \\ \qquad - C_2 y \operatorname{Im}\dfrac{a^2 w}{\sqrt{w^2(w^2 - a^2)^3}} \\ \tau_{xy} = (hC_1 - C_2)\operatorname{Im}\sqrt{\dfrac{w^2}{w^2 - a^2}} \\ \qquad + C_2 hy \operatorname{Re}\dfrac{a^2 w}{\sqrt{w^2(w^2 - a^2)^3}} \end{cases} \quad (8)$$

To meet the necessity of the stress boundary conditions as given in the previous section, the constants must be given by

$$C_1 = \sigma, \quad C_2 = \sigma h, \quad T = -\sigma h^2$$

Then, the stresses in equation (7) become as:

$$\begin{cases}\sigma_x = \sigma h^2(\operatorname{Re}\Phi'' - hy\operatorname{Im}\Phi''' - 1) \\ \sigma_y = \sigma(\operatorname{Re}\Phi'' + hy\operatorname{Im}\Phi''') \\ \tau_{xy} = -\sigma h^2 y\operatorname{Re}\Phi''' \end{cases} \quad (9)$$

For the convenience of the writing, another complex function $\psi = \psi(w)$ is introduced to replace above complex function. That is: $\psi = \Phi'' = \Phi''(w)$. Again mark that:

$$\eta = \sqrt{\cos\theta + ih\sin\theta}\,, \qquad \eta^2 = \cos\theta + ih\sin\theta$$

Then we have the following relations:

$$w - a = r(\cos\theta + ih\sin\theta) = r\eta^2$$

$$w^2 - a^2 = (2a + r\eta^2)r\eta^2$$

$$\begin{cases}\psi = \sqrt{\dfrac{w^2}{w^2 - a^2}} = \sqrt{\dfrac{(a + r\eta^2)^2}{(2a + r\eta^2)r\eta^2}} \\ \psi' = \dfrac{d\psi}{dw} = \dfrac{-a^2 w}{\sqrt{w^2(w^2 - a^2)^3}} \\ = \dfrac{-a^2(a + r\eta^2)}{\sqrt{(a + r\eta^2)^2(2a + r\eta^2)^3(r\eta^2)^3}} \end{cases} \quad (10)$$

$$\begin{cases}\sigma_x = \sigma h^2(\operatorname{Re}\psi - hy\operatorname{Im}\psi' - 1) \\ \sigma_y = \sigma(\operatorname{Re}\psi + hy\operatorname{Im}\psi') \\ \tau_{xy} = -\sigma h^2 y\operatorname{Re}\psi' \end{cases} \quad (11)$$

Now only we concentrate the key point on the right crack tip as shown in Figure I, and that is to say, $x > 0$, $r << a$. Some stress functions can be simplified by the following approximation ($x \approx a$, $w \approx a$):

$$\psi \approx \sqrt{\frac{a}{2r}}\frac{1}{\eta}, \qquad \psi' \approx -\frac{1}{2r}\sqrt{\frac{a}{2r}}\frac{1}{\eta^3}$$

Thus, the stress fields near to the crack tip are as:

$$\begin{cases}\dfrac{\sigma_x}{\sigma} = \sqrt{\dfrac{a}{2r}}h^2(\operatorname{Re}\dfrac{1}{\eta} + \dfrac{h\sin\theta}{2}\operatorname{Im}\dfrac{1}{\eta^3}) - h^2 \\ \dfrac{\sigma_y}{\sigma} = \sqrt{\dfrac{a}{2r}}(\operatorname{Re}\dfrac{1}{\eta} - \dfrac{h\sin\theta}{2}\operatorname{Im}\dfrac{1}{\eta^3}) \\ \dfrac{\tau_{xy}}{\sigma} = \sqrt{\dfrac{a}{2r}}\dfrac{h^2\sin\theta}{2}\operatorname{Re}\dfrac{1}{\eta^3} \end{cases} \quad (12)$$

Consequently, the mode I stress fields in the vicinity of the crack tip are obtained at present.

III. Strain Analysis

Suppose the principal elastic directions of the orthotropic plate coincide with the coordinate directions as shown in Figure I. And let the directions 1, 2 parallel to the axes x, y, respectively. Now consider the linear elastic strain-stress relations, the constitutive equations for the orthotropic materials are given as (plane stress state):

$$\begin{cases}\varepsilon_x = \dfrac{\sigma_x}{E_1} - \dfrac{\mu_{12}\sigma_y}{E_1} \\ \varepsilon_y = \dfrac{\sigma_y}{E_2} - \dfrac{\mu_{12}\sigma_x}{E_1} \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G_{12}} \end{cases} \quad (13)$$

By substituting the stresses in equation (11) into the constitutive equations, the strain components can be obtained as follows:

$$\begin{cases}\varepsilon_x = \dfrac{\sigma}{E_1}[(h^2 - \mu_{12})\operatorname{Re}\psi \\ \qquad - (h^2 + \mu_{12})hy\operatorname{Im}\psi' - h^2] \\ \varepsilon_y = \dfrac{\sigma}{E_1}h^2[(h^2 - \mu_{12})\operatorname{Re}\psi \\ \qquad + (h^2 + \mu_{12})hy\operatorname{Im}\psi' + \mu_{12}] \\ \gamma_{xy} = -\dfrac{\sigma}{G_{12}}h^2 y\operatorname{Re}\psi' \end{cases} \quad (14)$$

The partial derivatives of strains are as:

$$\begin{cases} \dfrac{\partial^2 \varepsilon_x}{\partial y^2} = \dfrac{\sigma}{E_1} h^2 [-(3h^2 + \mu_{12}) \operatorname{Re}\psi'' \\ \qquad + (h^2 + \mu_{12}) hy \operatorname{Im}\psi'''] \\ \dfrac{\partial^2 \varepsilon_y}{\partial x^2} = \dfrac{\sigma}{E_1} h^2 [(h^2 - \mu_{12}) \operatorname{Re}\psi'' \\ \qquad + (h^2 + \mu_{12}) hy \operatorname{Im}\psi'''] \\ \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = -\dfrac{\sigma}{G_{12}} h^2 \operatorname{Re}\psi'' + \dfrac{\sigma}{G_{12}} h^3 y \operatorname{Im}\psi''' \end{cases} \tag{15}$$

It is well known that the compatibility condition of three strains must be satisfied by:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{16}$$

Substituting the strain components into this equation, then we can obtain some characteristic equations. By solving these equations, the relations of material constants can be determined and that must be given as follows:

$$h^2 = \frac{E_1}{2G_{12}} - \mu_{12} = \sqrt{\frac{E_1}{E_2}} \tag{17}$$

Thus it can be seen that the strain components near to the crack tip ($r << a$) ought to be expressed by:

$$\begin{cases} \varepsilon_x = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma}{E_1} [(h^2 - \mu_{12}) \operatorname{Re}\dfrac{1}{\eta} \\ \quad + (h^3 + \mu_{12} h) \dfrac{\sin\theta}{2} \operatorname{Im}\dfrac{1}{\eta^3}] - \dfrac{\sigma}{E_1} h^2 \\ \varepsilon_y = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma}{E_1} h^2 [(h^2 - \mu_{12}) \operatorname{Re}\dfrac{1}{\eta} \\ \quad - (h^3 + \mu_{12} h) \dfrac{\sin\theta}{2} \operatorname{Im}\dfrac{1}{\eta^3}] + \mu_{12} \dfrac{\sigma}{E_1} h^2 \\ \gamma_{xy} = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma}{G_{12}} h^2 \dfrac{\sin\theta}{2} \operatorname{Re}\dfrac{1}{\eta^3} \end{cases} \tag{18}$$

This is the singular strain fields at the crack tip for orthotropic materials. By using the equation (17), the singular strain fields can be written as:

$$\begin{cases} \varepsilon_x = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma}{E_2} [(\dfrac{1}{h^2} - \dfrac{\mu_{12}}{h^4}) \operatorname{Re}\dfrac{1}{\eta} \\ \quad + (1 + \dfrac{\mu_{12}}{h^2}) \dfrac{\sin\theta}{2h} \operatorname{Im}\dfrac{1}{\eta^3}] - \dfrac{\sigma}{E_2 h^2} \\ \varepsilon_y = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma}{E_2} [(1 - \dfrac{\mu_{12}}{h^2}) \operatorname{Re}\dfrac{1}{\eta} \\ \quad - (1 + \dfrac{\mu_{12}}{h^2}) \dfrac{h \sin\theta}{2} \operatorname{Im}\dfrac{1}{\eta^3}] + \dfrac{\sigma \mu_{12}}{E_2 h^2} \\ \gamma_{xy} = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma h^2 \sin\theta}{2G_{12}} \operatorname{Re}\dfrac{1}{\eta^3} \end{cases} \tag{19}$$

For the isotropic materials ($h = 1$), then: $\eta = e^{i\theta/2}$. The singular strain fields can be given by:

$$\begin{cases} \varepsilon_x = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma}{E} \cos\dfrac{\theta}{2} [1 - \mu \\ \quad - (1 + \mu) \sin\dfrac{\theta}{2} \sin\dfrac{3\theta}{2}] - \dfrac{\sigma}{E} \\ \varepsilon_y = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma}{E} \cos\dfrac{\theta}{2} [1 - \mu \\ \quad + (1 + \mu) \sin\dfrac{\theta}{2} \sin\dfrac{3\theta}{2}] + \dfrac{\sigma\mu}{E} \\ \gamma_{xy} = \sqrt{\dfrac{a}{2r}} \dfrac{\sigma}{G} \sin\dfrac{\theta}{2} \cos\dfrac{\theta}{2} \cos\dfrac{3\theta}{2} \end{cases} \tag{20}$$

Where, E, μ and G are the elastic constants for common engineering materials. In a word, the singular strain fields at the crack tip can be determined by preceding equations.

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REFERENCES

[1] M. Fakoor. “Augmented strain energy release rate (ASER): A novel approach for investigation of mixed-mode I/II fracture of composite materials”. Eng Fract Mech, 2017, 179, pp.177-189.

[2] H. Zhang, P. Qiao. “A state-based peridynamic model for quantitative elastic and fracture analysis of orthotropic materials”. Eng Fract Mech, 2019, 206, pp.147-171.

[3] M.H. Sadd. “Elasticity, theory, applications and numerics”. Elsevier Inc., 2014.

[4] J.H. Kim, G.H. Paulino. “Mixed-mode fracture of orthotropic functionally graded materials using finite elements and the modified crack closure method”. Eng Fract Mech, 2002, 69, pp.1557-1586.

[5] M. Fakoor, N.M. Khansari. "Mixed mode I/II fracture criterion for orthotropic materials based on damage zone properties". Eng Fract Mech, 2016; 153, pp.407-420.

[6] X. Huang, Y. Liu, X. Huang. "Analytical characterizations of crack tip plastic zone size for central-cracked unstiffened and stiffend plates under biaxial loading". Eng Fract Mech, 2019, 206, pp.1-20.

[7] T. Nakmura, D.M. Parks. "Determination of elastic T-stress along three-dimensional crack fronts using an interaction integral". Int J Solids Struct, 1992, 29, pp.1597-1611.

[8] G.C. Sih. "Mechanics of fracture initiation and propagation". Kluwer Academic Publishers, 1991.

[9] Klaus Friedrich. "Application of fracture mechanics to composite materials". Elsevier Sci. Pub. B.V., 1989.

[10] K. Lu, T. Meshii. "A systematic investigation of T-stresses for a variety of center-cracked tension specimens". Theor Appl Fract Mech, 2015; 77, pp.74-81.

[11] M. Myatollahi, M. Pavier, D. Smith. "Mode I cracks subjected to large T-stresses". Int J Fract, 2002, 117, pp.159-174.

[12] N. Fallah, N. Nikraftar. "Meshless finite volume method for the analysis of fracture problems in orthotropic media". Eng Fract Mech, 2018; 204, pp.46-62.