

The Design Framework of Mathematical Proof Learning of Mathematics Education Department Students

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Abstract—The present research is intended to develop a learning design of the topic which then can be integrated into several subjects. Fundamental research suggests the causes and the types of students' difficulties in a mathematical proof, namely a lack of understanding of mathematical proof, mathematical symbols and logic, and mathematical evidentiary strategies. Seeing the difficulties of students in proving mathematics, it is necessary to develop a framework of learning design of mathematical proof, which will later become the basis for the design of mathematical proof learning. The purpose of the research was identifying the learning paths of mathematical proof learning, as a framework in developing learning scenarios based on those stages. The research was carried out for one year, derived by a study of the critical analysis of observations on mathematical proving activities including questions and answers based task. The findings of the study suggest that (1) the stages of the learning trajectory include: (a) experience and fundamentals, i.e. giving examples and strengthening the principles of proof, (b) proving with simple proof trajectory, that is proof of statement that can be solved by trajectory single thought, (c) existence proof, that is proof of the existence or non-existence of facts or concepts, (d) proving with a simple proof trajectory, and (e) proving through construction, (2) Students can visually create flowcharts to guide proof writing. However, there are difficulties in writing symbols and writing mathematical reasons that link the series of statements in proof.

Keywords—*design framework of learning, learning trajectory, mathematical proof*

I. INTRODUCTION

Fundamental research [1] found a number of students' difficulties in mathematical proofs, including difficulties in the symbolic meaning of which some students did not understand the meaning of mathematical symbols correctly which causes some parts of the proof they have built became lose meaning. In fact, the meaning of symbols in mathematics is very specific and unique meaning that their uses greatly influence the meaning of written sentences or statements. In addition, the difficulty of developing a proof strategy was also found. Some students experienced difficulties in solving several proofs assigned to them. Apparently, they have difficulty in deciding which strategy is more appropriate to prove some mathematics theorems.

Based on an inductive analysis, it is found that two categories cause students' difficulties in proving statements in mathematics, i.e., (1) understanding of mathematical

proof, and (2) understanding of mathematical concepts and principles. The students turned out to have difficulties in understanding what actual proof is in mathematics. Although heuristic proof with the help of computer software has now begun to be accepted in mathematics — it is still debate concerning its validity when being viewed from various rules — logic with an axiomatic deductive system still dominates the discussion of proof in mathematics.

The difficulty in proving mathematics shows the importance of developing a trajectory of mathematical proof learning that is in accordance with the characteristics of mathematics statements that are proven. Learning trajectory describes a learning sequence which must be taken and what concepts related to the material learned by students so that they will be able to learn thoroughly. [2] formulates that the trajectory of learning in pre-school age children can be used in teaching mathematics at an early age whereas Hadi in [3] formulated the hypothetical learning trajectory of fraction topic in elementary schools and the definite learning trajectory related to the learning design carried out in the classroom. The learning trajectory is hypothetical which is then called a hypothetical learning trajectory.

The trajectory of hypothetical learning is a guess about the range of activities that a student goes through in solving a problem or understanding a mathematics concept. While the learning path is a series of activities that are passed by a student in solving a problem or understanding a concept. In defining the trajectory of learning, the hypothetical learning trajectory is first formulated. In the implementation / trial of a hypothetical learning trajectory, it may experience some changes or improvements. The trajectory which is obtained from several revisions is called learning trajectory. In summary, the trajectory of learning is the result of revisions of the hypothetical learning trajectory based on experimented learning.

The learning trajectory meant in this study is seen as a series of activities conducted by students in solving mathematical proof problems. The formulated HLT for the topic comprehends of (1) experience and fundamentals, (2) proving with simple proof trajectory, (3) existence proof, (4) proving with a simple trajectory of proof and (5) proving through construction.

II. RESEARCH METHOD

The used research design related to the purpose of this study is design research proposed by [4], which is characterized by a cyclical process of preparing for the experiment, conducting an experiment, and retrospective analysis. The design research is also classified as development studies [5], because it is a systematic study of designing, developing, and evaluating educational interventions as a solution to solve complex problems in educational practice.

Design research initiated by [4] is focused on developing the order of the presentation of material in mathematics learning. It stems from thought experiment, i.e., thinking of the learning route / trajectory that students will go through. The results of the thought experiment are then applied in the classroom. By reflecting on the results of experiments in class, then the next thought experiment is conducted. In the long-term process, these two activities can be seen as a cumulative cyclical process as shown in the following Figure.

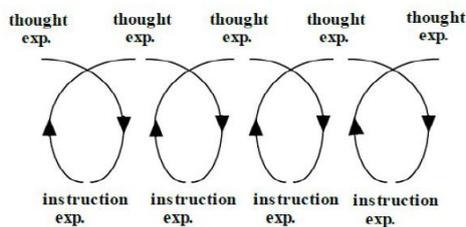


Fig. 1. Figure 1. Cumulative cyclical process

The following passage describes each phase of design research:

A. Preparation for the Experiment Phase

The main objective of the preparation stage is to formulate the HLT of mathematical proof that will be refined during the research process [6]. There are two main things that are conducted in this phase, namely: (1) reviewing various literature on mathematical proof and learning, (2) Designing HLT (a series of activities to complete mathematical proof.

B. Experiment Phase

The main purpose of the experiment phase is to test and increase the conjecture that has been made in the preparation stage and to develop an understanding of the way the design work is carried out [4]. In this phase the important component is HLT. HLT is used to observe the learning process and as a guide to teaching by providing proof problems. In this study, the learning stage in the classroom was carried out in two cycles. The first cycle was aimed to see how design can work and improve the next cycle. It was done in a small group of six students. The second cycle is the implementation of HLT that is carried out in the real class. The research subjects in the Cycle I and Cycle II were students of the mathematics education program of Universitas Negeri Makassar, one of higher education institution in Indonesia.

C. Retrospective Analysis

Retrospective analysis is a phase where a researcher carry out an evaluation of whether the applied HLT is in accordance with what is conjectured [7]. The learning trajectory plan used in the retrospective analysis is the main guide and reference in answering the formulation of the research problem. The main objective at this stage is to contribute to the development of HLT in supporting students' understanding. The role of HLT in this stage is to be a guide in determining the focus of analysis in research. The process of analysis focuses not only on the factors that support learning but also on some of the conjectures that do not run. The explanations from the analysis are used to draw conclusions and answer research questions

III. RESULTS AND DISCUSSION

The research findings in this study are based on the proving activity in the real analysis subject. Data is collected through observation as well as questioning and answering activities when students present the assignments. Based on the description of the HLT as revealed at the beginning of this paper, the following will be elaborated on the findings in the phases of the learning path of mathematical proof. This learning trajectory consists of several phases within a cycle of learning scenarios. One learning cycle consists of several meetings until all phases in the cycle have been carried out.

A. Preparation Phase

Based on the thought experiment, the hypothetical learning is sorted according to the difficulty level of mathematical proof. The researcher along with several mathematical proof learning experts propose several sequences of mathematical proof learning. Based on the results of the literature review and the experiences of the researchers in learning, the learning phases of mathematical proof are arranged based on the order of the difficulties, namely: (1) experience and fundamentals, (2) proving with simple proof trajectory, (3) existence proof, (4) proving with a simple trajectory of proof and (5) proving through construction.

B. Experiment and Retrospective Analysis Phase

The results of the designed HLT of mathematical proof are implemented at the experiment stage and evaluated at the retrospective analysis stage. The findings are described as follows

- Phase 1. Experience and Fundamentals

The results of the observations as well as discussion activities during the experiment indicate that the students need to understand the proofs through rigorous definitions and their negations, explain them visually. The order within the proof should be strictly taken into account based on the orders of previous concepts, definitions, and theorems.

The students are given daily problem experiences in the form of statements and how to prove the statements. In addition, students are still reminded of some basic understanding of conjunctions and quantification statements.

Specifically, the students are asked to explore (1) the basic definitions of mathematical concepts needed for the proof problem, (2) the connecting words and official

statements, which are used in the construction of a mathematical proof. The words are "and", "or", "if ... then ...", "... if and only if ...", "it is not true that ...". "For every ..." and "there ...". The examples of the mathematical statements that use these words are: For each number $\epsilon > 0$ there are real numbers x and y , $|xy| < 1$ so that $|f(x) - f(y)| < \epsilon$. Other examples, There is x number such that for each number $y > x$, $f(y) > f(x)$.

Then the students are asked to evaluate the truth or untruth of the statement when the context in which the statement has been stated, and write their negations. The results of the retrospective analysis show that this phase can be passed well by students.

• Phase 2. Proving with Simple proof trajectory

In this phase, the students are asked to prove mathematical statements that can be solved with a single-minded trajectory. In this phase, the students generally can grasp the idea of completion and be able to solve the problem of proof, either by using direct proof or by using indirect proof. For example, it will be proven that the statement, "if P then Q," the trajectory of the proof used by students is



Knowing that P is true, then knowing Q. The other proof is:



Knowing P is true and knowing Q is false. Deriving the implication that P is true and Q is false implying contradictions

The mistakes that usually appear at this stage are errors in writing notations, mistakes in choosing proof of writing sentences. These errors can be corrected through lecturer discussions with students. The results of the retrospective analysis show that this stage is considered quite successful by students.

• Phase 3. Proof of existence

In this phase, the students are asked to prove the existence and non-existence of a fact or concept in mathematics. One example used is to prove the non-existence of numbers $\sqrt{2}$ in a set of rational numbers and prove the existence of numbers $\sqrt{2}$ in the set of real numbers.

The proof begins by asking students to draw a trajectory diagram of the proof, then compiling a proof sentence that matches the previous diagram. The flowchart of proof really guides them in proving. However, the difficulties generally arise when writing / compiling proof based on the proof flow chart. The results of the retrospective analysis show that this phase is quite difficult for students.

• Phase 4. Proving with a simple trajectory of proof

There are many proofs that have a simple proof trajectory. However, in this study, it is simply investigated how students prove the statement of the implication form "If

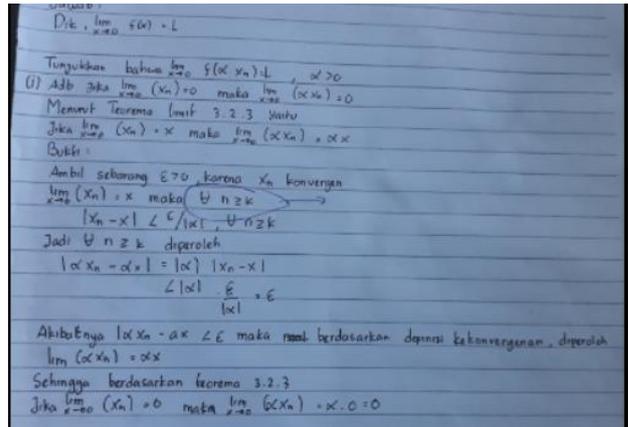
P then Q," but it is difficult to do directly, except using other theorems.

The flow chart used by students is presented below:



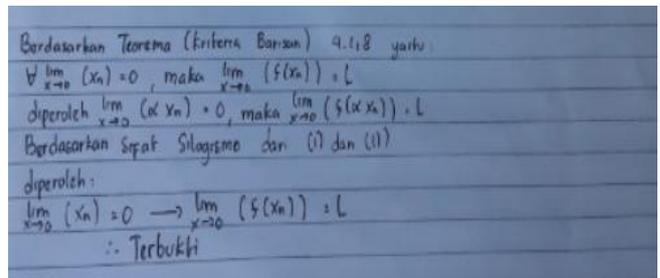
In the chart, P and P 'are equivalent statements stated by a theorem likewise Q and Q.' To prove the statement "If P then Q," simply indicate "If P 'then Q'."

One example of proof by students is presented below:



The proof above shows direct proof. Some parts of the proof are associated with incorrect logic, a series of unconnected arguments.

In that proof, the subject, at the end of the proof, successfully showing the final statement. However, there is a flaw in the proof (see the circled section), where the subject states the existence of a "k" number which is not previously stated.



From some of the mathematical proof phenomena presented above, it can be concluded that although the trajectory of proof can be made by students, the process of writing proof still leaves the difficulty of writing symbols and difficulty writing down mathematical reasons that link the series of statements in proof. In this case, the instructor is expected to provide scaffolding to help students complete the proofs.

• Phase 5. Proving through construction

In this phase, the students are asked to prove the statement by constructing a set, or a stage that is directed to obtain a statement that will be proven. The usual example is to prove the existence of a number $\sqrt{2}$ in a real number system. The proof begins with defining a set of positive numbers which are equal or less than 2. The existence of

numbers $\sqrt{2}$ is the smallest upper limit of the set constructed. In this section the students can make flowcharts of proof well. However, it is very difficult to show mathematical reasons for the series of statements in proof. Students must have a strict understanding of concept definitions related to statements that are being proven. The results of the retrospective analysis show that this phase requires scaffolding to help students complete proof.

IV. CONCLUSIONS

1. The stages of the learning path of mathematical proof are: (a) experience and fundamentals, namely giving examples of proof and strengthening the principles of proof, (b) proof with simple proof lines, that is proof of statements that can be solved with a single-minded flow, (c) proof of existence, that is proof of the existence or nonexistence of facts or concepts, (d) proof by simple flow of proof, and (e) proof through construction. The last three stages still require teaching scaffolding to help students complete mathematical proofs.
2. Students can make a flow chart of proof visually to guide the writing of proof. However, there are

difficulties in writing symbols and difficulties in writing mathematical reasons that link the series of statements in proof.

REFERENCES

- [1] S. Sabri and I. Minggu, "Students' Difficulties in Mathematics Proofs," in *Proceedings of the 1st International Conference on Science (ICOS-1)*, 2014, pp. 272–280.
- [2] M. E. Bardsley, *Pre-kindergarten teachers' use and understanding of hypothetical learning trajectories in mathematics education*. 2006.
- [3] A. R. Harini and A. H. Rosyidi, "Lintasan Belajar Siswa Pada Materi Jajargenjang Dengan Metode Penemuan Terbimbing Melalui Penelitian Desain," *J. Ilm. Pendidik. Mat.*, vol. 3, no. 5, 2016.
- [4] K. Gravemeijer and P. Cobb, "Design research from a learning design perspective," *Educ. Des. Res.*, pp. 17–51, 2006.
- [5] T. Plomp, "Educational design research: An introduction," *Educ. Des. Res.*, pp. 11–50, 2013.
- [6] K. Gravemeijer and D. van Eerde, "Design research as a means for building a knowledge base for teachers and teaching in mathematics education," *Elem. Sch. J.*, vol. 109, no. 5, pp. 510–524, 2009.
- [7] A. Bakker, "Design research in statistics education: On symbolizing and computer tools." Utrecht University, 2004.