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When is a sum of projections equal to a scalar operator?

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1 Introduction

Collections of self-adjoint operators that act on a separable complex Hilbert space H, $\dim H \leq \infty$, have their spectra, $\sigma(A_k)$, in given finite sets $M_k \subset \mathbb{R}$, $k = 1, \dots, n$, and are such that the sum of them is a multiple of the identity operator play an important role in analysis, algebraic geometry, representation theory, and mathematical physics (see [1, 2] and the bibliography therein). The problem of describing the set Σ_n of values of the parameter α for which there exists a Hilbert space H, n orthogonal projections on H, $P_1, \dots, P_n$, which are operators with the spectra in $\{0, 1\}$, and such that $\sum_{k=1}^n P_k = \alpha I_H$ has been studied in [3, 4, 5]. The latter condition is equivalent to the fact that the $*$-algebra

$$\mathcal{P}_{n,\alpha} = \mathbb{C}\langle p_1, \dots, p_n \mid p_k^2 = p_k^* = p_k (k = 1, \dots, n), \sum_{k=1}^n p_k = \alpha e\rangle$$

has $*$-representations on a Hilbert space. Since the dimension of H is not fixed (it could even be infinite), it is difficult to describe Σ_n by using Horn's inequalities, see [1, 2] and the bibliography therein.

In this survey, following [5], we describe the set Σ_n. For $n \leq 4$, the set Σ_n is discrete, and the description of Σ_n and the corresponding representations have become a part of the mathematical folklore (a survey of the main results and a bibliography can be found in [5]). However, it turns out that the set Σ_n contains a nonempty interval for $n \geq 5$. If $n \geq 4$, $\Sigma_n = \Lambda_n \cup [\frac{n-\sqrt{n^2-4n}}{2}, \frac{n+\sqrt{n^2-4n}}{2}] \cup (n - \Lambda_n)$, where Λ_n is a discrete set which is the union of the following two series:

$$\Lambda_n^1 = \left\{0, 1 + \frac{1}{n-1}, 1 + \frac{1}{n-2-\frac{1}{n-1}}, 1 + \frac{1}{n-2-\frac{1}{n-2-\frac{1}{n-1}}}, \dots\right\},$$

$$\Lambda_n^2 = \left\{1, 1 + \frac{1}{n-2}, 1 + \frac{1}{n-2-\frac{1}{n-2}}, 1 + \frac{1}{n-2-\frac{1}{n-2-\frac{1}{n-2}}}, \dots\right\}.$$

We also give the following expression for Λ_n:

$$\Lambda_n = \left\{ \frac{n - \sqrt{n^2 - 4n}\coth(k\mathrm{Arch}(\frac{\sqrt{n}}{2}))}{2} \mid k \in \mathbb{N} \right\}.$$

All points of the sets Σ_n were found with the help of an approach to the description of the sets Σ_n based on the introduction of two functors Φ^+ and Φ^- on the categories $\mathrm{Rep}\,\mathcal{P}_{n,\alpha}$ of $*$-representations of the algebras $\mathcal{P}_{n,\alpha}$, see [6]. The functors Φ^+ and Φ^- will be called Coxeter functors, because their structure and the role in the description of representations of the algebras $\mathcal{P}_{n,\alpha}$ are similar to those of the Coxeter functors in [7] in many respects.

Note that the problem of finding values of the parameter $\tau \in \mathbb{R}$ such that the $*$-algebra $\mathcal{TL}_{\infty,\tau} = \mathbb{C}\langle p_1, \ldots, p_n, \cdots \mid p_k^2 = p_k = p_k^*(k \in \mathbb{N}); p_k p_j = p_j p_k, |k-j| \geq 2; p_k p_{k\pm1} p_k = \tau p_k\rangle$ has at least one representation is similar and goes back to the famous series of works of V. Jones (see [8]).

2 A description of the set Σ_n.

2.1 Preliminaries.

2.1.1 Elementary properties of Σ_n.

Proposition 1. (a) $\Sigma_n \subset [0, n]$;

(b) $\{0, 1, \ldots, n\} \subset \Sigma_n$;

(c) $(0, 1) \cap \Sigma_n = \emptyset$;

(d) $(1, 1 + \frac{1}{n-1}) \cap \Sigma_n = \emptyset$;

(e) $\alpha \in \Sigma_n \iff n - \alpha \in \Sigma_n$.

Proof. (a) We have $0 \leq \alpha \leq n$, since the equivalent identities $\sum_{k=1}^n P_k = \alpha I$ and $\sum_{k=1}^n (I - P_k) = (n - \alpha)I$ have positive operators in the left-hand sides.

(b) If P_k are projections in a one-dimensional space such that m of them are identities and the other are zeros, then $\sum_{k=1}^n P_k = mI$.

(c) $\Sigma_n \cap (0, 1) = \emptyset$, since if $0 < \alpha < 1$ and $\sum_{k=1}^n P_k = \alpha I$, then at least one projection $P_j \neq 0$. Then $\sum_{k\neq j}^n P_k = \alpha I - P_j$. But there is a nonnegative operator in the left-hand side of this identity, whereas the right-hand side is an operator which is not nonnegative. A contradiction.

(d) Let us first give a simple proof assuming that $\dim H = m < \infty$. Let P_i, $i = 1, \ldots, n$, be projections in the space H, $0 < \epsilon$, and the sum of the projections equal $(1 + \epsilon)I$. Then $\forall k$, $k = 1, \ldots, n$, we have $\sum_{i\neq k}^n P_i = (1 + \epsilon)I - P_k$ and $\sum_{i\neq k}^n \mathrm{tr}(P_i) = (1 + \epsilon)m - \mathrm{tr}(P_k)$ (m is the dimension of H). Since $\sum_{i\neq k}^n \mathrm{tr}(P_i) \geq \mathrm{rank}(\sum_{i\neq k}^n (P_i)) = m$, we have that $\mathrm{tr}(P_k) \leq \epsilon m$. Because k is arbitrary, $(1 + \epsilon)m = \sum_{i=1}^n \mathrm{tr}(P_i) \leq \sum_{i=1}^n \epsilon m = mn\epsilon$, whence $\epsilon \geq \frac{1}{n-1}$.

If the space H is separable, to prove property (d) we will need the following lemmas on the spectrum of a sum of orthogonal projections.

Lemma 1. *Let a number* $1 \geq \tau > 0$ *and projections* P_1, P_2 *be given. Then if* $\lambda \in \sigma(\tau P_1 + P_2)$, $\lambda \neq 0, \tau, 1, 1+\tau$, *we have that* $1 + \tau - \lambda \in \sigma(\tau P_1 + P_2)$.

Proof. It is sufficient to check the statement of the lemma for irreducible pairs of orthogonal projections. Irreducible pairs of orthogonal projections can only be one and two-dimensional. For one-dimensional pairs of projections, $\lambda \in \sigma(\tau P_1 + P_2) \subset \{0, \tau, 1, 1+\tau\}$. Any two-dimensional pair of orthogonal projections is unitarily equivalent to the pair

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$$

for some $0 < \phi < \pi/2$, and the statement of the lemma for this pair is verified directly. ■

Corollary 1. *If* $0 < \epsilon < \tau \leq 1$ *and* $\tau P_1 + P_2 \leq (1+\epsilon)I$, *then* $\tau P_1 + P_2 \geq (\tau - \epsilon)P_{\operatorname{Im} P_1 + \operatorname{Im} P_2}$, *where* $P_{\operatorname{Im} P_1 + \operatorname{Im} P_2}$ *is the orthogonal projection onto the closed linear span of* $\operatorname{Im} P_1 + \operatorname{Im} P_2$.

Proof. Suppose that there exists a number $\lambda \in \sigma(\tau P_1 + P_2)$ such that $0 < \lambda < (\tau - \epsilon)$. Then $1 + \tau - \lambda > 1 + \epsilon$. However, by Lemma 1, $1 + \tau - \lambda \in \sigma(\tau P_1 + P_2)$, which contradicts the conditions of the corollary. ■

In the next lemma, we consider the case of a greater number of orthogonal projections.

Let $P_1, \ldots, P_k$ be projections on a Hilbert space. Define the subspaces $\mathfrak{H}_k = \operatorname{Im} P_1 + \cdots + \operatorname{Im} P_k$ in H as closed linear spans of $\operatorname{Im} P_1 + \cdots + \operatorname{Im} P_k$ in H.

Lemma 2. *Let* $0 < \epsilon < 1$ *and* $\sum_{k=1}^n P_k \leq (1+\epsilon)I$. *Then* $\sum_{k=1}^m P_k \geq (1 - (m-1)\epsilon)P_{\mathfrak{H}_m}$ *for all* $m = 1, 2, \ldots, n$.

Proof. We use induction on m. For $m = 2$, the statement of the lemma is directly deduced from Corollary 1, since $P_1 + P_2 \leq (1+\epsilon)I$. Let now $m > 2$ be fixed and $\sum_{k=1}^{m-1} P_k \geq (1 - (m-2)\epsilon)P_{\mathfrak{H}_{m-1}}$. Then $\sum_{k=1}^m P_k \leq (1+\epsilon)I$ and, by Corollary 1, $\sum_{k=1}^m P_k = \sum_{k=1}^{m-1} P_k + P_m \geq (1 - (m-2)\epsilon)P_{\mathfrak{H}_{m-1}} + P_m \geq (1 - (m-1)\epsilon)P_{\mathfrak{H}_m}$. ■

Let us now proceed with the proof of property (d). If P_k are projections on H, $\epsilon > 0$, and $\sum_{k=1}^n P_k = (1+\epsilon)I$, then the operator $\sum_{k=1}^{n-1} P_k = (1+\epsilon)I - P_n$ has the diagonal form in a certain basis, $\operatorname{diag}\{1+\epsilon, \ldots, 1+\epsilon, \ldots, \epsilon, \ldots \epsilon, \ldots\}$. This shows that the space $\mathfrak{H}_{n-1}$ coincides with the entire H and $\epsilon \in \sigma(P_1 + \cdots + P_{n-1})$. By applying Lemma 2 with $m = n-1$, we get $\epsilon \geq 1 - (n-2)\epsilon$, that is, $\epsilon \geq \frac{1}{n-1}$.

(e) If $P_1, \ldots, P_n$ are orthogonal projections on H such that $\sum_1^n P_k = \alpha I$, then $P_k^\perp$ are orthogonal projections on H such that $\sum_{k=1}^n P_k^\perp = \sum_{k=1}^n (I - P_k) = nI - \sum_{k=1}^n P_k = (n-\alpha)I$. Hence, $(n-\alpha) \in \Sigma_n$. ■

Remark 1. The $*$-algebras $\mathcal{P}_{n,\alpha}$ and $\mathcal{P}_{n,n-\alpha}$ are isomorphic. Therefore, the categories of their $*$-representations, $\operatorname{Rep}\mathcal{P}_{n,\alpha}$ and $\operatorname{Rep}\mathcal{P}_{n,n-\alpha}$, coincide. Indeed, let $\mathcal{P}_{n,\alpha} = \mathbb{C}\langle p_1, \ldots, p_n \mid p_k^2 = p_k^* = p_k, \sum_{k=1}^n p_k = \alpha e\rangle$, and $\mathcal{P}_{n,n-\alpha} = \mathbb{C}\langle \tilde{p}_1, \ldots, \tilde{p}_n \mid \tilde{p}_k^2 = \tilde{p}_k^* = \tilde{p}_k, \sum_{k=1}^n \tilde{p}_k = (n-\alpha)e\rangle$. Then the mapping $p_k \mapsto e - \tilde{p}_k$ defines a $*$-isomorphism of the $*$-algebras $\mathcal{P}_{n,\alpha}$ and $\mathcal{P}_{n,n-\alpha}$.

2.1.2 A description of Σ_n and $*$-representations of the $*$-algebras $\mathcal{P}_{n,\alpha}$, $\alpha \in \Sigma_n$ for $n \leq 4$.

Several papers deal with this problem (see [9, 10, 11, 12, 13, 14] e.a.) The following simple assertion holds.

Proposition 2. (a) $\Sigma_3 = \{0, 1, \frac{3}{2}, 2, 3\}$;

(b) $\mathcal{P}_{3,\alpha} = 0$, *if* $\alpha \notin \Sigma_3$;

(c) $\mathcal{P}_{3,0} = \mathcal{P}_{3,3} = \mathbb{C}^1$, $\mathcal{P}_{3,1} = \mathcal{P}_{3,2} = \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$, $\mathcal{P}_{3,3/2} = M_2(\mathbb{C}^1)$;

(d) *There exists a unique, up to a unitary equivalence, irreducible representation of the algebra* $\mathcal{P}_{3,3/2}$,

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}, \quad P_3 = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}.$$

All the algebras $\mathcal{P}_{4,\alpha}$ are already infinite dimensional; only the algebra $\mathcal{P}_{4,2}$ is a PI-algebra (see [15]). However, Σ_4 and the $*$-representations $\mathcal{P}_{4,\alpha}$, $\alpha \in \Sigma_4$, have a simple structure (see, for example, [14]).

Proposition 3. (a) $\Sigma_4 = \{0, 1, 1 + \frac{k}{k+2} (k \in \mathbb{N}), 2, 3 - \frac{k}{k+2} (k \in \mathbb{N}), 3, 4\}$;

(b) *The $*$-algebra $\mathcal{P}_{4,0}$ has a unique representation,* $P_1 = P_2 = P_3 = P_4 = 0$;

(c) *The $*$-algebra $\mathcal{P}_{4,1}$ has 4 irreducible (nonequivalent, one-dimensional) representations,* $P_1 = \cdots = P_{k-1} = P_{k+1} = \cdots = P_4 = 0$, $P_k = 1$, $k = 1, 2, 3, 4$;

(d) *For odd k, there exists a unique (up to an equivalence) $(k+2)$-dimensional irreducible representation of the $*$-algebra* $\mathcal{P}_{n,1+\frac{k}{k+2}}$;

(e) *For even $k = 2k_1$, there exist four nonequivalent $(k_1 + 1)$-dimensional irreducible representations of the $*$-algebra* $\mathcal{P}_{n,1+\frac{k}{k+2}}$;

(f) *The algebra $\mathcal{P}_{4,2}$ is a PI-algebra. The irreducible $*$-representations of $\mathcal{P}_{4,2}$ are one- and two-dimensional. There exist six nonequivalent one-dimensional representations of $\mathcal{P}_{4,2}$,— two projections equal zero and two projections equal the identity. Nonequivalent two-dimensional representations $\pi_{a,b,c}$ of the $*$-algebra $\mathcal{P}_{4,2}$ depend on points of the set* $\{(a,b,c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1, a > 0, b > 0, c \in [-1,1]$, *or* $a = 0, b > 0, c > 0$, *or* $a > 0, b = 0, c > 0\}$, *the operators of the representation are the following:*

$$\pi_{a,b,c}(p_1) = \frac{1}{2} \begin{pmatrix} 1+a & -b-ic \\ -b+ic & 1-c \end{pmatrix}, \; \pi_{a,b,c}(p_2) = \frac{1}{2} \begin{pmatrix} 1-a & b-ic \\ b+ic & 1+a \end{pmatrix},$$

$$\pi_{a,b,c}(p_3) = \frac{1}{2} \begin{pmatrix} 1-a & -b+ic \\ -b-ic & 1+a \end{pmatrix}, \; \pi_{a,b,c}(p_4) = \frac{1}{2} \begin{pmatrix} 1+a & b+ic \\ b-ic & 1-a \end{pmatrix}.$$

We remark that a proof of items (a) – (e) and the formulas for the operators of the irreducible representations of the $*$-algebras $\mathcal{P}_{4,\alpha}$, $\alpha \in \Sigma_4$, can be obtained from the constructions carried out below for the $*$-algebras $\mathcal{P}_{n,\alpha}$, where $n \geq 4$.

2.2 On Coxeter functors and their properties

2.2.1 Functors of linear and hyperbolic reflections

Let us construct a functor $T : \mathrm{Rep}\,\mathcal{P}_{n,\alpha} \longrightarrow \mathrm{Rep}\,\mathcal{P}_{n,n-\alpha}$, $\alpha < n$. If π is a representation in the category $\mathrm{Rep}\,\mathcal{P}_{n,\alpha}$ and $\pi(p_i) = P_i$ are projections on the space H, then, on the same space, the operators $P_i^{\perp} = I_H - P_i$ define a representation $T(\pi)$ in the category $\mathrm{Rep}\,\mathcal{P}_{n,n-\alpha}$. Functor T is identity on morphisms.

In the sequel, we will call the functor T the *linear reflection functor*. It is clear that $T^2 = Id$, where Id is the identity functor.

Construct now a functor $S : \mathrm{Rep}\,\mathcal{P}_{n,\alpha} \longrightarrow \mathrm{Rep}\,\mathcal{P}_{n,1+\frac{1}{\alpha-1}}$, $\alpha > 1$ (in the proceeding, we call it the *hyperbolic reflection functor*).

Let π be a representation of the algebra $\mathcal{P}_{n,\alpha}$, $\pi(p_i) = P_i$, where P_i are orthogonal projections on the space H. Consider the spaces $H_i = Im\ P_i$ and the natural isometries $\Gamma_i : H_i \longrightarrow H$. Then $\Gamma_i^* : H \longrightarrow H_i$ are epimorphisms and

$$\Gamma_i^*\Gamma_i = I_{H_i}, P_i = \Gamma_i\Gamma_i^*. \tag{2.1}$$

Let $\mathfrak{H} = H_1 \oplus H_2 \oplus \cdots \oplus H_n$. Define the linear operator $\Gamma : \mathfrak{H} \longrightarrow H$ by its Pierce decomposition, $\Gamma = [\Gamma_1, \Gamma_2, \ldots, \Gamma_n]$.

Since $\Gamma\Gamma^* = \sum_{i=1}^n \Gamma_i\Gamma_i^* = \sum_{i=1}^n P_i = \alpha I_H$, we have that $(\frac{1}{\sqrt{\alpha}}\Gamma)(\frac{1}{\sqrt{\alpha}}\Gamma^*) = I_H$, so that $\frac{1}{\sqrt{\alpha}}\Gamma^*$ is an isometry of the space H into $\mathfrak{H}$. Let $\hat{H}$ be the orthogonal complement to $\mathrm{Im}\ \Gamma^*$ in $\mathfrak{H}$.

Denote by $\sqrt{\frac{\alpha-1}{\alpha}}\triangle^*$ the natural isometry of $\hat{H}$ into $\mathfrak{H}$. Then $U^* = \left[\sqrt{\frac{\alpha-1}{\alpha}}\triangle^*, \frac{1}{\sqrt{\alpha}}\Gamma^*\right]$ is a unitary operator from the space $\hat{H} \oplus H$ onto the space $\mathfrak{H}$. Since $\mathfrak{H} = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, the operator U has the following Pierce decomposition:

$$U = \begin{bmatrix} \sqrt{\frac{\alpha-1}{\alpha}}\triangle_1 & \sqrt{\frac{\alpha-1}{\alpha}}\triangle_2 & \ldots & \sqrt{\frac{\alpha-1}{\alpha}}\triangle_n \\ \frac{1}{\sqrt{\alpha}}\Gamma_1 & \frac{1}{\sqrt{\alpha}}\Gamma_2 & \ldots & \frac{1}{\sqrt{\alpha}}\Gamma_n \end{bmatrix},$$

$U : \mathfrak{H} \longrightarrow \hat{H} \oplus H$, $\triangle_i : H_i \longrightarrow \hat{H}$, $\triangle_i^* : \hat{H} \longrightarrow H_i$. Since $U^*U = I_{\mathfrak{H}}$, we have that $\frac{\alpha-1}{\alpha}\triangle_i^*\triangle_i + \frac{1}{\alpha}\Gamma_i^*\Gamma_i = I_{H_i}$ or (since $\Gamma_i^*\Gamma_i = I_{H_i}$) $\triangle_i^*\triangle_i = I_{H_i}$ $(i = 1, \ldots, n)$. Moreover, $\frac{\alpha-1}{\alpha}\triangle_i^*\triangle_j + \frac{1}{\alpha}\Gamma_i^*\Gamma_j = 0$ for $i \neq j$, so that $\triangle_i^*\triangle_j = -\frac{1}{\alpha-1}\Gamma_i^*\Gamma_j$ for $i \neq j$. Since $UU^* = I_{\hat{H}\oplus H}$, we have that $\frac{\alpha-1}{\alpha}(\triangle_1\triangle_1^* + \cdots + \triangle_n\triangle_n^*) = I_{\hat{H}}$, or $\sum_{i=1}^n \triangle_i\triangle_i^* = \frac{\alpha}{\alpha-1}I_{\hat{H}}$. Besides, $\frac{\sqrt{\alpha-1}}{\alpha}\sum_{i=1}^n \triangle_i\Gamma_i^* = 0$, i.e., $\sum_{i=1}^n \triangle_i\Gamma_i^* = 0$. Hence, we have the following formulas:

$$\triangle_i^*\triangle_i = I_{H_i}, i = 1, \ldots, n; \tag{2.2a}$$

$$\sum_{i=1}^n \triangle_i\triangle_i^* = \frac{\alpha}{\alpha-1}I_{\hat{H}}; \tag{2.2b}$$

$$\triangle_i^*\triangle_j = -\frac{1}{\alpha-1}\Gamma_i^*\Gamma_j \text{ for } i \neq j; \tag{2.2c}$$

$$\sum_{i=1}^n \triangle_i\Gamma_i^* = 0. \tag{2.2d}$$

Define now the functor S as follows: $S(\pi) = \hat{\pi}$, where $\hat{\pi}(p_i) = \triangle_i \triangle_i^*$. It is easy to verify that identity (2.2a) implies that $\triangle_i \triangle_i^*$ are orthogonal projections which are denoted in the sequel by Q_i ($Q_i : \hat{H} \longrightarrow \hat{H}$). Identity (2.2b) means that $\sum_{i=1}^n Q_i = \frac{\alpha}{\alpha-1} I_{\hat{H}}$, that is, $\hat{\pi}$ is a representation of the algebra $\mathcal{P}_{n,1+\frac{1}{\alpha-1}}$.

Let C be a morphism from a representation π to a representation $\tilde{\pi}$, i.e., a mapping $C : H \longrightarrow \tilde{H}$ such that $C\pi(p_i) = \tilde{\pi}(p_i)C$. Denote by C_i the restriction of the operator C to the space H_i; C_i maps H_i into the space $\tilde{H}_i$. It is easy to see that

$$C\Gamma_i = \tilde{\Gamma}_i C_i; \tag{2.3a}$$

$$C_i \Gamma_i^* = \tilde{\Gamma}_i^* C. \tag{2.3b}$$

It follows from relations (2.3) that

$$C_i = \tilde{\Gamma}_i^* C \Gamma_i; \tag{2.4a}$$

$$C = \frac{1}{\alpha} \sum_{i=1}^n \tilde{\Gamma}_i C_i \Gamma_i^*. \tag{2.4b}$$

Using a formula similar to formula (2.4b) we set $\hat{C} = \frac{\alpha-1}{\alpha} \sum_{i=1}^n \tilde{\triangle}_i C_i \triangle_i^*$. Let us show that $\hat{C}$ is a morphism from the representation $\hat{\pi} = S(\pi)$ into the representation $\hat{\tilde{\pi}} = S(\tilde{\pi})$, i.e., $\hat{C}\hat{\pi}(p_i) = \hat{\tilde{\pi}}(p_i)\hat{C}$ or $\hat{C}Q_k = \tilde{Q}_k \hat{C}$ $(k = 1 \ldots, n)$.

It will suffice to prove that $\hat{C}\triangle_k = \tilde{\triangle}_k C_k$ and $C_k \triangle_k^* = \tilde{\triangle}_k^* \hat{C}$ (then $\hat{C}Q_k = \hat{C}\triangle_k\triangle_k^* = \tilde{\triangle}_k C_k \triangle_k^* = \tilde{\triangle}_k \tilde{\triangle}_k^* \hat{C} = Q_k \hat{C}$). We have $\hat{C}\triangle_k = \frac{\alpha-1}{\alpha} \sum_{i=1}^n \tilde{\triangle}_i C_i (\triangle_i^* \triangle_k)$. By using (2.2a) and (2.2d) we get $\hat{C}\triangle_k = -\frac{1}{\alpha} \sum_{i=1, i\neq k}^n \tilde{\triangle}_i (C_i \Gamma_i^*) \Gamma_k + \frac{\alpha-1}{\alpha} \tilde{\triangle}_k C_k$. It follows from (2.3b) that $\hat{C}\triangle_k = -\frac{1}{\alpha} \sum_{i=1, i\neq k}^n \tilde{\triangle}_i \tilde{\Gamma}_i^* (C\Gamma_k) + \frac{\alpha-1}{\alpha} \tilde{\triangle}_k C_k$, and (2.3a) yields $\hat{C}\triangle_k = -\frac{1}{\alpha} \sum_{i=1, i\neq k}^n \tilde{\triangle}_i \tilde{\Gamma}_i^* \tilde{\Gamma}_k C_k + \frac{\alpha-1}{\alpha} \tilde{\triangle}_k C_k$. Using (2.4b) we get $\hat{C}\triangle_k = \frac{1}{\alpha} \tilde{\triangle}_k \tilde{\Gamma}_k^* \tilde{\Gamma}_k C_k + \frac{\alpha-1}{\alpha} \tilde{\triangle}_k C_k = \frac{1}{\alpha} \tilde{\triangle}_k C_k + \frac{\alpha-1}{\alpha} \tilde{\triangle}_k C_k = \tilde{\triangle}_k C_k$, what was to be proved.

Similarly, $\tilde{\triangle}_k^* \hat{C} = \frac{\alpha-1}{\alpha} \sum_{i=1}^n \tilde{\triangle}_k^* \tilde{\triangle}_i C_i \triangle_i^* = -\frac{1}{\alpha} \sum_{i=1, i\neq k}^n \tilde{\Gamma}_k^* (\tilde{\Gamma}_i C_i) \triangle_i^* + \frac{\alpha-1}{\alpha} C_k \triangle_k^* = -\frac{1}{\alpha} \sum_{i=1, i\neq k}^n \tilde{\Gamma}_k^* C \Gamma_i \triangle_i^* + \frac{\alpha-1}{\alpha} C_k \triangle_k^* = -\frac{1}{\alpha} \sum_{i=1, i\neq k}^n C_k \Gamma_k^* \Gamma_i \triangle_i^* + \frac{\alpha-1}{\alpha} C_k \triangle_k^* = \frac{1}{\alpha} C_k \Gamma_k^* \Gamma_k \triangle_k^* + \frac{\alpha-1}{\alpha} C_k \triangle_k^* = \frac{1}{\alpha} C_k \triangle_k^* + \frac{\alpha-1}{\alpha} C_k \triangle_k^* = C_k \triangle_k^*$.

Define $S(C) = \hat{C}$. This completes the construction of the functor S.

Remark 2. A more precise notation for the functor S would include indices that indicate the category Rep $\mathcal{P}_{n,\alpha}$ on which it is defined, for example, $S_{n,\alpha}$. But it is more convenient for us to regard the functor S as being the same for each category Rep $\mathcal{P}_{n,\alpha}$, $\alpha > 1$.

Remark 3. The restriction of the constructed unitary operator $U : H_1 \oplus H_2 \oplus \cdots \oplus H_n \longrightarrow \hat{H} \oplus H$ to the subspace H_i is the isometry $\mathcal{B}_i = \begin{bmatrix} \sqrt{\frac{\alpha-1}{\alpha}} \triangle_i \\ \frac{1}{\sqrt{\alpha}} \Gamma_i \end{bmatrix}$ of the space H_i into $\hat{H} \oplus H$, so that the operator $\mathcal{P}_i = \mathcal{B}_i \mathcal{B}_i^*$ is an orthogonal projection in the space $\hat{H} \oplus H$,

$$\mathcal{P}_i = \begin{bmatrix} \frac{\alpha-1}{\alpha} \triangle_i \triangle_i^* & \frac{\sqrt{\alpha-1}}{\alpha} \triangle_i \Gamma_i^* \\ \frac{\sqrt{\alpha-1}}{\alpha} \Gamma_i \triangle_i^* & \frac{1}{\alpha} \Gamma_i \Gamma_i^* \end{bmatrix} = \begin{bmatrix} \frac{\alpha-1}{\alpha} Q_i & \frac{\sqrt{\alpha-1}}{\alpha} \triangle_i \Gamma_i^* \\ \frac{\sqrt{\alpha-1}}{\alpha} \Gamma_i \triangle_i^* & \frac{1}{\alpha} P_i \end{bmatrix}.$$

Using identities (2.2) it is easy to check that $\mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_n = I_{\hat{H} \oplus H}$. We thus have constructed a concrete "joint" dilatation of resolutions of the identity operators $I_{\hat{H}} =$

$\frac{\alpha-1}{\alpha}Q_1+\cdots+\frac{\alpha-1}{\alpha}Q_n$ and $I_H=\frac{1}{\alpha}P_1+\cdots+\frac{1}{\alpha}P_n$ in the spaces $\hat{H}$ and H, correspondingly, to a decomposition of the identity operator in the spaces $\hat{H}\oplus H$ into a sum of orthogonal projections.

Theorem 1. *We have* $S^2=\mathrm{Id}$ (*by* Id *we denote the identity functor on the corresponding category* Rep $\mathcal{P}_{n,\alpha}$). The functor S defines an equivalence between the categories Rep $\mathcal{P}_{n,\alpha}$ and Rep $\mathcal{P}_{n,1+\frac{1}{\alpha-1}}$.

Proof. Since $C_i=\tilde{\Gamma}_i^*C\Gamma_i$, $C_i=\tilde{\triangle}_i^*\hat{C}\triangle_i$, and $C=\frac{1}{\alpha}\sum_{i=1}^n\tilde{\Gamma}_iC_i\Gamma_i^*$, $\widehat{C}=\frac{\alpha-1}{\alpha}\sum_{i=1}^n\tilde{\triangle}_iC_i\triangle_i^*$, we have that the functor S is strict and full. Each representation $\hat{\pi}$ in the category Rep $\mathcal{P}_{n,1+\frac{1}{\alpha-1}}$ is equivalent to one of the representations $S(\pi)$ (for example, $S^2(\hat{\pi})$). The operators Γ_i, $\triangle_i$ enter the matrix U symmetrically, so that $S^2=\mathrm{Id}$. ∎

2.2.2 The Coxeter functors Φ^+ and Φ^-. The Coxeter mappings Φ^+ and Φ^- on Σ_n and on the dimensions of the representations.

Define now the functors Φ^+ and Φ^- as follows: $\Phi^+=ST$ for $\alpha<n-1$, $\Phi^-=TS$ for $\alpha>1$. In what follows, we call these functors the Coxeter functors on the set of categories Rep $\mathcal{P}_{n,\alpha}$.

Theorem 2. *The functors* Φ^+ : Rep $\mathcal{P}_{n,\alpha}\longrightarrow$ Rep $\mathcal{P}_{n,1+\frac{1}{n-1-\alpha}}$, Φ^- : Rep $\mathcal{P}_{n,\alpha}\longrightarrow$ Rep $\mathcal{P}_{n,n-1-\frac{1}{\alpha-1}}$ *define an equivalence of the corresponding categories;* $\Phi^+\Phi^-=\mathrm{Id}$, $\Phi^-\Phi^+=\mathrm{Id}$.

Proof. The proof follows in an evident way from Theorem 1 and a similar assertion for the functor T. ∎

The functors Φ^+, Φ^-, S, T give rise to mappings on the sets of dimensions of representations (in the case where the representations are finite dimensional) and on the set Σ_n. These mappings will be denoted with the same symbols as the functors.

By the generalized dimension of a representation π of the algebra $\mathcal{P}_{n,\alpha}$ on a space H, we will call the vector $(d;d_1,\dots,d_n)$, where $d=\dim H$, $d_i=\dim H_i$ ($H_i=\mathrm{Im}\ P_i$).

It is easy to see how the dimension changes when passing to the representations $S(\pi)$ and $T(\pi)$,

$$\begin{aligned}&S(d;d_1,d_2,\dots,d_n)=(\textstyle\sum_{i=1}^n d_i-d;d_1,d_2,\dots,d_n),\\&T(d;d_1,d_2,\dots,d_n)=(d;d-d_1,d-d_2,\dots,d-d_n).\end{aligned}\tag{2.5}$$

For the set of the generalized dimension, the mappings Φ^+, Φ^- are compositions of the mappings (2.5).

The number-valued mappings T, S, Φ^+, Φ^- on Σ_n are given by $T(\alpha)=n-\alpha$, $S(\alpha)=1+\frac{1}{\alpha-1}$, $\Phi^+(\alpha)=1+\frac{1}{n-1-\alpha}$, $\Phi^-(\alpha)=n-1-\frac{1}{\alpha-1}$. Denote $\Phi^{+k}(\alpha)=\Phi^+(\Phi^{+\ k-1}(\alpha))$ (Φ^{+0} is the identity mapping). Let $\Phi^{+k}(\alpha)=1+\frac{a_{k-1}}{a_k}$ ($k\in\mathbb{N}$). Then $\vec{a}=(a_0,a_1,a_2,\dots)$ is a linear recurrence sequence with the characteristic polynomial $F(x)=x^2-(n-2)x+1$ and the initial vector $(1,n-1-\alpha)$. As is well known, the linear space $L(F)$ of all linear recurrence sequences with a fixed characteristic polynomial $F(x)$ is a module over the polynomial ring $\mathbb{R}[x]$ if setting $x\vec{a}=(a_1,a_2,\dots)$ (a shift to the left by one position). Here, this module is cyclic with a generating element $\vec{e} = (0, 1, \dots)$ which is a recurrence sequence from $L(F)$ having $(0, 1)$ as the initial vector. A polynomial $\phi(x)$ satisfying $\vec{a} = \phi(x)\vec{e}$ is called a generator of the sequence $\vec{a}$.

It is easy to see that $\vec{a} = (1, n-1-\alpha, a_2, a_3, \dots) = (x+1-\alpha)\vec{e}$. Let us first construct the recurrence sequence $\vec{e}$, $\vec{e} = (b_0, b_1, b_2, \dots) \equiv (0, 1, n-2, (n-2)^2-1, (n-2)^3-2(n-2), \dots)$. One easily proves by induction that $b_k = \sum_{i=0}^{[\frac{k-1}{2}]}(-1)^i C_{k-1-i}^i (n-2)^{k-1-2i}$, $k \geq 2$, so that $a_k = b_{k+1} + (1-\alpha)b_k$ and $\Phi^+(\alpha) = 1 + \frac{1}{n-1-\alpha}$, $\Phi^{+2}(\alpha) = 1 + \frac{a_1}{a_2}$, $\Phi^{+k}(\alpha) = 1 + \frac{a_{k-1}}{a_k}$, that is, $\Phi^{+k}(\alpha) = 1 + \frac{\sum_{i=0}^{[\frac{k-1}{2}]}(-1)^i C_{k-1-i}^i (n-2)^{k-1-2i} + (1-\alpha)\sum_{i=0}^{[\frac{k-2}{2}]}(-1)^i C_{k-2-i}^i (n-2)^{k-2-2i}}{\sum_{i=0}^{[\frac{k}{2}]}(-1)^i C_{k-i}^i (n-2)^{k-2i} + (1-\alpha)\sum_{i=0}^{[\frac{k-1}{2}]}(-1)^i C_{k-1-i}^i (n-2)^{k-1-2i}}$.

2.3 About the set Σ_n.

Having constructed the functors S, T, Φ^+, Φ^-, we, at the same time, have proved the following lemma.

Lemma 3. *Let $\alpha \in \Sigma_n$. Then $T(\alpha) \in \Sigma_n$. If $\alpha > 1$, then the number $S(\alpha) \in \Sigma_n$ and the number $\Phi^-(\alpha) \in \Sigma_n$. If $\alpha < n-1$, then $\Phi^+(\alpha) \in \Sigma_n$.*

The following is the main theorem of this section.

Theorem 3. $\Sigma_n = \{\Lambda_n^1, \Lambda_n^2, [\frac{n-\sqrt{n^2-4n}}{2}, \frac{n+\sqrt{n^2-4n}}{2}], n - \Lambda_n^1, n - \Lambda_n^2\}$.

The proof of the theorem is split into two parts, — a description of points of the discrete spectrum and a description of points of the continuous spectrum.

Lemma 4. *The set $\Sigma_n \cap [0; \frac{n-\sqrt{n^2-4n}}{2})$ consists of two sequences of points. The first one is the sequence $x_k = \Phi^{+k}(0)$, $k = 0, 1, \dots$, that makes the set*

$$\Lambda_n^1 = \{0, 1 + \frac{1}{n-1}, 1 + \frac{1}{(n-2) - \frac{1}{n-1}} \dots, 1 + \frac{1}{(n-2) - \frac{1}{(n-2) - \frac{1}{\ddots - \frac{1}{(n-1)}}}}, \dots\}.$$

The second one is $y_k = \Phi^{+k}(1)$, $k = 0, 1, \dots$, that makes the set

$$\Lambda_n^2 = \{1, 1 + \frac{1}{n-2}, 1 + \frac{1}{(n-2) - \frac{1}{n-2}} \dots, 1 + \frac{1}{(n-2) - \frac{1}{(n-2) - \frac{1}{\ddots - \frac{1}{(n-2)}}}}, \dots\} .$$

These two sequences converge to the points $\beta_n = \frac{n-\sqrt{n^2-4n}}{2}$, as $n \to \infty$.

Proof. If $n \geq 4$, we have $\dots < \Phi^{+k}(0) < \Phi^{+k}(1) < \Phi^{+(k+1)}(0) < \Phi^{+(k+1)}(1) < \dots < \beta_n$ and $\lim_{k\to\infty} \Phi^{+k}(0) = \lim_{k\to\infty} \Phi^{+k}(1) = \beta_n$. The open intervals $(\Phi^{+k}(0), \Phi^{+k}(1))$ and $(\Phi^{+k}(1), \Phi^{+(k+1)}(0))$ do not contain the points Σ_n, since $(0,1) \cap \Sigma_n = (1, 1 + \frac{1}{n-1}) \cap \Sigma_n = \emptyset$. Hence, by Theorem 2, Φ^+ is a functor that gives the equivalence. Hence, $\Sigma_n \cap [0; \frac{n-\sqrt{n^2-4n}}{2}) = \Lambda_n^1 \cup \Lambda_n^2$. ■

Let us now prove that $(\beta_n, n-\beta_n) \subset \Sigma_n$. To do this, we will need the following lemmas.

Lemma 5. *If* $[3/2, 2] \subset \Sigma_5$, *then* $[2, n-2] \subset \Sigma_n$.

Proof. Let $[3/2, 2] \subset \Sigma_5$. By applying the functor Φ^- to the line segment $[3/2, 2]$, we get, by Lemma 3, that $[2, 3] \subset \Sigma_5$. Now, using induction on n we prove that $[2, n-2] \subset \Sigma_n$ for $n \geq 5$. Let $k \geq 5$ and $[2, k-2] \subset \Sigma_k$. Since $\Sigma_k \subset \Sigma_{k+1}$, we have that $[2, k-2] \subset \Sigma_{k+1}$. If $\alpha \subset [k-2, (k+1)-2]$, the number $(\alpha - 1) \subset \Sigma_k$, and so there is a representation $P_1 + \cdots + P_k = (\alpha - 1)I$, where P_i are certain projections from $L(H)$. By setting $P_{k+1} = I$, we get $P_1 + \cdots + P_{k+1} = \alpha I$. Hence, $[k-2, (k+1)-2] \subset \Sigma_{k+1}$ and, consequently, $[2, (k+1)-2] \subset \Sigma_{k+1}$. ■

Lemma 6. *If* $[2, n-2] \subset \Sigma_n$, *then* $(\frac{n-\sqrt{n^2-4n}}{2}, \frac{n+\sqrt{n^2-4n}}{2}) \subset \Sigma_n$.

Proof. The mapping Φ^+ is continuous. So, since $\Phi^+(2) = 1 + \frac{1}{n-3}$ and $\Phi^+(n-2) = 2$, we have that $[1 + \frac{1}{n-3}, 2] \subset \Sigma_n$. Now, $\Phi^+(1 + \frac{1}{n-3}) = 1 + \frac{1}{n-2-\frac{1}{n-3}}$ and $\Phi^+(2) = 1 + \frac{1}{n-3}$, that is, $[1 + \frac{1}{n-2-\frac{1}{n-3}}, 1 + \frac{1}{n-3}] \subset \Sigma_n$. By continuing this process, we see that $(\beta_n, 2] \subset \Sigma_n$, where $\beta_n = \lim_{k \longrightarrow \infty} \Phi^{+k}(2) = \frac{n-\sqrt{n^2-4n}}{2}$. Using the mapping T we get that Σ_n contains the interval $[n-2, n-\beta_n)$ and, consequently, $(\beta_n, n-\beta_n) \subset \Sigma_n$. ■

Lemma 7. $(3/2, 2) \subset \Sigma_5$.

Before proving the lemma, let us prove two auxiliary results. Everywhere in the sequel, $\alpha \in (3/2, 2)$ and $\epsilon = \alpha - 1$.

We will need the following definition.

Definition 1. By a *sewing* of the matrices

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & \dots & b_{1l} \\ \vdots & & \vdots \\ b_{l1} & \dots & b_{ll} \end{pmatrix},$$

we mean the matrix of the form

$$\begin{pmatrix} a_{11} & \dots & a_{1m-1} & a_{1m} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & \dots & a_{mm-1} & a_{mm} + b_{11} & b_{12} & \dots & b_{1l} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & b_{l1} & b_{l2} & \dots & b_{ll} \end{pmatrix},$$

which is denoted in the sequel by $A \tilde{+} B$.

It follows directly from the definition that if the matrices $P_1, P_2, \dots, P_k$ are projections, then the matrix $P_1 \tilde{+} P_2 \tilde{+} \dots \tilde{+} P_k$ is a sum of k projections (the matrix P_i is augmented with zero rows and zero columns if necessary as to get the needed dimension). In particular, if $0 \leq x \leq 2$ and $\tau = (x-1)^2$, then the matrix $(1) \tilde{+} \begin{pmatrix} \tau & \sqrt{\tau - \tau^2} \\ \sqrt{\tau - \tau^2} & 1 - \tau \end{pmatrix}$ with the spectrum $\{x, 2-x\}$ is a sum of the two projections, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} \tau & \sqrt{\tau - \tau^2} \\ \sqrt{\tau - \tau^2} & 1 - \tau \end{pmatrix}$.

Then the matrix $\begin{pmatrix} 1-\tau_1 & \sqrt{\tau_1-\tau_1^2} \\ \sqrt{\tau_1-\tau_1^2} & \tau_1 \end{pmatrix} \tilde{+} \begin{pmatrix} x & 0 \\ 0 & 2-x \end{pmatrix}$ is a sum of three projections. It is easy to check that if $\epsilon \le x \le \alpha$, $\tau_1 = \frac{\epsilon(\alpha-x)}{x}$, then it has the spectrum $\{x-\epsilon, \alpha, 2-x\}$.

Let us prove the following statement.

Proposition 4. *Let $\epsilon \le a \le \alpha$ and, for some $k \in \{0,1,2\}$, the inequalities $0 < 2-a-k\epsilon \le \epsilon$ hold. Then the matrix* $\operatorname{diag}\{a, \underbrace{\alpha,\dots,\alpha}_{k \text{ times}}, 2-a-k\epsilon\}$ *is a sum of three projections.*

Proof. The cases where $k = 0$ and $k = 1$ have been considered above. Let $k = 2$ and $0 < 2-a-k\epsilon \le \epsilon$. Set

$$Q = \begin{pmatrix} (a+\epsilon)/2 & \frac{1}{2}\sqrt{2a+2\epsilon-(a+\epsilon)^2} \\ \frac{1}{2}\sqrt{2a+2\epsilon-(a+\epsilon)^2} & 1-(a+\epsilon)/2 \end{pmatrix},$$

$$R = \begin{pmatrix} (a+\epsilon)/2 & -\frac{1}{2}\sqrt{2a+2\epsilon-(a+\epsilon)^2} \\ -\frac{1}{2}\sqrt{2a+2\epsilon-(a+\epsilon)^2} & 1-(a+\epsilon)/2 \end{pmatrix},$$

$\tau_1 = \frac{\epsilon(1-a)}{a+\epsilon}$, $\tau_2 = \frac{\epsilon(a+2\epsilon-1)}{2-a-\epsilon}$. Then the spectrum of the matrix

$$D = \begin{pmatrix} 1-\tau_1 & \sqrt{\tau_1-\tau_1^2} \\ \sqrt{\tau_1-\tau_1^2} & \tau_1 \end{pmatrix} \tilde{+} (Q+R) \tilde{+} \begin{pmatrix} \tau_2 & \sqrt{\tau_2-\tau_2^2} \\ \sqrt{\tau_2-\tau_2^2} & 1-\tau_2 \end{pmatrix}$$

consists of the points a, α, α, $2-a-2\epsilon$, counting the multiplicity. Since

$$\begin{pmatrix} 1-\tau_1 & \sqrt{\tau_1-\tau_1^2} \\ \sqrt{\tau_1-\tau_1^2} & \tau_1 \end{pmatrix} \oplus \begin{pmatrix} \tau_2 & \sqrt{\tau_2-\tau_2^2} \\ \sqrt{\tau_2-\tau_2^2} & 1-\tau_2 \end{pmatrix}$$

is a projection, the matrix D, as well as any matrix that is equivalent to it, can also be represented as a sum of three projections. ∎

Let us now consider sums of five projections.

Proposition 5. *Let $1 \le b \le \alpha$. Then for some $k \in \{1,2,3\}$, we have $0 < 3-b-k\epsilon \le \epsilon$ and there exist five projections $P_1,\dots,P_5$ such that* $\sum_1^5 P_i = \operatorname{diag}\{b, \underbrace{\alpha,\dots,\alpha}_{k \text{ times}}, 3-b-k\epsilon\}$.

Proof. Since $1 \le b \le \alpha$, we have that $\epsilon < 3-\alpha \le 3-b \le 2 \le 4\epsilon$. Whence, $0 < 3-b-k\epsilon \le \epsilon$ for some $k \in \{1,2,3\}$. Let now b and k be fixed and $0 < 3-b-k\epsilon \le \epsilon$. Define

$$Q_1 = \begin{pmatrix} b/2 & \sqrt{b/2-b^2/4} \\ \sqrt{b/2-b^2/4} & 1-b/2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} b/2 & -\sqrt{b/2-b^2/4} \\ -\sqrt{b/2-b^2/4} & 1-b/2 \end{pmatrix}.$$

It follows from Proposition 4 that for the number $a = \alpha-(2-b)$ there exist projections Q_3, Q_4, and Q_5 such that

$$Q_3+Q_4+Q_5 = \operatorname{diag}\{a, \underbrace{\alpha,\dots,\alpha}_{k-1 \text{ times}}, 2-a-(k-1)\epsilon\}.$$

The matrix $D = (Q_1+Q_2)\tilde{+}(Q_3+Q_4+Q_5)$ can be represented as a sum of five projections $P_1,\dots,P_5$ constructed from the matrices $Q_1,\dots,Q_5$ using the sewing operation. At the same time, $D = \operatorname{diag}\{b, \underbrace{\alpha,\dots,\alpha}_{k \text{ times}}, 3-b-k\epsilon\}$. ∎

Remark 4. The matrices $P_1, \ldots, P_5$ from Proposition 5 satisfy the following condition:

$$P_1 \operatorname{diag}\{0, \ldots, 0, 1\} = P_2 \operatorname{diag}\{0, \ldots, 0, 1\} = 0,$$

$$P_3 \operatorname{diag}\{1, 0, \ldots, 0\} = P_4 \operatorname{diag}\{1, 0, \ldots, 0\} = P_5 \operatorname{diag}\{1, 0, \ldots, 0\} = 0.$$

Hence, the matrices $\underbrace{P_i \tilde{+} \ldots \tilde{+} P_i}_{l \text{ times}}$, $l \in \mathbb{N} \cup \infty$, are also projections for $i = 1, \ldots, 5$.

Proof of Lemma 7. Let $b_1 = \alpha$. It follows from Proposition 5 that there exist a number k_1 and projections $P_1^1, \ldots, P_5^1$ such that $\sum_1^5 P_i^1 = \operatorname{diag}\{b_1, \underbrace{\alpha, \ldots, \alpha}_{k_1 \text{ times}}, 3 - b_1 - k_1\epsilon\}$. By choosing $b_2 = \alpha - (3 - b_1 - k_1\epsilon)$ (clearly, $1 \le b_2 \le \alpha$) and using the constructions in Proposition 5, we find projections $P_1^2, \ldots, P_5^2$ such that $\sum_1^5 P_i^2 = \operatorname{diag}\{b_2, \underbrace{\alpha, \ldots, \alpha}_{k_2 \text{ times}}, 3 - b_2 - k_2\epsilon\}$. Continuing this process we choose b_s by the formula $b_s = \alpha - (3 - b_{s-1} - k_{s-1}\epsilon)$ and find a sequence of projections $P_1^s, \ldots, P_5^s$, $s = 1, 2, 3, \ldots$. It follows from Remark 4 that $P_i = P_i^1 \tilde{+} P_i^2 \tilde{+} P_i^3 \tilde{+} \ldots$ are projections in l_2 for each $i \in \{1, 2, 3, 4, 5\}$. Moreover, by the construction, $\sum_1^5 P_i = \alpha I$. ■

Remark 5. Let us note that the inclusion $[3/2; 5/2] \subset \Sigma_5$ is proved in [16] by using another method.

Lemmas 5, 6, and 7 give $(\frac{n-\sqrt{n^2-4n}}{2}, \frac{n+\sqrt{n^2-4n}}{2}) \subset \Sigma_n$.

The proof of the theorem is concluded by the following lemma proved by V. S. Shulman [17] in a more general situation. We will give a proof of the lemma.

Lemma 8. *The set Σ_n is closed.*

Proof. Let $\alpha_k \in \Sigma_n$, $P_j^{(k)} \in L(H_k)$ $(j = 1, \ldots, n)$ be projections such that $\sum_{j=1}^n P_j^{(k)} = \alpha_k I_{H_k}$ and α_k converges to α. Consider the C^*-algebra $\mathcal{A}$ of uniformly norm bounded sequences of operators $X_k \in L(H_k)$ and a closed two-sided $*$-ideal $\mathcal{J}$ of sequences converging to zero with respect to the norm. Consider also the C^*-algebra $\mathcal{B} = \mathcal{A}/\mathcal{J}$. Denote by π the quotient mapping $\mathcal{A} \longrightarrow \mathcal{B}$. Then the projections $P_j = (P_j^{(1)}, P_j^{(2)}, P_j^{(3)} \ldots) \in \mathcal{A}$ define the projections $\pi(P_j) \in \mathcal{B}$. Here $\sum_{j=1}^n \pi(P_j) = \alpha I_{\mathcal{B}}$. Since the abstract C^*-algebra $\mathcal{B}$ is isomorphic to a concrete C^*-algebra of operators, the lemma is proved. ■

References

[1] Klyachko A. A. Stable bundles, representation theory and Hermitian operators. *Selecta Math.* **4** (1998), 419–445.

[2] Fulton W. Eigenvalues, invariant factors, highest weights and Schubert calculus. *Bull. of the AMS* **37** No. 3 (2000), 209-249.

[3] Rabanovich V. I., Samoĭlenko Yu. S. When the sum of idempotents or projectionss is a multiple of the identity. *Funkts. Anal. Prilozhen.* **34**, no. 4 (2000), 91–93; English transl. Funct. Anal. Appl. **34**, no. 4 (2000), 311–313.

[4] Rabanovich V. I., Samoĭlenko Yu. S. When a scalar operator can ber represented as a sum of projections. *Ukr. Mat. Zh.* **53**, no. 7 (2001), 939–952; English transl. Ukr. Math. J. **53**, no. 7 (2001), 1116–1133.

[5] Kruglyak S. A., Rabanovich V. I., Samĭlenko Yu. S. *Funktsional'nyi Analiz i ego Prilozheniya* **36**, No 3 (2002), 20–35.

[6] Kruglyak S. A. Coxeter functors for a certain class of ∗-quivers. *Ukraĭn Mat. Zh.* **54** No. 6 (2002), 789–797 (Russian); translation in *Ukrainian Math. J.* **53** No. 7 (2001).

[7] Bernshtein I. I., Gel'fand I. M., Ponomarev V. A. Coxeter functors an a theorem of Gabriel. *Uspekhi Matematicheskikh Nauk*, Vol. XXVIII, No. 2 (1973) 19–33. (Russian).

[8] Jones V. Index for subfactors. *Invent. Math.* **72** (1983), 1–25.

[9] Fillmore P. A. On sums of projections. *J. Funct. Anal.* **4** (1969), 146–152.

[10] Wang J.-H. Decomposition of operators into quadratic type, Ph.D. dissertation., National Chiao Tung Univ., Hsinchu, Taiwan, 1991.

[11] Wu P. Y. Additive combinations of special operators. In: Functional Analysis and Operator Theory, Banach Center Publ., vol. 30, Warszawa, 1994, pp. 337–361.

[12] Bespalov Yu. V. Collections of orthogonal projections satisfying relations. *Ukraĭn Mat. Zh.* **44**, no. 3 (1992), 309–317 (Russian); translation in *Ukrainian Math. J.* **44**, no. 3 (1992).

[13] Ehrhardt T. Sums of four idempotents. (preprint), Chemnitz, 1995.

[14] Ostrovskiĭ V., Samoĭlenko Yu. Introduction to the Theory Representation of Finitely Presented $*-$algebras. 1. Representations by bounded operators. *Rev. Math. & Math Phys.* vol. 11 1–261, Gordon and Breach (1999).

[15] Strelets A. V., Rabanovich V. I., Samoĭlenko Yu. S. On identities in the algebras $\mathcal{Q}_{n,\lambda}$ generated by idempotents. *Ukraĭn Mat. Zh.* **53**, no. 10 (2001), 1380–1390 (Russian); translation in *Ukrainian Math. J.* **53** no. 10 (2001), 1380–1390.

[16] Ehrhardt T. On sums of five projections equal to a multiple of the identity. (preprint), Chemnitz, 2000.

[17] Shulman V. S. On representations of limit relations. *Meth. Func. Anal. Topology* **7**, no. 4 (2002), 85–87.