On the partial stochastic stability of stochastic differential delay equations with Markovian switching

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Abstract: In the paper, we are concerned with the partial asymptotic stochastic stability (stability in probability) of stochastic differential delay equations with Markovian switching(SDDEwMSs), the sufficient conditions for partial asymptotic stability in probability have been given and we have generalized some results of Sharov and Ignatyev to cover a class of much more general SDDEwMSs.

Introduction

Stochastic differential equations (SDEs) have attracted much attention since they are not only academically challenging but also of practical importance, and have played an important role in many ways such as in insurance, finance, population dynamic. Much more papers are concerned with the global stability of SDDEwMSs with respect to sample paths or moments. However, in many practical systems, such stability is sometimes too strong to be satisfied. So the notion of the partial stability (e.g. Peiffer and Rouche (1969), Rouche et al. (1977) ect.)has been involved, and the Second Method of Lyapunov as an indispensable tool has been used to investigated the partial stability (Sontag and Wang (2001), Vorotnikov (1998), Vorotnikov and Rumyantsev (2001)). In the process of investigating the qualitative properties of equilibria and boundedness properties of motions of dynamical systems, the partial stability plays the key role, and the systems are often determined by all kinds of equations, including stochastic differential equations, of course.

To the best of our knowledge, the results of the partial asymptotic stability in probability of stochastic differential delay equations are beautiful. However, another problem appears on the work, as we all known, the dynamical systems often jump from this state into another state with the probability, so the markov chain becomes very popular in recent years, because it is extensively used to deal with the jump phenomena to obtain the good results, and the investigation of SDDEwMSs is quite necessary. In this paper, motivated by the previously mentioned problems, we make the attempt to study this topic to fill the gap.

This paper is organized as follows. In section 2, we present some basic preliminaries and the form of stochastic differential delay equations with Markovian switching. In section 3, the sufficient conditions for partial asymptotic stability in probability have been obtained and proof has been given. In section 4, some well-known results are generalized in the remarks.

Preliminaries and definitions

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P\}$ be a complete probability space with a filtration satisfying the usual conditions, i.e., the filtration is continuous on the right and $\mathcal{F}_0$ contains all P-zero sets. $B(t)$ is a standard Brownian motion defined on the probability space. Let $C([-\tau,0];R)$ denote the family of functions φ from $[-\tau,0]$ to R that are right-continuous and have limits on the left. $C([-\tau,0];R)$ is equipped with the norm $\|\varphi\| = \sup_{-\tau\le s\le 0} |\varphi(s)|$ and $\|x\| = \sqrt{x^T x}$ for any $x \in R^n$. If A is a vector or matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^T A)}$, while its operator norm is denoted by $\|A\| = sup\{|Ax|:|x|=1\}$ .Denote by $C^b_{\mathcal{F}_0}([-\tau,0];R)$ the family of all bounded, $\mathcal{F}_0$ measurable, $C([-\tau,0];R^n)$-valued random variables. Let $p>0, t\ge 0$, $L^p_{\mathcal{F}_t}([-\tau,0];R)$ denote by the family of all $\mathcal{F}_t$ measurable, $C([-\tau,0];R)$ -valued random variables $\varphi = \{\varphi(\theta): -\tau \le \theta \le 0\}$,and $\sup_{-\tau\le\theta\le 0} E|\varphi(\theta)|^p < \infty$.

Let $\{r(t), t \in R_+ = [0,+\infty)\}$ be a right-continuous Markov chain on the probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\ge 0}, P\}$ taking values in a finite state space $S = \{1,2,...,N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P(r(t+\Delta) = j \mid r(t) = i) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & if \;\; i \ne j \\ 1+\gamma_{ii}\Delta + o(\Delta), & if \;\; i = j \end{cases}.$$

where $\Delta > 0$,Here $\gamma_{ij} \ge 0$ is the transition rate from i to j ,if $i \ne j$ .while $\gamma_{ii} = -\sum_{j\ne i}\gamma_{ij}$.We assume that Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$ .It is known that almost every sample path of $r(t)$ is right continuous step function with a finite number of simple jumps in any finite sub-interval of R_+.

Consider the following stochastic differential delay equations with markovian switching: for $\tau > 0$

$$dX(t) = f(t, X(t), X(t-\tau), r(t))dt + g(t, X(t), X(t-\tau), r(t))dB(t), t > 0 \qquad (1)$$

with the initial condition $X_0 = \xi = (\xi_1, \xi_2)^T \in C([-\tau,0];R^n)$,where $\xi_1 \in R^k$ and $\xi_2 \in R^p$, $k+p=n$, which is independent of $B(\cdot)$. Here, we furthermore assume that:

$$f: R_+ \times R^n \times R^n \times S \to R^n, g: R_+ \times R^n \times R^n \times S \to R^{n\times m}.$$

For our purpose, Let f and g satisfy local Lipschitz and linear growth condition, which can ensure the existence and uniqueness of solution, denoted by $X(t)$ on $t>0$ for Eq.(1).

Denote $X = (X_1, X_2)^T \in R^n$, where $X_1 \in R^k$ and $X_2 \in R^p$, $k+p=n$ .The domain $B_K = \{X \in R^n : \|X_1\| < K, \|X_2\| < \infty\}$, and the stopping time σ_B is the first exit time from the B_K of the sample path of the process $X(t)$ .Denote the set of functions $\mathcal{C} := \{\varphi: R^+ \to R^+$, continuous, monotonically increasing and $\varphi(0) = 0\}$.

Denote by $C^2(R_+ \times R^n \times S; R_+)$ the family of all non-negative functions $V(t,x,i)$ on $R_+ \times R^n \times S$, which are twice continuously differential with respect to x .For any $(t,x,y,i) \in R_+ \times R^n \times R^n \times S$, define an operator LV by

$$LV(t,x,y,i) = V_t(t,x,i) + V_x(t,x,i)f(t,x,y,i) + \frac{1}{2}trace[g^T(t,x,y,i)V_{xx}(t,x,i)g(t,x,y,i)] + \sum_{j=1}^{N}\gamma_{ij}V(t,x,j),$$

where

$$V_t(t,x,i) = \frac{\partial V(t,x,i)}{\partial t}, V_x(t,x,i) = (\frac{\partial V(t,x,i)}{\partial x_1}, \cdots, \frac{\partial V(t,x,i)}{\partial x_n}), V_{xx}(t,x,i) = (\frac{\partial^2 V(t,x,i)}{\partial x_i \partial x_j})_{n\times n}.$$

Definition 2.1. The trivial solution $X(t)$ of Eq.(1) is said to be partially stability in probability with respect to $X_1(t)$, if for any $\epsilon > 0, \epsilon_1 > 0$ and $t_0 \ge 0$, there exists $\delta = \delta(\epsilon, \epsilon_1, t_0) > 0$ such that $P\{\sup_{t\ge t_0}\|X_1(t)\| > \epsilon\} < \epsilon_1$, whenever $\|\xi\| < \delta$.

Definition 2.2. The trivial solution $X(t)$ of Eq.(1) is said to be partial asymptotic stability in probability with respect to $X_1(t)$, if it is the $X_1(t)-$ stable in probability and $\lim_{\xi\to 0} P\{\lim_{t\to\infty}\|X_1(t)\| = 0\} = 1.$

Main results

*Theorem 3.1.*Let there exists a nonnegative functional $V(t,x,i) \in C^2(R_+ \times B_K \times S; R_+)$, such that
(I) $\varphi_1(\|X_1\|) \le V(t,X,i) \le \varphi_2(\|X_1\|)$, where $\varphi_1, \varphi_2 \in \mathcal{C}$;
(II) $LV \le 0$;
(III) For any sufficiently small $\eta > 0$ and $\theta > 0$,any solution $X(t)$ of Eq.(1), beginning in the domain $\{\eta < \|X_1(t)\| < \theta\}$,such that $\rho = inf\{t : X_1(t) = \eta\} < \infty, a.s.$
Then the trivial solution of Eq.(1) is said to be partial asymptotic stability in probability with respect to $X_1(t)$.

Now, before giving proof of Theorem 3.1, we should present a lemma.

*Lemma 3.1.*Let $V(t,x,i)$ be a function in class $C^2(R_+ \times B_K \times S; R_+)$, bounded in the domain $(R_+ \times B_K \times S; R_+)$, and suppose that $LV \le 0$ in this domain. Then the process $V(\sigma_B \wedge t, X(\sigma_B \wedge t), r(t))$ is a super-martingale, so that $EV(\sigma_B \wedge t, X(\sigma_B \wedge t), r(t))$, $\le V(t_0, X(t_0), i_0)$ for $X(t) \in B_K$.

Proof. The proof is very similar with the Lemma appeared in the Has'minskii(1980), so we omit it.

Proof of Theorem 3.1. As the lemma 3.1 and Doob(1953),the process $V(\sigma_B \wedge t, X(\sigma_B \wedge t), r(t))$ is a super-martingale, and we can get

$$\lim_{t\to\infty} V(\sigma_B \wedge t, X(\sigma_B \wedge t), r(t)) = c, a.s.. \qquad (2)$$

Denote $\Omega_1 = \{\omega : \sigma_B = \infty\}$, owing to the $LV \le 0$ and the theorem(Sharov 1978), the solution of Eq.(1) is the partially stability in probability with respect to $X_1(t)$, so we have $\lim_{\xi\to 0} P(\Omega_1) = 1.$ Combining with condition(III) and all the paths of Ω_1 ,then $\inf_{t>0}\|X_1(t)\| = 0.$ By the lemma of Has'minskii(1980), that is, the coefficients of Eq.(1) satisfy the local Lipschitz and linear growth condition in every domain bounded, and the process $X(t)$ is regular, then the set $\{\omega : X_1(t,\omega) = 0\}$ is inaccessible to any sample path of the process if $\xi_1 \ne 0$, where $\xi = (\xi_1, \xi_2)^T$. So we get

$$\liminf_{t\to\infty} \|X_1(t)\| = 0. \qquad (3)$$

Combining with condition(I) and Eq.(3), it follows that

$$\liminf_{t\to\infty} V(t, X(t), r(t)) = 0. \qquad (4)$$

As the Eq.(2), we have obtained

$$\lim_{t\to\infty} V(\sigma_B \wedge t, X(\sigma_B \wedge t), r(t)) = \lim_{t\to\infty} V(t, X(t), r(t)) = c, a.s.. \qquad (5)$$

By the Eq.(4) and Eq.(5), we have the limit of above equation $c = 0$, since the condition (I) and all the paths of Ω_1, we can implies that

$$\lim_{t\to\infty} \|X_1(t)\| = 0, a.s.. \qquad (6)$$

So we have obtained the trivial solution of Eq.(1) is partial asymptotic stability in probability with respect to X_1 with $\lim_{\xi\to 0} P(\Omega_1) = 1$ and Eq.(6).

Remarks

*Remark 4.1.*When $\tau \equiv 0$ and $r(t) \equiv 0$, equation (1) reduces to

$$dX(t) = f(t, X(t))ds + g(t, X(t))dB(t), t > 0 \qquad (7)$$

which is recently studied in Ignatyev(2009), that is to say, the theorem 3.1 of Ignatyev(2009) has been generalized.

Remark 4.2. The operator LV of equation Eq.(7) of Sharov(1978) must be negative, but we can obtain the partial asymptotic stability in probability of Eq.(1) under LV being non-positive($LV \le 0$), which makes the conditions of solution be more feasible to be satisfied.

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References

[1]Doob, J.L.. Stochastic Processes. Chapman and Hall, New York, 1953.

[2]Sharov,V.F.. Partial stability and stabilization of stochastic systems. Avtom. Telemekh.11(1978), 63-71 (in Russian).

[3]Has'minskii,R.Z.. Stochastic Stability of Differential Equations. Sijthoand Noordho,Rockville, Maryland, 1980.

[4]Mao,X.,Yuan,C.. Stochastic Differential Equations with Markovian Switching. Imperical College Press, 2006.

[5]Kolmanovskii, V., Koroleva, N., Maizenberg, T., Mao, X., Matasov, A.. Neutral stochastic differential delay equations with Markovian switching. Stoch. Anal. Appl. 21(2003), 819-847.

[6]Ignatyev, O.A.. Partial asymptotic stability in probability of stochastic differential equations. Statistics and Probability Letters. 79(2009) ,597-601.

[7]Liu D,Yang Y. Doubly perturbed neutral diffusion processes with Markovian switching and Poisson jumps. Applied Mathematics Letters,23(2010),1141-1146.