Construction of compactly supported orthonormal symmetric and antisymmetric two-direction refinable functions and its corresponding wavelet with scale M

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Abstract: In this paper, the concepts and related conclusions of two-direction multiresolution analysis and two-direction refinable functions are introduced; two-direction wavelet with scale 2 is researched and is extended to orthonormal symmetric and antisymmetric two-direction wavelet with scale M and relevant properties are obtained. A condition that L^2 stable solution of two-direction refinable functions can create for MRA is given; finally construction of two-direction wavelet by two-direction compactly orthonormal refinable functions is discussed.

Introduction

Wavelet analysis theory originated in the 1970 s, it is a rapid development of new fields. At present, it has been widely used in signal analysis, image processing, Quantum mechanics, theoretical physics; Military electronic countermeasures and weapons intelligent; Computer classification and recognition; Music and language of synthetic; Medical imaging and diagnosis; Seismic data processing; Large mechanical fault diagnosis, etc [1-4]. The different wavelet has been constructed according to the actual needs. Daubechies [5] have proved that in addition to haar wavelet beyond, for with orthogonal, symmetry and compact support with 2-band wavelet was not exist. In order to solve this problem, People have taken 2-band wavelet spread to multi- wavelet, multidimensional wavelets and made a deep research and gave the structure method. However, due to the Multiwavelets its limitations, there may be problems in the practical application. Literature [10-11] have detailed explore the multiwavelets balance problems.

Yang S Z [12-13] put forward the concept of two-direction refinable functions, and introduces the bidirectional orthogonal scaling function and wavelet function, and then got a few more practical theory and application results. Two-direction refinement equation is the promotion of refinement equation, So based on the bidirectional orthogonal refinement function can get good nature of the two-direction orthogonal wavelet. Xie C Z [14-15] researched the nature of the two-direction wavelet, and gives the structure two-direction wavelet method and example. This text combining with the two-direction wavelet, symmetric antisymmetric and promote literature 2-scale orthogonal symmetric wavelet to M-scale orthogonal symmetric and antisymmetric. Symmetric and antisymmetric two direction wavelet as a kind of new wavelet, it can be more widely used in more fields in real life.

Orthonormal Two-Direction Refinable Function and Two-Direction Multi-Resolution Analysis

Suppose $\phi \in L_2(R)$ is two-direction refinable functions, satisfies

$$\phi(x) = \sum_{k\in Z} p_k^+ \phi(mx-k) + p_k^- \phi(k-mx), \tag{1}$$

In this formula $\{p_k^+\}$ is $m(m \geq 2)$ forward scale sequence, $\{p_k^+\}$ is m negative scale sequence.

For (1) on both sides simultaneously Fourier transform has:

$$\hat{\phi}(\omega) = P^{+}(e^{-i\omega/m})\hat{\phi}(\omega/m) + P^{-}(e^{-i\omega/m})\hat{\phi}(\omega/m), \tag{2}$$

where $p^{+}(z) = (1/m)\sum_{k\in Z} p_k^{+} z^k$, is called forward two scale sequence symbol , and $p^{-}(z) = (1/m)\sum_{k\in Z} p_k^{-} z^k$ is called negative two scale sequence symbol.

Definition 1 If there are two positive constant A and B, such that

$$A\sum_{k\in Z} c_k c_k^{*} \le \| \sum_{k\in Z} c_1^k \phi(x-k) + \sum_{k\in Z} c_2^k \phi(k-x) \|_2^2 \le B\sum_{k\in Z} c_k c_k^{*} \tag{3}$$

for any sequences $\{c_k\}_{k\in Z} = \{[c_1^k, c_2^k]\}_{k\in Z} \in l^2(Z^2)$. In order to study the solution existence of (1), Though the deformation of (1), we get that

$$\phi(-x) = \sum_{k\in Z} p_k^{+}\phi(-mx-k) + p_k^{-}\phi(k+mx). \tag{4}$$

Let

$$\Phi(x) = \begin{bmatrix} \phi(x) \\ \phi(-x) \end{bmatrix} = \sum_{k\in Z} \begin{bmatrix} p_k^{+} & p_k^{-} \\ p_{-k}^{-} & p_{-k}^{+} \end{bmatrix} \begin{bmatrix} \phi(mx-k) \\ \phi(k-mx) \end{bmatrix} = \sum_{k\in Z} \begin{bmatrix} p_k^{+} & p_k^{-} \\ p_{-k}^{-} & p_{-k}^{+} \end{bmatrix} \Phi(mx-k) \tag{5}$$

Now, we introduce autocorrelation matrix symbol of the two-direction refinable function $\phi(x)$:

$$. \quad \Omega(\omega) = \sum_{k\in Z} \begin{bmatrix} <\phi(x),\phi(x-k)> & <\phi(x),\phi(k-x)> \\ <\phi(-x),\phi(x-k)> & <\phi(-x),\phi(k-x> \end{bmatrix} e^{-ikx} \tag{6}$$

If 1is a characteristic value of a matrix or operator, and module of other characteristic value is less than 1, the matrix or operator satisfies the condition E .

Define sequences of subspaces $\{V_j\} \subset L^2(R)$, where

$$V_j = close_{L^2(R)} < m^{j/2}\phi(m^j x - k), m^{j/2}\phi(l - m^j x) > \tag{7}$$

and, then can generate a two-direction multiresolution analysis if and only if of (7) satisfy the following conditions:

(i) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$; (ii) $close_{L^2(R)}\{\bigcup_{j\in Z} V_j)\} = L^2(R)$;

(iii) $\bigcap_{j\in Z}(V_j) = \{0\}$; (iv) $f(x) \in V_j$ if and only if $f(mx) \in V_j$;

(v) $\{\phi(x-k), \phi(n-x), k, n \in Z\}$ be a Riesz base of V_0.

Definition 2 Let $\phi(x)$ is compact support bidirectional refinable function, we call that $\phi(x)$ is two-direction orthogonal refinable function if

$$<\phi(x),\phi(x-k)> = \delta_{0,k}, \quad <\phi(x),\phi(n-x)> = 0, \quad k,n \in Z ,$$

hold. Multi-resolution analysis which a two-direction orthogonal refinable function $\phi(x)$ generates is called two-direction orthogonal multiresolution analysis.

Definition 3 If the function $g(x)$ is symmetric (antyisymmetric) about b , then $g(b+x) = g(b-x)$ ($g(b+x) = -g(b-x)$) holds for all .

By the above definition, For two-direction wavelet functions and two-direction refinable functions

$$\psi(x) = \sum_{k\in Z} q_k^{+}\phi(mx-k) + \sum_{k\in Z} q_k^{-}\phi(k-mx), \phi(x) = \sum_{k\in Z} p_k^{+}\phi(mx-k) + \sum_{k\in Z} p_k^{-}\phi(k-mx).$$

They are symmetric or antyisymmetric if and only if $\phi(mx-k)$ and $\phi(k-mx)$ are symmetric or antyisymmetric.

Theorem 1 The orthogonal compactly supported two-direction refinable function $\phi(x)$ is antisymmetric about point a , and such that $\phi(x) = \sum_{k\in Z} p_k^{+}\phi(mx-k) + \sum_{k\in Z} p_k^{-}\phi(k-mx).$

Then we have $p_k^{+} = -p_{ma-k}^{+}, p_k^{-} = -p_{ma-k}^{-}$.

Proof. Since $\phi(x)$ is antisymmetric about point a, we know that $\phi(x+a)=-\phi(a-x)$,

i.e, $\phi(x+a)=\sum_{k\in Z} p_k^+\phi(mx+ma-k)+\sum_{k\in Z} p_k^-\phi(k-mx-ma).$

$\phi(a-x)=\sum_{k\in Z} p_k^+\phi(ma-ma-k)+\sum_{k\in Z} p_k^-\phi(k-ma+mx).$

Since $\phi(mx-k)$ and $\phi(k-mx)$ are both antisymmetric, combined with Lemma 4[16], we easily that get that $p_k^+=-p_{ma-k}^+, p_k^-=-p_{ma-k}^-$, .

In the same way, For symmetrical and refinable function $\phi(x)$, we have $p_k^+=p_{ma-k}^+, p_k^-=p_{ma-k}^-$. Analog Theorem 2, we can get to meet the two-direction wavelet function $\psi(x)=\sum_{k\in Z} q_k^+\phi(mx-k)+\sum_{k\in Z} q_k^-\phi(k-mx).$ Then $\psi(x)$ is symmetric or antisymmetric about point a ,we have

$$q_k^+=q_{ma-k}^+, q_k^-=q_{ma-k}^-, q_k^-=q_{ma-k}^-;\text{or } q_k^+=-q_{ma-k}^+, q_k^-=-q_{ma-k}^-.$$

Theorm 2 Let $\phi(x)$ is a orthogonal two-direction and refinable function compactly supported symmetric and antisymmetric ,and such that

$$\phi(x)=\sum_{k=0}^{n} p_k^+\phi(mx-k)+\sum_{k=0}^{n} p_k^-\phi(k-mx)$$

Define $\Phi(x)=(\phi(mx),\phi(mx-1),...,\phi(mx-(m-1)))^T$, and satisfies the conditions of the Theorem 1 , then $\Phi(x)$ is a orthogonal Compactly supported two-direction refinable function, and the support set $\Phi(x)$ is the subset of

$$[-\frac{n}{(m-1)m},\frac{n}{(m-1)m}+\frac{m-1}{m}].$$

Proof:

$<\Phi(x),\Phi(x-n)>=\int_R \Phi(x)\overline{\Phi(x-n)dx}.$

$=\int_R [\phi(mx),...,\phi(mx-(m-1)]^T[\phi(mx-mn),...,\phi(mx-mn-(m-1)]dx=\delta_{0,n}I.$

That completes Orthogonality.

Symmetry Since every $\phi(mx),...,\phi(mx-(m-1))$ is symmetric or antisymmetric, we get easily that $\Phi(x)$ is symmetric or antisymmetric.

Compacted support By reference [13] , we know that the support set of $\phi(x)$ is a subset of $[-\frac{n}{m-1},\frac{n}{m-1}]$, so the support set of $\phi(mx)$ is $[-\frac{n}{(m-1)m},\frac{n}{(m-1)m}]$, and the support set of $\phi(mx-(m-1))$ is $[-\frac{n}{(m-1)m}+\frac{m-1}{m},\frac{n}{(m-1)m}+\frac{m-1}{m}]$. Therefore, the support interval of $\Phi(x)$ is the subset of $[-\frac{n}{(m-1)m},\frac{n}{(m-1)m}+\frac{m-1}{m}]$.

According to Theorem 2 , we can similarly construct two-direction orthogonal symmetric and antisymmetric compactly supported wavelet functions.

Conclusion

In this paper, two-direction and refinable function and two-direction multi-resolution analysis concept are introduced, and properties of he m scale compactly supported orthogonal symmetric and antisymmetric two-direction wave are studied . In addition, we get general methods of construction of the compactly supported orthogonal symmetric and antisymmetric two-direction and refinable function corresponding bidirectional wavelet As for concrete construction methods and actual applications remain to be further research.

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