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Scattering and Spectral Singularities for some Dissipative Operators of Mathematical Physics [1]

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Abstract

Analogies in the spectral study of dissipative Schrödinger operator and Boltzmann transport operator are analyzed. Scattering theory technique together with functional model approach are applied to construct spectral representations for these operators.

1. The topic of this survey is scattering and spectral analysis of some non selfadjoint operators of mathematical physics appearing in the study of processes with radiation or dissipation of energy. The properties of the corresponding dynamical semigroups are determined by the spectral structure of non selfadjoint Hamiltonians while scattering approach is the way to describe the time asymptotic behavior of solutions to corresponding evolutionary equations.

The following questions to be discussed are of special interest in this context:

- localization of the spectrum of non selfadjoint Schrödinger-type operators;
- the so-called separation of spectral components and closely related spectral decomposition problem;
- spectral singularities and absolute continuity of continuous spectrum component;
- construction of wave operators and their stationary representation (i.e. time-independent description).

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Some recent results in these directions will be presented here for the case of two operator models, namely *dissipative Schrödinger operator* and *Boltzmann integro-differential operator* appearing in the theory of particles transport.

First of all the pioneering work [1] by T. Kato on non-stationary scattering theory for non selfadjoint operators must be mentioned. As an application of his general approach Kato considered in $\mathcal{H} = \mathrm{L}^2(0,\infty)$ Schrödinger operator

$$L \,=\, L_0 + V \,=\, -\frac{d^2}{dx^2} \,+\, V(x)$$

corresponding to Dirichlet boundary condition at zero. Complex potential $V(x)$ was supposed to be small enough in the following sense

$$\int_0^\infty x\,|V(x)|\,dx \,<\, 1\,. \tag{$*$}$$

Under this assumption operator L has purely continuous spectrum $\sigma_{\mathrm{c}}(L) = \mathbb{R}_+$ and moreover L is equivalent to selfadjoint operator L_0. If condition $(*)$ is not fulfilled operator L may have eigenvalues and in this setting the question arises: when space $\mathcal{H}$ can be decomposed into the direct sum of invariant subspaces

$$\mathcal{H} \,=\, \mathcal{H}_d \,+\, \mathcal{H}_{\mathrm{c}}$$

so that spectrum of L in $\mathcal{H}_d$ is discrete while restriction of L to the subspace $\mathcal{H}_{\mathrm{c}}$, L_{c} for notation, is similar to L_0.

The second operator model in question comes from the theory of particles transport and is related to the linearized Boltzmann equation. In the space $\mathrm{L}^2(\mathbb{R} \times [-1,1])$ consider the operator

$$(\,L = L_0 + V \,=\,)\;\, i\,\mu\,\frac{\partial}{\partial x} \,+\, i\,c(x)\int_{-1}^{1} \cdot\; d\mu$$

The so-called propagation coefficient $c(x)$, $x \in \mathbb{R}$, is a bounded nonnegative function. This operator setting is due to K. Friedrichs (and is called Friedrichs model in particles transport theory). The spectral structure of L in the case when $c(x)$ is proportional to the indicator of a segment was studied in detail by J. Lehner and G. Wing [2].

I hope that it cannot bring us to misunderstanding if we shall use the same notation $L = L_0 + V$ in this situation as well. In what follows some of the results can be formulated in a common way for Schrödinger and Boltzmann operators and moreover those results are obtained by similar methods. This is the reason for such a usage of notation. Anyway it will be specified which of the two operators is ment.

2. Let us first discuss the structure of the spectrum and localization of spectral components for the operators introduced in section 1.

To deal with Schrödinger operator introduce Jost solution $e(x,k)$ of the equation

$$-y'' + V(x)\,y = k^2\,y$$

with asymptotic behavior $e(x,k) \sim e^{ikx}$, $x \to \infty$, and denote $e(k) := e(0,k)$. Function $e(k)$ is analytic and bounded in the upper half-plane $\mathbb{C}_+$ provided

$$\int_0^\infty |V(x)|\,dx < \infty\,.$$

Under this assumption the spectrum of the operator $L = -\dfrac{d^2}{dx^2} + V(x)$ consists of two components:

$$\text{continuous } \sigma_{\rm c}(L) = \mathbb{R}_+ \quad \& \quad \text{discrete } \sigma_d(L) = \{k^2 : e(k) = 0\}.$$

The set of eigenvalues is bounded and its accumulation points (if any) belong to $\mathbb{R}_+$. There are no positive eigenvalues anyway, while real zeroes of $e(k)$ give rise to the so-called *spectral singularities* (cf. [3]). In the case of compactly supported potential $V(x)$ spectral singularities are the poles of analytic continuation of the resolvent kernel; when λ approaches spectral singularity the resolvent $R(\lambda) = (L - \lambda I)^{-1}$ grows in norm faster than $\big(\mathrm{dist}\{\lambda, \sigma(L)\}\big)^{-1}$.

Proposition 1. *Under the condition*

$$\int_0^\infty x\,|V(x)|\,dx < \infty$$

function $e(k)$ is continuous in the closed upper half-plane and discrete spectrum $\sigma_d(L)$ is finite provided operator L has no spectral singularities.

Now the location of the spectral components for Boltzmann transport operator $L = i\,\mu\,\dfrac{\partial}{\partial x} + i\,c(x)\displaystyle\int_{-1}^{1} \cdot \; d\mu$ will be described. Assume that $c(x) \in \mathrm{L}^\infty(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ and $c(x) \geq 0$. Then

$$\sigma(L) = \sigma_{\rm c}(L) \cup \sigma_d(L)\,, \quad \sigma_{\rm c}(L) = \mathbb{R}\,, \quad \sigma_d(L) \subset i\,\mathbb{R}_+\,.$$

The eigenvalues in the upper half-plane correspond to unstable modes of the initial problem for the linearized Boltzmann equation. This is why the estimate of their number is of particular interest.

Proposition 2. *All eigenvalues of Boltzmann operator are semi-simple* (*i.e. corresponding Jordan blocks are diagonal*)*. The total multiplicity $N(c)$ of eigenvalues admits the following estimate*

$$N(c) \;\leq\; 1 + \iint c(x)\,\ln^2|x-y|\,c(y)\,dx\,dy\,.$$

Remark. The right-hand side here is finite if $c(x) = O(|x|^{-\alpha})$, $|x| \to \infty$, $\alpha > 1$, and finiteness of the right-hand side implies that $c(x) \in \mathrm{L}^1(\mathbb{R})$. The estimate for $N(c)$ is obtained by a variant of the method known in the case of Schrödinger operator as *Birman-Schwinger principle* [4].

Now let us consider a one-parameter family $L_0 + \tau V$. When τ increases new eigenvalues appear from zero and their number $N(\tau c)$ increases. At the very moment when the new eigenvalue appears there is a spectral singularity at zero which is not an eigenvalue however. A somewhat similar picture can be seen when the family of Schrödinger operators $L_0 + \tau V$ is considered. Provided

$$\int_0^\infty x\,|V(x)|\,dx \;<\; \infty$$

for τ sufficiently small Kato condition $(*)$ is satisfied and there are no eigenvalues at all. As τ increases eigenvalues arise from continuous spectrum; moreover those points of the continuous spectrum from where the eigenvalues appear are just spectral singularities.

3. Now we pass to consideration of continuous spectrum components for the operators in question. Denote by $\mathcal{P}$ a projection onto the linear span of root vectors of L given by the formula

$$\mathcal{P} \;=\; \frac{1}{2\pi i}\int_\Gamma R(\lambda)\,d\lambda$$

where the contour Γ separates σ_d and σ_{c}. Projection $\mathcal{P}$ induces the decomposition

$$\mathcal{H} \;=\; \mathcal{H}_d \,+\, \mathcal{H}_{\mathrm{c}}\,.$$

Subspace $\mathcal{H}_d = \mathcal{P}\mathcal{H}$ corresponds to $\sigma_d(L)$ in the sense that restriction L_d of L to $\mathcal{H}_d$ has discrete spectrum $\sigma(L_d) = \sigma_d(L)$.

Part L_{c} (we are interested in) is the restriction of L to the subspace $\mathcal{H}_{\mathrm{c}} = (I - \mathcal{P})\mathcal{H}$, called *continuous spectrum subspace* since $\sigma(L_{\mathrm{c}}) = \sigma_{\mathrm{c}}(L)$.

For studying operator L in the subspace $\mathcal{H}_{\mathrm{c}}$ it makes sense to apply non-stationary scattering theory technique. It enables us to answer the following question due to F. Berezin (see [5]): whether there is an equivalence of L_{c} to the unperturbed operator L_0 and, if so, construct the wave operators implementing this equivalence. The first step to this end is

Theorem 1. *Suppose that potential $V(x)$ is bounded and $\operatorname{Im} V(x) \geq 0$. If the integral*

$$\int^\infty x\,\ln^\alpha x\,|V(x)|^2\,dx \tag{1}$$

converges for $\alpha > 5/2$ then there exists a bounded direct wave operator

$$\Omega \;=\; s\text{-}\lim_{t\to\infty}\;\exp(itL)\,\exp(-itL_0)$$

that intertwines L and L_0 : $\;L\,\Omega = \Omega\,L_0$.

In order to establish the existence of the strong limit in this proposition (following the procedure known as *Cook's criterion*) it suffices to show that the derivative

$$\frac{d}{dt}\exp(itL)\exp(-itL_0)\,\varphi \;=\; i\,\exp(itL)\,V\exp(-itL_0)\,\varphi$$

is integrable. Since the operators $\exp(itL)$ are contractive for $t \geqslant 0$ it is enough to verify that the integral

$$\int^{\infty}\big\|V\exp(-itL_0)\,\varphi\big\|\,dt$$

converges for the elements φ of some subset dense in $\mathcal{H}$. Such a set exists provided condition (1) is fulfilled.

The same approach applies to Boltzmann transport operator $L = L_0 + V$.

Theorem 2. *Direct wave operator* $\Omega = \operatorname*{s-lim}_{t\to\infty}\exp(itL)\exp(-itL_0)$ *exists provided*

$$\int^{\infty}\frac{dt}{\sqrt{t}}\Big(\int_{|x|>t}c(x)^2\,dx\Big)^{1/2} \;<\; \infty\,.$$

The hypotheses concerning function $c(x)$ mentioned above (Theorem 2 and Proposition 2) as well as conditions imposed below in Theorem 3 are fulfilled if for a certain $\alpha > 1$

$$c(x) \;=\; O\big(|x|^{-\alpha}\big) \;\text{ as }\; |x| \to \infty\,.$$

In particular under latter condition $N(c) < \infty$, hence $\sigma_d(L)$ is finite, and moreover the direct wave operator Ω is well defined. For the elements φ of an appropriate set dense in $\mathcal{H}$ and arbitrary $\psi \in \mathcal{H}$ one has

$$\big(\Omega\varphi,\psi\big) \;=\; \big(\varphi,\psi\big) \;-\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\big(\sqrt{V}R_0(\omega+i0)\varphi,\sqrt{V}R^*(\omega-i0)\psi\big)\,d\omega\,.$$

4. The further steps aim to establish the so-called *completeness relationship*

$$\Omega\,\mathcal{H} \;=\; \mathcal{H}_{\mathrm{c}}$$

and to construct inverse wave operator

$$\tilde{\Omega} = \operatorname*{s-lim}_{t\to\infty}\;\exp(itL_0)\,\exp(-itL)\,.$$

Theorem 3. *In the hypothesis of Theorem* 2 *suppose that* $c(x) \neq 0$ *and besides* $c(x)\big(\ln|x-y|\big)^2 c(y) \in \mathrm{L}^1(\mathbb{R}\times\mathbb{R})$. *If zero is not a spectral singularity for corresponding Boltzmann operator* $L = L_0 + V$ *then completeness relationship holds and there exists wave operator* $\tilde{\Omega}$ *on the subspace* $\mathcal{H}_{\mathrm{c}}$. *Moreover*

$$L_{\mathrm{c}} \;=\; \Omega\,L_0\,\tilde{\Omega}\,,$$

where $\tilde{\Omega}$ *is left-inverse to* Ω *and its right-inverse in the subspace* $\mathcal{H}_{\mathrm{c}}$.

This gives a solution to Friedrichs' problem about spectral decomposition for the Boltzmann transport operator in the following sense: in terms of direct and inverse wave operators one explicitly constructs the spectral representation of the operator $L = L_0 + V$.

Now we turn back to the dissipative ($\operatorname{Im} V(x) \geq 0$) Schrödinger operator $L = L_0 + V$ and discuss the similar question regarding its continuous spectrum subspace $\mathcal{H}_c$. For the sake of simplicity let us restrict ourselves to the case of purely imaginary potential $V(x) = iQ(x)$, function $Q(x)$ being real-valued and nonnegative.

Theorem 4. *Assume that*

$$\int_0^\infty x\, Q(x)\, dx \, < \, \infty$$

and there exists an interval $\Delta \subset \mathbb{R}_+$ *such that* $Q(x) \neq 0,\, x \in \Delta$. *If operator* $L = L_0 + iQ$ *has no spectral singularities then completeness relationship holds. On the subspace* $\mathcal{H}_c$ *inverse wave operator* $\tilde{\Omega}$ *exists and*

$$L_c \, = \, \Omega\, L_0\, \tilde{\Omega}$$

where $\tilde{\Omega}\,\Omega = I$ *and* $\Omega\,\tilde{\Omega}\,\varphi = \varphi$ *for* $\varphi \in \mathcal{H}_c$.

In this situation continuous spectrum of non selfadjoint operator L is in fact *absolutely continuous* in the (natural) sense that L_c is similar to selfadjoint operator L_0 possessing absolutely continuous spectrum. This similarity is realized by direct and inverse wave operators Ω and $\tilde{\Omega}$.

The most essential part of this scheme is completeness relationship $\Omega\mathcal{H} = \mathcal{H}_c$ to which an outline of the proof will be given below. First one has to establish that the range of Ω coincides with the outer subspace $\mathcal{N}_e$ of the operator L :

$$\Omega\,\mathcal{H} \, = \, \mathcal{N}_e \, = \, \operatorname{clos}_{\mathcal{H}}\bigl\{\psi \in \mathcal{H} : \sup_{\varepsilon > 0} \int_{\mathbb{R}} \|\sqrt{Q} R(\omega + i\varepsilon)\, \psi\|^2 \, d\omega < \infty\bigr\}.$$

The so-called inner subspace $\mathcal{N}_i$ consists of vectors $\psi \in \mathcal{H}$ such that

$$\mathcal{N}_i \, = \, \bigl\{\psi \in \mathcal{H} : \bigl((R(\omega + i\varepsilon) - R(\omega - i\varepsilon))\psi, \varphi\bigr) \to 0,\, \varepsilon \downarrow 0,\, \text{a.e.}\, \omega \in \mathbb{R},\, \forall \varphi \in \mathcal{H}\bigr\}.$$

In the representation of functional model (see [6]) one can verify that subspaces $\mathcal{N}_e$ and $\mathcal{N}_i$ are mutually complementary and thus

$$\mathcal{H} \, = \, \mathcal{N}_e \, + \, \mathcal{N}_i\, .$$

Since $\mathcal{N}_e \subset \mathcal{H}_c$ and $\mathcal{H}_d \subset \mathcal{N}_i$ the space $\mathcal{H}$ admits the decomposition

$$\mathcal{H} \, = \, \mathcal{H}_d \, + \, \mathcal{N}_e \, + \, \bigl(\mathcal{H}_c \cap \mathcal{N}_i\bigr)\, .$$

So in order to establish completeness relationship it suffices to show that

$$\mathcal{N}_e \, = \, \mathcal{H}_c \;\; \Leftrightarrow \;\; \mathcal{H}_c \cap \mathcal{N}_i = \{0\}\, .$$

5. Sometimes it turns out possible to reduce the latter question to the study of certain scalar analytic function. To this end some preliminary considerations are required. The so-called characteristic function of the operator L is given by the formula

$$S(\lambda) \;=\; I \,+\, 2\,i\,\sqrt{Q}\,(L^* - \lambda I)^{-1}\sqrt{Q}\,.$$

Operator function $S(\lambda)$ is analytic in $\mathbb{C}_+$ and is invertible provided λ is not an eigenvalue:

$$S^{-1}(\lambda) \;=\; I \,-\, 2\,i\,\sqrt{Q}\,(L - \lambda I)^{-1}\sqrt{Q}\,, \quad \lambda \notin \sigma_d(L)\,.$$

Analytic function $m(\lambda) \not\equiv 0$ is called a scalar multiple for $S(\lambda)$ if there exists an operator function $\Sigma(\lambda)$ bounded and analytic in $\mathbb{C}_+$ such that

$$\Sigma(\lambda)\,S(\lambda) \;=\; S(\lambda)\,\Sigma(\lambda) \;=\; m(\lambda)\,I\,.$$

For the Boltzmann transport operator (as well as in some other cases) such a scalar multiple is constructed in the form of an appropriate Fredholm determinant. For one-dimensional Schrödinger operator the scalar multiple can be given explicitly: $m(\lambda) = e(\sqrt{\lambda})$. Note that

$$m(\lambda) = 0 \ \text{ in } \ \mathbb{C}_+ \quad \Leftrightarrow \quad \lambda \in \sigma_d(L)\,.$$

It is well known that every bounded analytic in $\mathbb{C}_+$ function, in particular $m(\lambda)$, admits the so-called Nevanlinna-Riesz factorization:

$$m(\lambda) \;=\; m_1(\lambda)\cdot m_2(\lambda)\cdot m_3(\lambda)\,.$$

The first factor $m_1(\lambda)$ is the Blaschke product

$$m_1(\lambda) \;=\; \prod \frac{\lambda - \lambda_n}{\lambda - \overline{\lambda}_n}\,, \qquad \{\lambda_n\} = \sigma_d(L)\,,$$

where λ_n are the zeroes of $m(\lambda)$ in $\mathbb{C}_+$ that coincide with the eigenvalues of L. The second factor, called outer function, has the form

$$m_2(\lambda) \;=\; e^{ib}\,\exp\Big(\frac{i}{\pi}\int_{\mathbb{R}} \ln|m(t)|\,\frac{t\lambda+1}{\lambda-t}\,\frac{dt}{1+t^2}\Big)\,, \quad b\in\mathbb{R}\,.$$

The third one is given by the formula

$$m_3(\lambda) \;=\; e^{ia\lambda}\,\exp\Big(i\int_{\mathbb{R}} \frac{t\lambda-1}{\lambda+t}\,d\mu(t)\Big)\,,$$

where $a \geq 0$ and μ is a singular measure. This factor is called singular inner function.

Proposition 3. *Suppose that characteristic function* $S(\lambda)$ *has a scalar multiple* $m(\lambda) = m_1(\lambda) \cdot m_2(\lambda)$ *such that the singular inner factor in Nevanlinna-Riesz factorization is trivial* $m_3(\lambda) \equiv 1$. *Then the corresponding subspace* $\mathcal{H}_{\rm c} \cap \mathcal{N}_i$ *is zero:*

$$\mathcal{H}_{\rm c} \cap \mathcal{N}_i \;=\; \{0\} \;\;\Rightarrow\;\; \mathcal{H}_{\rm c} \;=\; \mathcal{N}_{\rm e}\,.$$

Thus, continuous spectrum subspace admits the following characterization: $\mathcal{H}_{\rm c}$ consists of those elements $\varphi \in \mathcal{H}$ for which vector valued function $\sqrt{Q}R(\lambda)\,\varphi$ belongs to the Hardy class $\mathrm{H}^2(\mathbb{C}_+)$. Remark that similar condition is known to appear in scattering theory for selfadjoint operators and is related to what is called *Kato smoothness property.* Note that in our setting (provided there exists such an interval $\Delta \subset \mathbb{R}_+$ that $Q(x) \neq 0$ for $x \in \Delta$) operator $L = L_0 + iQ$ is completely non selfadjoint, i.e. it has no nontrivial invariant subspaces on which L induces a selfadjoint operator.

6. The question when the singular inner factor in the factorization of the scalar multiple is trivial reduces to the study of analytic properties of the resolvent near the real axis. It turns out that in both problems concerning Boltzmann and Schrödinger equations the answer to the question above is affirmative provided there are *no spectral singularities.*

In order to study this question for the Schrödinger operator L one can apply the results of the analytic functions theory to the scalar multiple $m(\lambda) = e(\sqrt{\lambda})$. As it was already pointed out (Proposition 1) function $m(\lambda)$ is continuous in the closed upper half-plane if

$$\int_0^\infty x\,|V(x)|\,dx \;<\; \infty$$

and the same holds true for its outer factor $m_2(\lambda)$. By virtue of the equality $|m_2(\kappa + i0)| = |m(\kappa + i0)|$ and provided L has no spectral singularities (i.e. $m(\lambda)$ has no real zeroes) function $m_2(\lambda)$ does not vanish at points of the real axis. A finite Blaschke product $m_1(\lambda)$ does not take zero value on the real axis too. This implies immediately that

$$m_3(\lambda) \;=\; \frac{m(\lambda)}{m_1(\lambda) \cdot m_2(\lambda)}$$

is continuous in the closed upper half-plane. Now we make use of the following assertion (see [7]): singular inner function can be extended from $\mathbb{C}_+$ by continuity to those points of $\mathbb{R}$ that does not belong to the support of measure μ. By this fact and continuity of $m_3(\lambda)$ up to the real axis it follows that $\operatorname{supp}\mu = \varnothing$ and therefore $m_3(\lambda) = e^{ia\lambda}$, $a \geq 0$. Taking into account that $m(i\tau) \to 1$ as $\tau \to \infty$ we have $a = 0$. In general it may happen that the singular factor $m_3(\lambda)$ in Nevanlinna-Riesz factorization of scalar multiple is nontrivial and in this case the *singular continuous spectrum* of L is naturally understood as the support of measure μ.

In conclusion stationary representations of wave operators Ω and $\tilde{\Omega}$ related to eigenfunction expansions for operator L must be mentioned (for details see [8]). Denote by $\psi(x,k)$ a solution to equation

$$-y'' + V(x)\,y = k^2\,y$$

satisfying initial conditions $\psi(0,k) = 0$, $\psi'(0,k) = k$. In the case $V(x) = 0$ the solution $\psi(x,k)$ is just $\sin kx$. Let Φ be the standard Fourier sine-transform and consider a transformation

$$\Psi\, f(\kappa) = \int_0^\infty f(x)\,\psi(x,\kappa)\,dx\,,$$

that maps $\mathrm{L}^2(0,\infty)$ onto itself provided

$$\int_0^\infty (1+x^2)\,|V(x)|\,dx < \infty\,.$$

Transformation Ψ vanishes on the subspace $\mathcal{H}_d$ while on the subspace $\mathcal{H}_c$ it is invertible and its inverse is given by the formula

$$\Psi^{-1}g(x) = \frac{2}{\pi}\int_0^\infty g(\kappa)\,\psi(x,\kappa)\,\frac{d\kappa}{e(\kappa)e(-\kappa)}\,.$$

Proposition 4. *In terms of transforms Ψ and Φ wave operators can be expressed as follows:*

$$\Omega = \Psi^{-1}e(\kappa)\;\Phi\,, \quad \tilde{\Omega} = \Phi^{-1}e(\kappa)^{-1}\Psi$$

provided L has no spectral singularities.

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