A Global Optimization Approach for Solving the D.C. Multiplicative Programming Problem

Xue-gang Zhou
Department of Applied Mathematics, Guangdong University of Finance,
Guangzhou, Guangdong, 510521,China
xgzhou@yahoo.cn

Keywords: D.C. multiplicative programming, global optimization, cutting plane, conical partition

Abstract. In this paper, we present a global optimization algorithm for solving the D.C. multiplicative programming (DCMP) over a convex compact subset. By introducing auxiliary variables, we give a transformation under which both the objective and the feasible region turn to be d.c.Then we solve equivalent D.C. programming problem by branch and bound method and outer approximation algorithm.

Introduction

The multiplicative programming (MP) come from many fields, for example, financial ptimization[1], VLISI chip design[2], and so on. In the past 20 years, many solution algorithms have been proposed for globally solving the problem (MP). The methods can be classified as parameterization based methods [3], branch-and-bound methods [4], ect. In this paper, we consider d.c. multiplicative programming problems as follows:

$$\text{(DCMP)} \quad v=\max \quad f(x)g(x) \qquad \text{s.t.} \quad x\in X=\{x\in R^{n} | h(x) \le 0\},$$

where $0\le f:R^n\to R$ and $0<g:R^n\to R$ are d.c. (difference of convex) function, $h:R^n\to R$ is a convex function, X is a convex, compact, nonempty subset of R^n. Suppose that $f=f_1-f_2$ and $g=g_1-g_2$ for some convex functions $f_i:R^n\to R$ and $g_i:R^n\to R$ with $i=1,2$.

Equivalent problem

For function $f(x)$, we consider an epi-multiple function $F(x,\lambda)$ as follows:

$$F(x,\lambda):=\begin{cases}\lambda f(\lambda^{-1}x) & \text{if } \lambda>0,\\ 0 & \text{if } \lambda=0, x=0,\\ -\infty & \text{otherwise.}\end{cases} \tag{1}$$

For g and h, we similarly define G and H.

Lemma 1 [5]. If f is a convex (concave) function, then so is the function F for $\lambda>0$.

Lemma 2 [6]. If $F(x,\lambda)$ is positively homogeneous, then, for any $\alpha_2>\alpha_1>0$ and $F(x,\lambda)\neq0$, $F(\alpha_2 (x,\lambda))= \alpha_2F((x,\lambda))> \alpha_1F((x,\lambda))= F(\alpha_1 (x,\lambda))$.

To transform the problem (DCMP) into a d.c. programming, let $\beta(x):=g(x)$, $y(x):= x\beta$. Since $g(x)>0$ for all $x\in X$, we have $\beta(x)>0$ for any $x\in X$. Given a point $x^0\in X$, we can obtain $(y^0, \beta^0)\in R^{n+1}$, where $y^0=y(x^0)$,$\beta^0=\beta(x^0)$. Define $S:=\{(y, \beta)\in R^{n+1} | \exists x\in X$ such that $\beta=g(x)$, $y= x\beta\}$. Then, for any $(y,\beta)\in S$, we have

$$F(y, \beta)= \beta f(y/\beta)= \beta f(x)= f(x)\, g(x), \quad G(y, \beta)= \beta g(y/\beta)= \beta g(x)= \beta^2. \tag{2}$$

Let β^l:=min{$g(x)$| $x\in X$ },β^u:=max{$g(x)$| $x\in X$ }, then we get $S\subseteq\{(y, \beta)\in R^{n+1}| H(y, \beta)\leq 0,\beta^l\leq\beta\leq\beta^u\}\cap\{(y, \beta)| \beta^2-G(y, \beta)\leq 0\}$.Now we consider the following d.c. programming problem:

$$\text{(P1)} \quad \max \quad F(y, \beta)$$
$$\text{s.t.} \quad \beta^2-G(y, \beta)\leq 0, \qquad (3)$$
$$H(y, \beta)\leq 0,\beta^l\leq\beta\leq\beta^u$$

From (3) and problems (DCMP) and (P1). it is easy to see that max (DCMP) ≤ max(P1). Moreover, the following theorem shows that (DCMP) is equivalent to (P1). From Lemma 1, we see that the function H is convex and F and G are d.c. function. Let $F_1(y, \beta)= \beta f_1(y/\beta)$, $F_2(y, \beta)= \beta f_2(y/\beta)$, then F_1, F_2 are convex. Denote $G_1(y, \beta)= \beta g_1(y/\beta)$ and $G_2(y, \beta)= \beta g_2(y/\beta)$, then are convex. Then problem (P1) can be rewritten as

$$\text{(P1)} \quad \max \quad F_1(y, \beta)-F_2(y, \beta)$$
$$\text{s.t.} \quad \beta^2-G_1(y, \beta)+G_2(y, \beta)\leq 0, \qquad (4)$$
$$H(y, \beta)\leq 0,\beta^l\leq\beta\leq\beta^u$$

Theorem 3. If (y^*,β^*) is an optimal solution of (P1) then y^*/β^* is an optimal solution of (DCMP). If x^* is an optimal solution of (DCMP), then $(x^*g(x^*), g(x^*))$ is an optimal solution of (P1).

Proof. Let (y^*,β^*) is an optimal solution of (P1), then (y^*,β^*) is a feasible solution of the problem (P1). So $\beta^*\geq\beta^l>0$ and $\beta^*h(y^*/\beta^*)\leq 0$ due to $H(y^*,\beta^*)\leq 0$.It follows that y^*/β^* is a feasible solution of the problem (DCMP). Let (y,β) be any feadible point of (P1), then we have

$$Bf(y/\beta)= F(y, \beta) \leq F(y^*,\beta^*)=\beta^*f(y^*/\beta^*). \qquad (5)$$

Assume that y^*/β^* is not an optimal solution of (DCMP), then there exists $x^0\in X$ such that $f(x^0)g(x^0) > f(y^*/\beta^*)g(y^*/\beta^*)$. Let $\beta^0=g(x^0)$,$y^0=\beta^0x^0$. Then $(y^0,\beta^0)$is a feasible point of (P1). Since $(\beta^*)^2\leq G(y^*,\beta^*)$, we see $\beta^0f(y^0/\beta^0)= f(x^0)g(x^0)> f(y^*/\beta^*)g(y^*/\beta^*)=(1/\beta^*) f(y^*/\beta^*)\beta^*g(y^*/\beta^*)\geq\beta^*f(y^*/\beta^*)$. It contradicts (5), that is, y^*/β^* is an optimal solution of (DCMP). Now, let x^* is an optimal solution of (DCMP). Then, for all $x\in X$, $f(x^*)g(x^*)\geq f(x)g(x)$ and $(x^*g(x^*),g(x^*))$ is a feasible point of (P1). If $(x^*g(x^*),g(x^*))$ is not an optimal solution of (DCMP), there exists a feasible point (y^0,β^0) of (P1) such that$\beta^0f(y^0/\beta^0)> f(x^*)g(x^*)$. We see $(\beta^0)^2\leq G(y^0,\beta^0)$ and $H(y^0,\beta^0)\leq 0$. It follows that $y^0/\beta^0\in X$ from $h(y^0,\beta^0)\leq 0$, and from $\beta^0\leq g(y^0/\beta^0)$, we have $f(y^0/\beta^0)g(y^0/\beta^0)\geq\beta^0f(y^0/\beta^0)$. So, $f(y^0/\beta^0)g(y^0/\beta^0)\geq\beta^0f(y^0/\beta^0)>f(x^*)g(x^*)$. It contradicts that is an optimal solution of (DCMP).

Then, we see that the constraint$\beta^l\leq\beta\leq\beta^u$ of (P1) can be simply replaced by $\beta>0$. Denoted by D the feasible region of (P1) and $bd(D)$ the boundary of D.

Theorem 4. If (y^*,β^*) is an optimal solution of (P1), then $(y^*,\beta^*)\in bd(D)$.

Proof. Let (y^*,β^*) is an optimal solution of (P1), and $(y^*,\beta^*)\notin bd(D)$. Then there exists a neighborhood $N_\varepsilon(y^*,\beta^*)$ with a radius of $\varepsilon>0$ such that $N_\varepsilon(y^*,\beta^*)$. So, $(1+\varepsilon_0)(y^*,\beta^*)\in N_\varepsilon(y^*,\beta^*)\in D$, where $\varepsilon_0=\varepsilon/(2\|(y^*,\beta^*)\|)$. From lemma 2, we see $F((1+\varepsilon_0)(y^*,\beta^*))> F((y^*,\beta^*))$. This is a contradiction of the maximality of (y^*,β^*).

Similarly, we have the following corollary.

Corollary 5. If(y^*,β^*) is an optimal solution of (P1), then $G(y^*,\beta^*)=(\beta^*)^2$.

By introducing two additional variables μ and v, we transformed the problem (P1) into a equivalent programming as follows:

$$(\text{P}_{\text{main}}) \quad \max \quad F_1(y, \beta)-\mu$$
$$\text{s.t.} \quad F_2(y, \beta)-\mu\leq 0,\beta^2+G_2(y, \beta)-\delta\leq 0, \qquad (6)$$
$$\delta-G_1(y, \beta) \leq 0,H(y, \beta)\leq 0,\beta^l\leq\beta\leq\beta^u.$$

Moreover, we denote that $F=\{(y,\beta,\mu,\delta)\in R^{n+3}| F_2(y,\beta)-\mu\leq 0\}$, $G_1=\{(y,\beta,\mu,\delta)\in R^{n+3}| \delta-G_1(y, \beta)$

$\leq 0\}$, $G_2=\{(y,\beta,\mu,\delta)\in R^{n+3}|\beta^2+G_2(y,\beta)-\delta\leq 0\}$, $H=\{(y,\beta,\mu,\delta)\in R^{n+3}|\ H(y,\ \beta)\leq 0,\beta^l\leq\beta\leq\beta^u\ \}$.Then the feasible region of the problem (P_{main}) is $(F\cap G_2\cap H)\backslash G_1$, The constraint is reverse convex constraint since F, G_1, G_2, H are convex sets.

Algorithm and convergence

To establish an outer approximation algorithm for problem (P_{main}), we suppose:(i) int(Ω)≠∅, where $\Omega:=(F\cap G_2\cap H)\backslash G_1$. (ii) there exists $(y^0,\beta^0,\mu^0,\delta^0)\in$ int$(F\cap G_2\cap H\cap G_1)$. Denote $\Omega_1:=(F\cap G_2\cap H)$. So, Ω_1 is convex. Then there exists convex function F_Ω such that $\Omega_1=\{(y,\beta,\mu,\delta)\in R^{n+3}|\ F_\Omega(y,\beta,\mu,\delta)\leq 0\}$. In our algorithm, we must generate a sequence of conical partition sets $\{T_k\}_{k\in I}$ such that $\Omega_1\subseteq T_k$ and $T_{k+1}\subseteq T_k$ for all $i\in I$. Let a polytope T_0 is with n+3 dimensions be an initial one. We can get a simplex containing Ω_1 as T_0. Suppose that T_k is given at step k, we consider problem as follows:

$$(P_k)\quad \max\quad F_1(y,\beta)-\mu$$
$$\text{s.t.}\quad (y,\beta,\mu,\delta)\in T_k.$$

Obviously, the optimal value of the problem(P_k) is an upper bound of (P_{main}). Assume that T_k includeseveral polyhedral convex cones $C^{ki}(i=1,2,\cdots,k_q)$ having n+3 edges which emanate from a point $(y^0,\beta^0,\mu^0,\delta^0)$ of assumption (ii). Let $z^0=(y^0,\beta^0,\mu^0,\delta^0)$ and omit the subscript of C. Hence we see that there exist n+4 affinely independent points z^0, z^1, $\cdots$, z^{n+3} such that $C=\{z\in R^{n+3}|z=\sum_{i=1}^{n+3}\eta_i(z^i-z^0)+z^0,\eta_i\geq 0\}$. For all $i=1,2,\cdots,n+3$, we assume without loss of generality that $\|z^i\|=1$, $\theta_i=\sup\{\theta\in R|z^0+\theta(z^i-z^0)\in G_2\cap H\cap G_1\}$, $w^i=z^0+\theta_i(z^i-z^0)$, $U=(w^1-z^0,\cdots,w^{n+3}-z^0)$ and $L_2:=\{z\in R^{n+3}|z=z^0+U\eta,e^T\eta\geq 1\}$, where $\eta=(\eta_1,\cdots,\eta_{n+3})$,e=$(1,1,\cdots,1)^T$. Because z^0, $z^1,\cdots,z^{n+3}$ are affinely independent, U is a nonsingular matrix. So, $L_2:=\{z\in R^{n+3}|e^TU(z-z^0)\geq 1\}$. It is easy to get the following result.

Lemma 6 $\Omega\cap C\subseteq L_2\cap C$ and $\Omega\cap C\subseteq(L_2\cap\{z|\tau(z)\leq 0\})\cap C$.

Base on Lemma 6, we can get an upper bound of (P_{main}). Denote

$$u:=\max\{F_1(y,\beta)-\mu|(y,\beta,\mu,\delta)\in(L_2\cap\{z|\tau(z)\leq 0\})\cap C\}. \quad (7)$$

Then$(L_2\cap\{z|\tau(z)\leq 0\})\cap C$ is polyhedral. If $(L_2\cap\{z|\tau(z)\leq 0\})\cap C\neq\emptyset$, the value s attained at one of its vertices, otherwise, let $u:=-\infty$. For any $i=1,2,\cdots,n+3$, let $\bar\theta_i:=\sup\{\theta\,|\,z^0+\theta(z^i-z^0)\in\Omega\}$. If $\theta_i\leq\bar\theta_i$, then $z^0+\theta_i(z^i-z^0)$ and $z^0+\bar\theta_i(z^i-z^0)$ are feasible point of (P_{main}). Therefore,

$$l:=\max\{F(y,\beta)-\mu\,|\,(y,\beta,\mu,\nu)=z^0+\theta(z^i-z^0),\theta\in[\theta_i,\bar\theta],i=1,2,\cdots,n+3\} \quad (8)$$

is lower bound of (P_{main}) on $\Omega\cap C$. Assume that the lower bound l is get at $(y^i,\beta^i,\mu^i,\delta^i)$. Then we can obtain that $F(\lambda y^i,\lambda\beta^i,\mu^i,\delta^i)>\lambda F(y^i,\beta^i,\mu^i,\delta^i)$ for all $\lambda>1$ form Lemma 2. So,

$$l_{\geq}:=\max\{F(\lambda y^i,\lambda\beta^i,\mu^i,\delta^i)|\ F(\lambda y^i,\lambda\beta^i,\mu^i,\delta^i)\in\Omega,\ \lambda>1\}, \quad (9)$$

greater than or equal to l.

Now, we establish a global algorithms solving the problem (P_{main}). We use a cuttingplane method to approximate Ω, and use a conical partition to fulfill an exhaustive process. During each iteration k, let $T_k:=\{C^{k1},\cdots,C^{kq}\}$ be a conical partition of Ω. For $C^{ki}\in T_k$, we compute u_{ki} by (7) and a polytope $L_2^{k_i}\cap\{z\,|\,\ell^{k_i}(z)\leq 0\}\cap C^{k_i}$ which approximates $\Omega^{ki}\cap C^{ki}$ from Lemma 6. Compute l_{ki} by (8) and (9). After knowing $u_k=\min\{u_{ki}\}$ and $l_k=\max\{l_{ki}\}$, the algorithm to be proposed chooses one cone $C^{ki}\in T_k$ as a candidate to be divided into smaller, and repeats the process.

Algorithm:

Step 0: Let T_0 is a initial conical partition emanating from z^0 such that $\Omega\in T_0$, $l_0=-1$, $u_0=\infty$, $k=0$.

Step 1: For any $C^{ki}\in T_k$, compute u_{ki} and l_{ki}, and get the solutions $(\bar y^{k_i},\bar\beta^{k_i},\bar\mu^{k_i},\bar v^{k_i})$ and

$(\underline{y}^{k_i},\underline{\beta}^{k_i},\underline{\mu}^{k_i},\underline{v}^{k_i})$, respectively. Set $C^{k_i}:=L_2^{k_i}\bigcap\{z\mid\ell^{k_i}(z)\le 0\}\bigcap C^{k_i}$. If there exists i such that $l_{ki}\ge l_k$, then $l_k:=l_{ki}$.If there exists i such that $u_{ki}< u_k$ set $u_k:=u_{ki}$.

Step 2: If u_k-l_k=0, then terminate. Otherwise M:=$\{C^{ki}|u_{ki}\ge l_k\}$. Choose $C\in\{C^{ki}\mid u_{ki}=u_k\}$, create a conical partition Y of C.

Step 3: T_{k+1}:=(M\C)$\cup Y$, $l_{k+1}:=l_k$, $u_{k+1}:=u_k$, $k:=k+1$, go to step 1.

Theorem 7 Suppose that a conical partition generated by algorithm is exhaustive. If algorithm does not terminate after a finite number of iterations, then every accumulation point of the sequence $(\bar{y}^{k_i},\bar{\beta}^{k_i},\bar{\mu}^{k_i},\bar{v}^{k_i})$ of the algorithm is an optimal solution of the problem (P_{main}).

Proof. It is obviously.

References

[1] C.D. Maranas, I.P. Androulakis, C.A. Floudas, A.J. Berger, J.M. Mulvey, Solving long-term financial planning problems via global optimization. Journal of Economic Dynamics and Control. 21 (1997) 1405-1425.

[2] M.C. Dorneich, N.V. Sahinidis, Global optimization algorithms for chip design and compaction. Engineering Optimization. 25(2) (1995) 131-154.

[3] H. Konno, T. Kuno, Y. Yajima, Global optimization of a generalized convex multiplicative function. Journal of Global Optimization. 4 (1994) 47-62.

[4] X.G. Zhou, K. Wu, A method of acceleration for a class of multiplicative programming problems with exponent, Journal of Computational and Applied Mathematics, 223 (2009) 975-982.

[5] R.T. Rockafellar, R.J.-R.Wets, Variational Analysis, Springer, 1998.

[6] Konno, H., and Yamashita, H., Minimizing Sums and Products of Linear Fractional Functions ouer a Polytope, Naval Research Logistics, 46 (1999) 583-596.