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Geometry of differential operators, odd Laplacians, and homotopy algebras

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Abstract

We give a complete description of differential operators generating a given bracket. In particular we consider the case of Jacobi-type identities for odd operators and brackets. This is related with homotopy algebras using the derived bracket construction.

1 Introduction

In this paper we give a survey of results of our works [1], [2], [3] (see also [4]). Their original motivation was to give a clear geometric picture of the relation between an odd bracket and its "generating operator" in the Batalin-Vilkovisky formalism.

In the pioneer paper [5], for the needs of the Lagrangian quantization of gauge theories, Batalin and Vilkovisky constructed a remarkable second-order operator on the "phase space" of fields and antifields. One of us (H.K.) suggested a mathematical framework in which the Batalin-Vilkovisky operator is interpreted as an invariant operator acting on functions on an odd Poisson manifold equipped with a volume form [6]. It is nothing but an "odd Laplace operator" associated with an odd Poisson structure in the same way as the usual Laplacian on a Riemannian manifold is associated with a Riemannian metric, the main difference being in the fact that on a Riemannian manifold there is a natural volume form, while for an odd Poisson case, even if the bracket is non-degenerate (i.e., for odd symplectic manifolds), no natural volume form exists and a volume form should be introduced as an extra piece of data. (In a similar situation in the even Poisson geometry a vector field rather than a second-order operator arises [7].) As it has turned out, the bracket of functions can be recovered from the odd Laplacian, as the failure of the Leibniz property, so the odd Laplacian 'generates' the odd Poisson structure. Other constructions of a generating operator, based on a connection, have been proposed [8]. On the other hand, physical motivations require an operator acting on half-densities (semidensities) rather than on functions. A canonical operator on half-densities on an odd symplectic manifold was discovered in [9, 10]. It does not require any extra structure such as a connection or volume form. As it has turned out, on general odd Poisson manifolds the situation is more complicated [1]. There is no longer a unique operator on half-densities, though the construction of a Laplace operator using a volume form is still 'more canonical' for half-densities than for functions or densities of any other weight, since for half-densities this operator, as was shown in [1], depends only on the orbit of a volume form under the action of a certain groupoid. This groupoid (called the "master groupoid" in [1]) consists of transformations of volume forms $\boldsymbol{\rho} \mapsto \boldsymbol{\rho}' = e^f \boldsymbol{\rho}$ satisfying the "master equations" $\Delta_{\boldsymbol{\rho}}\, e^{f/2} = 0$, thusly revealing their composition property. (One should note preliminary results in this direction in [11], which in a hindsight can be interpreted as pointing at half-densities.)

The results of [1] quite unexpectedly showed a similarity between odd Poisson geometry and the usual Riemannian geometry. Certain formal properties of the Laplacian on half-densities or densities of other weight, happen to be the same regardless of parity of the 'bracket' (if one views a Riemannian metric as a bracket). Physically, this points at a formal similarity between the BV master equation on half-densities and the Schroedinger equation, the classical master equation being analogous to the eikonal or Hamilton-Jacobi equation [1]. In fact, for the four theories (odd/even Poisson, odd/even Riemannian structure) defined by a rank two tensor T^{ab} on M, analogies come in pairs, see [1].

Such a viewpoint helps, in a way, to 'trivialize' the original problem of the relation between a bracket and a generating operator, by seeing it as the problem of describing all second-order operators with a given principal symbol. (No confusion should be with the quantization problem: the question is about one individual symbol, not about a construction for all symbols.) In such form, a solution is immediate. One only has to formulate it geometrically. For second-order operators on functions, a piece of information necessary to recover an operator from its principal symbol is the so called subprincipal symbol. A good way of looking on it is to view it as a sort of connection, more precisely, an "upper connection" or "contravariant derivative" on volume forms, i.e., an operator mapping them to vector fields. In a 'bracket setup', it can also be viewed as an extension of the 'bracket' of functions to a 'bracket' between functions and volume forms, or, using the Leibniz rule, between functions and densities of arbitrary weight. This gives a complete description of generating operators acting on functions.

While complete, this solution is not entirely satisfying aesthetically, since it lacks symmetry: to describe operators just on functions one has to consider brackets involving densities. Also, operators, say, on half-densities must have their part in the picture.

As we showed in [2], passing from the algebra of functions to the *algebra of densities* $\mathfrak{V}(M)$ on a (super)manifold M solves all problems. The reason for this is a natural invariant scalar product in the algebra $\mathfrak{V}(M)$. It is possible to establish a natural one-to-one correspondence between 'brackets' in this algebra and second-order operators. From the viewpoint of M, a second-order operator in $\mathfrak{V}(M)$ is a quadratic pencil of operators Δ_w acting on densities of weight $w \in \mathbb{R}$. A bracket in $\mathfrak{V}(M)$ is specified by a bracket of functions, a corresponding to it upper connection on volume forms, and one extra piece of data similar to the familiar to physicists Brans-Dicke field of the Kaluza-Klein type models. These results contain all previous formulae, e.g. for functions and half-densities, and explain them.

As we said, up to a certain point there is no difference whatsoever between the case of even operators and even brackets (which can be considered on a usual manifold) and that of odd operators and odd brackets, necessarily on a supermanifold. Parity of a bracket or of its generating operator becomes essential when one wants to obtain something like a Lie algebra. One can show that there is no way of having a Jacobi-type identity for a symmetric even bracket ('brackets' coming from operators are always symmetric, possibly in a graded sense, because they are just polarizations of quadratic forms, the principal symbols). Therefore to make a progress in this direction one should focus on the odd case, thus bringing us back to 'Batalin-Vilkovisky operators'. Our technique allows to give an exhaustive answer to the questions about the possible "Jacobi identities" involving an odd operator together with the corresponding bracket.

From the algebraic viewpoint, identities that arise here are a particular case of more general identities. This prompts to investigate the situation further. The bracket generated by a second-order operator is an example of 'derived brackets' (see [12]; for a recent survey see [13]). Taken with the generating operator itself, it should be viewed as a part of a sequence of n-ary 'brackets', which can be defined for any operator of order N, so that $n = 0, \dots, N$. (This works for even and odd operators, and this sequence of higher brackets has the property that the $k+1$ bracket is equal to the discrepancy of the Leibniz identity for the k-th bracket, see [14].) If one asks for suitable Jacobi identities, it is natural do it in an abstract setup. Such a setup, called "higher derived brackets", has been suggested in [3]. We have shown there how simple data such as a Lie superalgebra with a projector on an Abelian subalgebra and an odd element Δ can produce a strongly homotopy Lie algebra; in fact, we show how the 'higher Jacobi identities' are controlled by Δ^2. The situation with differential operators and the corresponding brackets is just one particular instance of that.

The algebraic constructions of [3] make the first step of generalizing the results of [2] to higher order operators. This should be a subject of further studies.

This paper is organized as follows. In Section 2 we formulate the problem and give examples. In particular we review the properties of the operator $\Delta_{\boldsymbol{\rho}}$. In Section 3 we introduce the algebra of densities $\mathfrak{V}(M)$ and state the main theorem about the one-to-one correspondence between operators and brackets. In Section 4 we consider the case of odd operators and brackets. In Section 5 we give the main results concerning higher derived brackets. Throughout the paper we use the supermathematical conventions about commutators, derivations, etc.; tilde is used to denote parity.

2 Main problem: operators and brackets

2.1 Setup

Let M be a supermanifold and Δ an arbitrary second-order differential operator acting on $C^\infty(M)$. In local coordinates:

$$\Delta = \frac{1}{2}\, S^{ab}\, \partial_b \partial_a + T^a\, \partial_a + R. \tag{2.1}$$

The principal symbol of Δ is the symmetric tensor field S^{ab}, or the quadratic function $S = \frac{1}{2}\, S^{ab} p_b p_a$ on T^*M. The principal symbol can be alternatively understood as a symmetric bilinear operation on functions:

$$\{f, g\} := \Delta(fg) - (\Delta f)\, g - (-1)^{\varepsilon \tilde{f}} f\, (\Delta g) + \Delta(1)\, fg,$$

where $\varepsilon = \tilde{\Delta}$ is the parity of the operator Δ. In coordinates

$$\{f, g\} = S^{ab}\partial_b f\, \partial_a g(-1)^{\tilde{a}\tilde{f}}.$$

That $\{f, g\}$ is a bi-derivation can be formally deduced from the fact that $\operatorname{ord}\Delta \leqslant 2$. In the following by a *bracket* in a commutative algebra we mean an arbitrary symmetric bi-derivation. We say that Δ is a *generating operator* for the bracket $\{f, g\}$. Notice also that $\{f, g\} = [[\Delta, f], g]$, where $[\ ,\ ]$ denotes the commutator of operators.

Problem: construct a generating operator for a given bracket, i.e., reconstruct Δ for a given symmetric tensor field S^{ab}; describe all generating operators Δ.

2.2 Examples

Example 2.1. Let $(S^{ab}) = (g^{ab}) = (g_{ab})^{-1}$, where (g_{ab}) is a Riemannian metric. The Laplace-Beltrami operator

$$\Delta f = \operatorname{div} \operatorname{grad} f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} \left(\sqrt{g} g^{ab} \frac{\partial f}{\partial x^b} \right),$$

where $g = \det(g_{ab})$, is a generating operator for the “bracket” $\{f, g\} = \nabla f \cdot \nabla g = g^{ab}\partial_a f \partial_b g$ (up to a factor of $\frac{1}{2}$). An arbitrary generating operator for this bracket has the appearance $\frac{1}{2}\Delta + X + R$, where X is a vector field, R is a scalar function.

For a degenerate g^{ab}, one has to replace $\sqrt{g}$ by some volume density ρ. In the following example this is always the case, even for non-degenerate matrices.

Example 2.2. Suppose now (S^{ab}) is odd and specifies an odd Poisson bracket. Take an arbitrary volume form $\boldsymbol{\rho} = \rho Dx$. Then the operator introduced in [6],

$$\Delta_{\boldsymbol{\rho}} f = \operatorname{div}_{\boldsymbol{\rho}} X_f = \frac{1}{\rho} \frac{\partial}{\partial x^a} \left(\rho S^{ab} \frac{\partial f}{\partial x^b} \right), \tag{2.2}$$

is, up to $\frac{1}{2}$, a generating operator for the bracket. (We denote by X_f the Hamiltonian vector field corresponding to f.) $\Delta_{\boldsymbol{\rho}}$ mimics the Laplace-Beltrami operator.

Unlike the classical Laplace-Beltrami above, the operator (2.2) requires a choice of $\boldsymbol{\rho}$. We shall study $\Delta_{\boldsymbol{\rho}}$ in more detail below, as well as give a description of all generating operators for a given odd Poisson bracket.

Example 2.3. Rather, a counterexample. If we try to apply the same construction to an *even* Poisson bracket, specified by a Poisson tensor (bivector) (P^{ab}), which is antisymmetric, then the second-order terms cancel, and we get a first-order operator

$$\Delta_{\boldsymbol{\rho}} f = \operatorname{div}_{\boldsymbol{\rho}} X_f = \frac{1}{\rho} \frac{\partial}{\partial x^a} \left(\rho P^{ab} \right) \frac{\partial f}{\partial x^b}, \tag{2.3}$$

known as a *modular vector field* of an even Poisson bracket (see [7]). It does not generate the bracket.

2.3 Properties of $\Delta_{\boldsymbol{\rho}}$

The operator $\Delta_{\boldsymbol{\rho}}$ acting on functions on an odd Poisson manifold and depending on a choice of volume form was introduced in [6]. (In [6], everything was formulated for odd symplectic manifolds, but the results hold for the general Poisson case. See also [8].) From now on we redefine $\Delta_{\boldsymbol{\rho}}$ by inserting the factor $\frac{1}{2}$ into formula (2.2). The properties of $\Delta_{\boldsymbol{\rho}}$ are as follows. First,

$$\Delta_{\boldsymbol{\rho}}(fg) = (\Delta_{\boldsymbol{\rho}} f)g + (-1)^{\tilde{f}} f(\Delta_{\boldsymbol{\rho}} g) + (-1)^{\tilde{f}+1}\{f,g\}\,.$$

If we change $\boldsymbol{\rho}$, $\boldsymbol{\rho} \mapsto \boldsymbol{\rho}' = e^{\sigma}\boldsymbol{\rho}$, then $\Delta_{\boldsymbol{\rho}'} = \Delta_{\boldsymbol{\rho}} + \frac{1}{2} X_{\sigma}$. (Similar properties hold for an even $S^{ab} = g^{ab}$.) The following two properties are peculiar for an odd bracket. Holds

$$\Delta_{\boldsymbol{\rho}}\{f,g\} = \{\Delta_{\boldsymbol{\rho}} f, g\} + (-1)^{\tilde{f}+1}\{f, \Delta_{\boldsymbol{\rho}} g\}.$$

The operator $\Delta_{\boldsymbol{\rho}}^2$ is a Poisson vector field, and $\Delta_{\boldsymbol{\rho}'}^2 = \Delta_{\boldsymbol{\rho}}^2 - X_{H(\boldsymbol{\rho}',\boldsymbol{\rho})}$, where $H(\boldsymbol{\rho}',\boldsymbol{\rho}) = e^{-\sigma/2}\,\Delta_{\boldsymbol{\rho}}(e^{\sigma/2})$. See [1]. It follows that $\Delta_{\boldsymbol{\rho}}^2$ gives a well-defined cohomology class, which we call the *modular class* for an odd Poisson bracket.

2.4 $\Delta_{\boldsymbol{\rho}}$ on half-densities and the master groupoid

The action of $\Delta_{\boldsymbol{\rho}}$ extends to densities of arbitrary weight by setting $\Delta_{\boldsymbol{\rho}}\boldsymbol{\psi} := \boldsymbol{\rho}^{w}\,\Delta_{\boldsymbol{\rho}}(\boldsymbol{\rho}^{-w}\boldsymbol{\psi})$ on w-densities. As it turns out, the case of half-densities ($w = \frac{1}{2}$) is distinguished. We have

$$[\Delta_{\boldsymbol{\rho}}, f] = \mathcal{L}_f + \frac{1}{2}\,(1-2w)\;\Delta_{\boldsymbol{\rho}} f$$

and

$$\Delta_{\boldsymbol{\rho}'} = \Delta_{\boldsymbol{\rho}} + \frac{1}{2}\,(1-2w)\mathcal{L}_{\sigma} - w(1-w)\,H(\boldsymbol{\rho}',\boldsymbol{\rho})\,,$$

where $H(\boldsymbol{\rho}',\boldsymbol{\rho}) = e^{-\sigma/2}\,\Delta_{\boldsymbol{\rho}}(e^{\sigma/2})$ as above and $\mathcal{L}_f$ denotes the Lie derivative along a Hamiltonian vector field X_f. Clearly $w = 0, 1, \frac{1}{2}$ are singular values. The case $w = \frac{1}{2}$ is distinguished by a particularly simple transformation law. On half-densities $\Delta_{\boldsymbol{\rho}'} = \Delta_{\boldsymbol{\rho}} + \frac{1}{4}\,H(\boldsymbol{\rho}',\boldsymbol{\rho})$. It follows that the solutions of the 'master equations' $\Delta_{\boldsymbol{\rho}}\,e^{\sigma/2} = 0$, for various $\boldsymbol{\rho}$, can be composed, making a groupoid, which we call the *master groupoid* of an odd Poisson manifold. On an orbit of the master groupoid, the operator $\Delta_{\boldsymbol{\rho}}$ acting on half-densities does not depend on a volume form $\boldsymbol{\rho}$.

In the odd symplectic case, one can show that the coordinate volume forms corresponding to all Darboux coordinate systems belong to the same orbit, giving a distinguished orbit [10], [1]. We call this the 'Batalin-Vilkovisky Lemma' (compare [15]). It is a proper replacement of the Liouville theorem, which is no longer valid in the odd case. Therefore on odd symplectic manifolds, though there is no natural volume form (and no such a form invariant under all canonical transformations can exist), there exists a canonical Δ-operator on half-densities independent of a choice of volume form.

For operators on functions, the square $\Delta_{\boldsymbol{\rho}}^2$, which is a Poisson vector field in general, also does not change on an orbit of the master groupoid. In particular, if for some $\boldsymbol{\rho}$ it happens that $\Delta_{\boldsymbol{\rho}}^2 = 0$, then $\Delta_{\boldsymbol{\rho}'}^2 = 0$ for all $\boldsymbol{\rho}' = e^{\sigma}\boldsymbol{\rho}$ such that $\Delta_{\boldsymbol{\rho}}\,e^{\sigma/2} = 0$. In the symplectic case, such is the "Darboux coordinate orbit". In the general odd Poisson case, the existence of such an orbit with $\Delta_{\boldsymbol{\rho}}^2 = 0$ is an open question.

3 Operators in the algebra of densities

3.1 Subprincipal symbol as upper connection

Come back to a general operator (2.1) acting on functions. We can set $R = \Delta 1$ to zero without loss of generality. Thus a reconstruction of Δ amounts to recovering the coefficients T^a. Recall the notion of Hörmander's subprincipal symbol; in our case $\operatorname{sub}\Delta = (\partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} - 2T^a)p_a$. Unlike the principal symbol, $\operatorname{sub}\Delta$ is coordinate-dependent. Precisely, $\gamma^a = \partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} - 2T^a$ has the transformation law $\gamma^{a'} = \left(\gamma^a + S^{ab}\,\partial_b \ln J\right)\frac{\partial x^{a'}}{\partial x^a}$, where $J = \frac{Dx'}{Dx}$ is the Jacobian. Thus $\operatorname{sub}\Delta$ can be interpreted as an "upper connection" in the bundle $\operatorname{Vol} M$, i.e., specifying a *contravariant derivative* $\nabla^a\rho = (S^{ab}\partial_b + \gamma^a)\rho$ on volume forms. The coordinate-dependent Hamiltonian $\gamma = \operatorname{sub}\Delta = \gamma^a p_a$ plays the role of a local connection form. If the matrix S^{ab} is invertible, then we can lower the index a to get a usual connection. (Notice that Δ acts on scalar functions, and no extra structure is assumed on our manifold a priori. Geometry arises just from the operator Δ.) Thus, a second-order operator Δ on functions (normalized by $\Delta 1 = 0$) is equivalent to a set of data: a bracket on functions and an associated upper connection in $\operatorname{Vol} M$.

Alternatively, such an upper connection in the bundle $\operatorname{Vol} M$ can be viewed as an extension of the bracket of functions $\{f, g\}$ to a 'long' bracket $\{f, \boldsymbol{\psi}\}$ where the second argument is a volume form: $\{f, \boldsymbol{\psi}\} = \left(S^{ab}\partial_b f\,\partial_a\psi(-1)^{\tilde{a}\tilde{f}} + \gamma^a\partial_a f\,\psi\right)Dx$. Thus, Δ on functions is equivalent to a bracket on functions equipped with an extension of it to volume forms (for one argument). Now we want to make the situation more symmetric.

3.2 Interlude: the algebra of densities

For a (super)manifold M the *algebra of densities* $\mathfrak{V}(M)$ consists of formal linear combinations of densities of arbitrary weights $w \in \mathbb{R}$. It contains the algebra of functions. The multiplication is the usual tensor product. We can specify elements $\boldsymbol{\psi} \in \mathfrak{V}(M)$ by generating functions $\boldsymbol{\psi}(x,t)$, which are defined on the total space of a one-dimensional bundle $\hat{M}$ over M. The algebra $\mathfrak{V}(M)$ possesses a unit 1 and a natural invariant scalar product given by the formula: $\langle\boldsymbol{\psi}, \boldsymbol{\chi}\rangle = \int_M \operatorname{Res}(t^{-2}\boldsymbol{\psi}(x,t)\boldsymbol{\chi}(x,t))\,Dx$. It can be viewed as a natural generalized volume form on the supermanifold $\hat{M}$. Hence there is a canonical divergence operator for derivations of the $\mathbb{R}$-graded algebra $\mathfrak{V}(M)$. Using it, one can classify derivations by decomposing them into the divergence-free part and the "scalar" part (see [2]).

3.3 Main theorem

Consider brackets in the algebra $\mathfrak{V}(M)$, i.e., symmetric bi-derivations. A bracket of weight zero in $\mathfrak{V}(M)$ is given by a symmetric tensor $(\hat{S}^{\hat{a}\hat{b}}) = \begin{pmatrix} S^{ab} & t\gamma^a \\ t\gamma^a & t^2\theta \end{pmatrix}$ on $\hat{M}$. From the viewpoint of M, the blocks of this matrix have the following meaning: S^{ab} specifies a bracket of functions, γ^a gives an associated upper connection in $\operatorname{Vol} M$ as above, and a new bit of data θ allows to consider brackets between volume forms, so that $\{Dx, Dx\} = \theta(Dx)^2$. It is a third-order geometric object (the transformation law involves the third derivatives of coordinates, see [2], and depends on S^{ab} and γ^a), in the same way as γ^a is a second-order object with the transformation law depending on S^{ab}. So we have something like a flag.

The component θ is completely analogous to the Brans-Dicke field g^{55} in Kaluza-Klein type models.

On the other hand, let us consider differential operators in the algebra $\mathfrak{V}(M)$. This is stronger than simply take operators acting on densities of various weights independently. In particular, a second-order operator of weight zero in the algebra $\mathfrak{V}(M)$ is a quadratic pencil of operators on w-densities of the form $\Delta_w = \Delta_0 + wA + w^2B$ where Δ_0 is a well-defined second-order operator on functions, A and B have orders one and zero, respectively (they do not have invariant meaning separately from Δ_0). Since the algebra $\mathfrak{V}(M)$ has a natural invariant scalar product, it makes sense to consider self-adjoint operators. A pencil Δ_w corresponds to a self-adjoint operator in $\mathfrak{V}(M)$ if $(\Delta_w)^* = \Delta_{1-w}$.

Theorem 1. *There is a one-to-one correspondence between operators and brackets in $\mathfrak{V}(M)$. Every second-order operator generates a bracket and, conversely, for a given bracket a generating operator always exists and can be uniquely specified by the conditions of normalization $\Delta 1 = 0$ and self-adjointness. An operator pencil Δ_w canonically corresponding to a bracket in $\mathfrak{V}(M)$ is given by the formula*

$$\Delta_w = \frac{1}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} + (2w-1)\gamma^a\right)\partial_a + \ w\,\partial_a\gamma^a(-1)^{\tilde{a}(\varepsilon+1)} + w(w-1)\,\theta\right).$$

The operator pencil defined in Theorem 1 will be shortly called the *canonical pencil* (for a given bracket in $\mathfrak{V}(M)$).

Example 3.1. The pencil considered in Section 2 and defined using a volume form $\boldsymbol{\rho}$, has $\gamma^a = S^{ab}\gamma_b$ and $\theta = \gamma^a\gamma_a$, where $\gamma_a = \partial_a \log\rho$. Let us call it the *Laplace-Beltrami pencil.*

The algebra $\mathfrak{V}(M)$ allows to link results for functions (when there is a unit, but no scalar product) and for half-densities (when there is a scalar product, but no unit). The canonical pencil corresponding to a bracket in $\mathfrak{V}(M)$ is nothing but the operator $\frac{1}{2}\operatorname{div}\operatorname{grad}$ in $\mathfrak{V}(M)$, where grad is given by a bracket and div is the canonical divergence (see above). Using this description, one can give the transformation law of the canonical pencil Δ_w under a change of γ^a and θ (see [2]), generalizing the formula for $\Delta_{\boldsymbol{\rho}}$. Taking the Laplace-Beltrami pencil as a convenient 'origin', one can get from there a useful parameterization of all canonical pencils with a given S^{ab}, see [2]. An interesting question is about the specialization map $\Delta_w \mapsto \Delta_{w_0}$ from pencils to operators on w_0-densities with a particular w_0. The values $w_0 = 0, 1, \frac{1}{2}$ are singular. For other w_0, one can find a unique canonical pencil such that Δ_{w_0} coincides with a given second-order operator on w_0-densities (no restrictions). If $w_0 = \frac{1}{2}$, the image of the specialization map consists of all self-adjoint operators on half-densities and the kernel consists of pencils $(2w-1)\mathcal{L}_X$, where X are vector fields. If $w_0 = 0$, the image of the specialization map consists of all operators vanishing on constants and the kernel is the subspace $\{w(w-1)f\}$.

4 Odd case: Jacobi identities

4.1 Algebraic statements

Suppose Δ is an odd second-order differential operator in some algebra A, generating an odd bracket in A, which we shall denote $\{\ ,\ \}$. Set for simplicity $\Delta 1 = 0$ (we assume that there is a unit). Notice that automatically $\operatorname{ord}\Delta^2 \leqslant 3$, because $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$. It is not difficult to check the following assertions:

$\operatorname{ord}\Delta^2 \leqslant 2$ is equivalent to the Jacobi condition $\sum \pm\{\{f,g\},h\} = 0$; we shall refer to it as to $\underline{Jacobi_3}$;

$\operatorname{ord}\Delta^2 \leqslant 1$ is equivalent to the $\underline{Jacobi_3}$ plus extra two conditions, which are equivalent: Δ^2 is a derivation of the associative product, Δ is a derivation of the bracket; we shall refer to the latter as to $\underline{Jacobi_2}$;

$\operatorname{ord}\Delta^2 \leqslant 0$, finally, is equivalent to all above plus $\Delta^2 = 0$, to which we shall refer as to $\underline{Jacobi_1}$.

Hence, in the notation $\underline{Jacobi_n}$, the number n stands for the number of arguments.

4.2 Geometric meaning

We can apply this to operators and brackets in $C^\infty(M)$ or $\mathfrak{V}(M)$. In the latter case we know that there is a canonical operator generating a given bracket (the canonical pencil). Keeping the above notation, we have the following (parentheses stand for the canonical Poisson bracket on T^*M):

Theorem 2. *For operators on functions:*

$$\operatorname{ord}\Delta^2 \leqslant 2 \quad \Leftrightarrow \quad \underline{Jacobi_3} \quad \Leftrightarrow \quad (S,S) = 0,$$

hence $D = (S,\ )$ *is a differential;*

$$\operatorname{ord}\Delta^2 \leqslant 1 \quad \Leftrightarrow \quad \underline{Jacobi_3} + \underline{Jacobi_2} \quad \Leftrightarrow \quad (S,S) = 0,\ (S,\gamma) = 0,$$

i.e., γ *is flat,* $D\gamma = 0$.

Notice that $D\gamma = (S,\gamma)$ plays the role of curvature for an upper connection γ; it only makes sense with $D^2 = 0$, i.e., when $\{\ ,\ \}$ is a genuine odd Poisson bracket.

A further condition that $\Delta^2 = 0$ might be seen as a version of a 'Batalin-Vilkovisky equation' for a pair S, γ of an odd Poisson bracket and a flat upper connection, but we prefer to relate 'BV equations' with changes of γ, like for $\boldsymbol{\rho}$ in Section 2.

Theorem 3. *For a unique operator corresponding to a bracket in* $\mathfrak{V}(M)$*: the conditions*

$$\operatorname{ord}\Delta^2 \leqslant 2 \quad \Leftrightarrow \quad \underline{Jacobi_3} \ \text{ in } \ \mathfrak{V}(M)$$

automatically imply the conditions

$$\operatorname{ord}\Delta^2 \leqslant 1 \quad \Leftrightarrow \quad \underline{Jacobi_3} + \underline{Jacobi_2} \ \text{ in } \ \mathfrak{V}(M)$$

and are equivalent to the equations

$$\begin{aligned} (S,S) = 0,\ (S,\gamma) &= 0 \\ (S,\theta) + (\gamma,\gamma) &= 0 \\ (\gamma,\theta) &= 0 \end{aligned}$$

We can get an interesting corollary from here. Suppose the bracket generated on functions is non-degenerate (an odd symplectic structure). Then the equations for γ, θ are solved uniquely (if we ignore constants) giving $\gamma^a = S^{ab}\gamma_b$ where $\gamma_b = -\partial_b \log \mathcal{A} = e^{\mathcal{A}}\partial_b e^{-\mathcal{A}}$ for some volume form $\boldsymbol{\rho} = e^{-\mathcal{A}}Dx$, where $\mathcal{A}$ can be interpreted as "action", and $\theta = \gamma^a\gamma_a$. Thus in this case the Jacobi conditions in the algebra of densities bring us back to the operator $\Delta_{\boldsymbol{\rho}}$.

5 Homotopy algebras

5.1 Brackets generated by operators of higher order

The following construction was essentially given by Koszul [14]. Let Δ be an operator in an arbitrary commutative $\mathbb{Z}_2$-graded algebra with a unit. Define

$$\begin{aligned} \{a\} &= [\Delta, a]1, \\ \{a,b\} &= [[\Delta, a], b]1, \\ \{a,b,c\} &= [[[\Delta, a], b], c]1, \\ &\text{etc.}; \end{aligned}$$

the n-ary bracket is obtained by taking the $n+1$ consequent commutators. One might add an 0-ary bracket, which is simply the element $\Delta 1$. Δ is a differential operator of order $\leqslant N$ if all brackets with more than N arguments vanish. In such case the top nonzero bracket is a multi-derivation. It is the polarization of the symbol of Δ. For $N = 2$ we return to the situation considered above. One can check that all brackets obtained in this way are symmetric and that the $(k+1)$-fold bracket appears as the obstruction to the Leibniz rule for the k-fold bracket. This holds for even and odd operators.

5.2 Case of odd operators

Suppose that Δ is odd. Hence all the brackets generated by Δ are also odd. Can we get, in addition to those linked 'Leibniz identities' holding automatically, some sort of Jacobi identities for these brackets?

Let $a_1, \ldots, a_n$ be elements of A. We do not need multiplication here, so for a moment one can think of A just as of a vector space. Define the *n-th Jacobiator* on $a_1, \ldots, a_n$ by the formula

$$J^n(a_1, \ldots, a_n) = \sum_{k+l=n} \sum_{(k,l)\text{-shuffles}} (-1)^{\alpha} \{\{a_{\sigma(1)}, \ldots, a_{\sigma(k)}\}, a_{\sigma(k+1)}, \ldots, a_{\sigma(k+l)}\}.$$

Here $(-1)^{\alpha}$ is the sign prescribed by the sign rule for a permutation of homogeneous elements $a_1, \ldots, a_n \in A$. This definition assumes the existence in A of odd brackets of all orders between 0 and n. The *n-th Jacobi identity* for a sequence of odd brackets in A is the equality $J^n = 0$ identically for all arguments.

Up to sign conventions, this is essentially the definition used in Stasheff's theory of strongly homotopy algebras: a vector space endowed with a sequence of (super)symmetric odd brackets such that all Jacobi identities hold, for all $n = 0, 1, 2, 3, \ldots$, is a *strongly homotopy Lie algebra* (an *L_∞-algebra*). Very often it is assumed that the distinguished element given by the 0-ary bracket and called a *background* vanishes. See, e.g., [16], [17], where slightly different conventions are used.

Coming back to our situation when the brackets are generated by an odd operator Δ, we have the following remarkable statement [3].

Theorem 4. *The following conditions are equivalent:*

(1) $\operatorname{ord} \Delta^2 \leqslant r$;

(2) *all Jacobi identities with the number of arguments $> r$ are satisfied.*

In particular, if $\Delta^2 = 0$, we get an L_∞-algebra structure in A besides the associative multiplication (compare also [18]). This is a particular example of "homotopy Batalin-Vilkovisky algebras".

Theorem 4 is a corollary of the general Theorem 5 below, valid in the following abstract axiomatic setting suggested in [3]. Consider a Lie superalgebra $\mathfrak{g}$ endowed with a projector P such that $\operatorname{Im} P$ and $\operatorname{Ker} P$ are subalgebras and $\operatorname{Im} P$ is Abelian. Let $D \in \mathfrak{g}$ be an arbitrary element. The *n-th derived bracket* of D is a symmetric multilinear operation on the space $V = \operatorname{Im} P$ defined by the formula

$$\{a_1, \ldots, a_n\}_\Delta := P[\ldots[[\Delta, a_1], a_2], \ldots, a_n],$$

where $a_i \in V$.

Theorem 5. *Consider an odd element $\Delta \in \mathfrak{g}$. The n-th Jacobiator of the derived brackets of Δ is equal to the n-th derived bracket of the even element Δ^2:*

$$J^n_\Delta(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}_{\Delta^2}.$$

In our example the Lie superalgebra $\mathfrak{g}$ consists of all operators in a commutative associative algebra with a unit A, the projector P is the evaluation on the unit element, $\operatorname{Im} P$ being A. Theorem 4 immediately follows.

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References

[1] H. Khudaverdian and Th. Voronov. On odd Laplace operators. *Lett. Math. Phys.*, 62:127–142, 2002, `arXiv:math.DG/0205202`.

[2] H. Khudaverdian and Th. Voronov. On odd Laplace operators. II. Contemp. Math. Amer. Math. Soc., Providence, RI, 2004 (to appear), `arXiv:math.DG/0212311`.

[3] Th. Voronov. Higher derived brackets and homotopy algebras. *J. of Pure and Appl. Algebra* (to appear), `arXiv:math.QA/0304038`.

[4] H. Khudaverdian and Th. Voronov. Geometry of differential operators, and odd Laplace operators. *Russian Math. Surveys*, 58:197–198, 2003, `arXiv:math.DG/0301236`.

[5] I. A. Batalin and G. A. Vilkovisky. Gauge algebra and quantization. *Phys. Lett.*, 102B:27–31, 1981.

[6] O.[H.] M. Khudaverdian. Geometry of superspace with even and odd brackets. Preprint of the Geneva University, UGVA-DPT 1989/05-613, 1989. Published in: *J. Math. Phys.* 32 (1991), 1934–1937.

[7] A. Weinstein. The modular automorphism group of a Poisson manifold. *J. Geom. Phys.*, 23(3-4):379–394, 1997.

[8] Y. Kosmann-Schwarzbach and J. Monterde. Divergence operators and odd Poisson brackets. *Ann. Inst. Fourier*, 52:419–456, 2002, `arXiv:math.QA/0002209`.

[9] O.[H.] M. Khudaverdian. Δ-operator on semidensities and integral invariants in the Batalin–Vilkovisky geometry. Preprint 1999/135, Max-Planck-Institut für Mathematik Bonn, 19 p., 1999, arXiv:math.DG/9909117.

[10] H. Khudaverdian. Semidensities on odd symplectic supermanifold. *Comm. Math. Phys.*, 2004 (to appear), arXiv:math.DG/0012256.

[11] A. Schwarz. Symmetry transformations in Batalin-Vilkovisky formalism. *Lett. Math. Phys.*, 31(4):299–301, 1994.

[12] Y. Kosmann-Schwarzbach. From Poisson algebras to Gerstenhaber algebras. *Ann. Inst. Fourier, Grenoble*, 46:1243–1274, 1996.

[13] Y. Kosmann-Schwarzbach. Derived brackets, arXiv:math.DG/0312524.

[14] J.-L. Koszul. Crochet de Schouten-Nijenhuis et cohomologie. *Astérisque*, (Numero Hors Serie):257–271, 1985. The mathematical heritage of Élie Cartan (Lyon, 1984).

[15] I. A. Batalin and G. A. Vilkovisky. Closure of the gauge algebra, generalized Lie equations and Feynman rules. *Nucl. Phys.*, B234:106–124, 1984.

[16] T. Lada and J. Stasheff. Introduction to SH Lie algebras for physicists. *Internat. J. Theoret. Phys.*, 32(7):1087–1103, 1993, arXiv:hep-th/9209099.

[17] T. Lada and M. Markl. Strongly homotopy Lie algebras. *Comm. Algebra*, 23(6):2147–2161, 1995, arXiv:hep-th/9406095.

[18] K. Bering, P. H. Damgaard, and J. Alfaro. Algebra of higher antibrackets. *Nuclear Phys. B*, 478(1-2):459–503, 1996, arXiv:hep-th/9604027.