A Family of Linearizations of Autonomous Ordinary Differential Equations with Scalar Nonlinearity

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Abstract

This paper deals with a method for the linearization of nonlinear autonomous differential equations with a scalar nonlinearity. The method consists of a family of approximations which are time independent, but depend on the initial state. The family of linearizations can be used to approximate the derivative of the nonlinear vector field, especially at equilibrium points, which are of particular interest, it can be used also to determine the asymptotic stability of equilibrium point, especially in the non-hyperbolic case. Using numerical experiments, we show that the method presents good agreement with the nonlinear system even in the case of highly nonlinear systems.

1 Introduction

Many physical systems are modeled by nonlinear autonomous differential equations of the following form

$$\frac{d^k x(t)}{dt^k} + b_{k-1}\frac{d^{k-1}x(t)}{dt^{k-1}} + ... + b_0 x(t) = f(x(t)) \qquad (1.1)$$

where $f : I\!R \longrightarrow I\!R$ is a nonlinear function, b_i are real numbers, k is the order of the system. If we put $x(t) = x_1(t), \frac{dx(t)}{dt} = x_2(t), ..., \frac{d^{k-1}x(t)}{dt^{k-1}} = x_k(t)$, system (1.1) is written equivalently as

$$\begin{cases} \frac{dx_i(t)}{dt} = x_{i+1}(t), i = 1, ..., k-1 \\ \frac{dx_k(t)}{dt} = -b_0 x_1(t) - b_1 x_2(t) - ... - b_{k-1} x_k(t) + f(x_1(t)) \end{cases} \qquad (1.2)$$

We associate to system (1.2) an initial state $x_1(t_0) = x_{10}, ..., x_k(t_0) = x_{k0}$. We assume that conditions on f for which the solution of the initial value problem exists and is unique are satisfied. Note that in general, the nonlinear vector field is a function of all the state variables, system (1.2) is a particular case. Of course, the dynamics of the system depend crucially on the scalar nonlinear function. System (1.2) associated with an initial state has the following general form

$$\begin{cases} \frac{dX(t)}{dt} = F\left(X\left(t\right)\right) \\ X\left(t_0\right) = X_0 \end{cases} \tag{1.3}$$

where $X = [x_1, x_2, ..., x_k]^T$ is the state vector, $F : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is the vector field.

In general, when dealing with the dynamics of nonlinear systems, linearization plays an important role; since it provides simple models which are locally equivalent to nonlinear systems. In fact, it turns out in many situations that nonlinear systems can be approximated in some regions of operation by linear systems. Furthermore, the theory of linear systems is more complete comparing to nonlinear systems. A classical example is the asymptotic stability, where the method the most used for the determination of the stability type of an equilibrium point is based on the linearization around the equilibrium.

The most classical linearization is based on Fréchet derivative at the equilibrium point. Recall that an equilibrium point X_{eq} of system (1.3) satisfies $F\left(X_{eq}\right) = 0$, where $X_{eq} = [x_{eq1}, x_{eq2}, ..., x_{eqk}]^T$. In the particular case of system (1.2), the equilibrium is given by $X_{eq} = [x_{eq1}, 0, ..., 0]^T$. The classical linearization near the equilibrium point X_{eq} is given by

$$DF\left(X_{eq}\right) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ r_0 & -b_1 & -b_2 & \cdots & -b_{k-1} \end{bmatrix} \tag{1.4}$$

where $r_0 = -b_0 + f^{'}\left(x_{eq1}\right)$. $DF\left(X_{eq1}\right)$ is the Jacobian matrix at the equilibrium point. Unfortunately the classical linearization near an equilibrium point X_{eq} has the following drawbacks

1. When $DF\left(X_{eq}\right)$ presents at least a zero eigenvalue or a pair of purely imaginary eigenvalues, there is no equivalence between the linear and the nonlinear systems, i.e., the behavior of the linearized system can be very different from the nonlinear system. This statement results from the Hartman-Grobman theorem.

2. The classical linearization is a first order approximation, the quality of the approximation degenerates for highly nonlinear systems.

3. In many cases, it happens that the Jacobian matrix at the equilibrium point does not exist. Classical linearization theorems do not apply in this case.

As a result to these drawbacks, various complementary methods have been suggested to be used for some specific problems, and especially to overcome the drawbacks of the classical linearization.

A least squares linearization method was suggested in [2] and [3] to approximate nonlinear circuits equations. Benouaz and Arino [4] gave to the method the mathematical validity and they show the applicability of the method to solve stability problems. In fact they have shown that in the first order case, the nonlinear system described by

$$\begin{cases} \frac{dx(t)}{dt} = -b_0 x\left(t\right) + f\left(x\left(t\right)\right) \\ x\left(t_0\right) = x_0 \end{cases} \tag{1.5}$$

can be approximated near the origin by the following linear system

$$\begin{cases} \frac{dx(t)}{dt} = ax(t) \\ x(t_0) = x_0 \end{cases} \tag{1.6}$$

where a is a real number given by

$$a = -b_0 + \frac{2}{x_0^2} \int_0^{x_0} f(x)\, dx \tag{1.7}$$

where the approximation is not defined for $x_0 = 0$. The real number a was computed under the assumption that the nonlinear vector field presents negative spectrum around the origin; this is an important restriction. In this paper, our aim is to make some progress regarding the approximation. We first relax the assumption of the negative spectrum and second, we introduce a family of linearizations which is a generalization of equation (1.7). We show that the method allows to approximate the derivative of the nonlinear vector field. We also establish a comparison with the classical linearization.

2 Generalization of the approximation

In this section, we relax the assumption on the spectrum of f in the first order case, we generalize the method to a family of approximations, and we show that the generalization to systems of order k is straightforward. Consider the first order nonlinear system given by (1.5) with $f(0) = 0$. We associate to system (1.5) a linear system such as (1.6). In [4], the linearized system was determined by the minimization of the following cost function

$$\Phi(a) = \int_{t_0}^{+\infty} \left| -b_0 x(t) + f(x(t)) - ax(t) \right|^2 dt \tag{2.1}$$

under the assumption that $f'(x)$ is negative around the origin. The solution obtained by the minimization of the cost function (2.1) is the following

$$a_{t\geq 0} = -b_0 + \frac{\int_{t_0}^{+\infty} [f(x(t))\, x(t)]\, dt}{\int_{t_0}^{+\infty} x^2(t)\, dt} \tag{2.2}$$

where $a_{t\geq 0}$ denotes the real number a obtained by the minimization of the cost function (2.1) for forward time.

The assumption of negative spectrum is necessary for the convergence of the solution, however it is easy to see that the convergence problem can be solved when $f'(x)$ is positive near the origin by considering the backward evolution of time. We denote by $a_{t\leq 0}$ the real number a obtained by the minimization of the cost function (2.1) for backward time. We get in this case

$$a_{t\leq 0} = -b_0 + \frac{\int_{+\infty}^{t_0} [f(x(t))\, x(t)]\, dt}{\int_{+\infty}^{t_0} x^2(t)\, dt} \tag{2.3}$$

By inverting the bounds of the integration in (2.3), it turns out that $a_{t\geq 0}$ and $a_{t\leq 0}$ are computed using the same formula. After a variable change $t \to x(t)$; where $x(t)$ is the solution of the linear system, with $dt = \frac{dx}{ax(t)}$, we get

$$a = -b_0 + \frac{\int_0^{x_0} f(x)\,dx}{\int_0^{x_0} x dx} = -b_0 + \frac{2}{x_0^2}\int_0^{x_0} f(x)\,dx \tag{2.4}$$

Equation (2.4) is valid for both $a_{t\geq 0}$ and $a_{t\leq 0}$, and the approximation is unique and optimal in the least squares sense.

As a generalization of the approximation given by equation (2.4), we define the following family of approximations

$$a_n = -b_0 + \frac{n+2}{x_0^{n+2}}\int_0^{x_0} x^n f(x)\,dx \tag{2.5}$$

where n is an integer which satisfies $n \geq 0$. a_n are not defined for $x_0 = 0$. Qualitatively, there is no difference between different a_n, but we will see in the application that there exists some difference in terms of the error due to the approximations. In fact, for higher values of n, the error in the neighborhood of the initial state decreases considerably, and increases in the neighborhood of the origin. The approximations a_n can be seen as the result of the minimization of the following cost functions

$$\begin{aligned} \Phi(a_n) &= \int_{t_0}^{+\infty} \left| x^n(t)\left[-b_0 x(t) + f(x(t)) - a_n x(t)\right]\right|^2 dt \\ n &= 0, 1, 2, ... \end{aligned} \tag{2.6}$$

Equation (2.4) corresponds to the particular case when $n = 0$. When the initial state is small, the following linear systems

$$\begin{cases} \frac{dx}{dt} = a_n x(t), n = 0, 1, 2, ... \\ x(t_0) = x_0 \end{cases} \tag{2.7}$$

approximate the nonlinear vector field near the origin, and the solutions of the linear systems which are given by $x(t) = e^{a_n(t-t_0)}x_0; (n = 0, 1, 2...)$ approximate the nonlinear solution. Note that a_n are time invariant but they depend on the initial state. This makes an important difference with the classical linearization which is independent of the initial state.

Similar results can be obtained for systems of order k. We associate the following linear system to equation (1.1)

$$\begin{cases} \frac{d^k x(t)}{dt^k} + b_{k-1}\frac{d^{k-1}x(t)}{dt^{k-1}} + ... + b_1\frac{dx(t)}{dt} = ax(t) \\ X(t_0) = X_0 \end{cases} \tag{2.8}$$

where a is a real number to be determined.

We assume that the origin is an equilibrium point for the nonlinear vector field, i.e., $F(0) = 0$ which means that $f(0) = 0$. This is not a restriction since it can be performed by a simple shift of coordinates. The classical linearized system near the origin has the following form

$$\begin{cases} \frac{d^k x(t)}{dt^k} + b_{k-1}\frac{d^{k-1}x(t)}{dt^{k-1}} + ... + b_1\frac{dx(t)}{dt} = \left[-b_0 + f'(0)\right] x(t) \\ X(t_0) = X_0 \end{cases} \tag{2.9}$$

The linear equivalent system (2.8) is obtained by straightforward generalization of the first order case, and we get the following family of linear systems ($n = 0, 1, 2...$)

$$\begin{cases} \frac{d^k x(t)}{dt^k} + b_{k-1}\frac{d^{k-1}x(t)}{dt^{k-1}} + ... + b_1\frac{dx(t)}{dt} = a_n x(t) \\ X(t_0) = X_0 \end{cases} \tag{2.10}$$

where a_n are given by

$$a_n = -b_0 + \frac{n+2}{x_0^{n+2}} \int_0^{x_0} x^n f(x)\,dx \tag{2.11}$$

The second term in the expression of a_n is due to the nonlinear function. This term will be denoted $\tilde{a}_n$ and is given by

$$\tilde{a}_n = \frac{n+2}{x_0^{n+2}} \int_0^{x_0} x^n f(x)\,dx \tag{2.12}$$

The coefficients $\tilde{a}_n$ allow us to approximate the area under $f(x)$ in the interval $x \in [0, x_0]$. This can be seen easily in the particular case when $n = 0$. The area under f is given by

$$S = \int_0^{x_0} f(x)\,dx \tag{2.13}$$

It is easy to see from the formula of a_n that S is given by

$$S = a_0 \frac{x_0^2}{2} \tag{2.14}$$

This property is illustrated in figure 1 for $f(x) = x^3, x_0 = 4$. The area under $f(x)$ is compared with the area under the lines of slopes $\tilde{a}_0$, $\tilde{a}_1$ and $\tilde{a}_2$, with $\tilde{a}_0 = 8$, $\tilde{a}_1 = \frac{48}{5}$, $\tilde{a}_2 = \frac{16}{3}$.
In the next section, we discuss the relation between the approximation and the derivative of the nonlinear vector field.

3 Approximation of the derivative of the nonlinear vector field

In this section, we discuss the relation of the family of approximations a_n with the classical linearization near the origin, which is assumed to be an equilibrium point for the nonlinear system i.e., $f(0) = 0$. In fact, it can be proven using a simple L'Hopital's rule that

$$f'(0) = \lim a_n(x_0) + b_0; x_0 \to 0 \tag{3.1}$$

So, if x_0 is very small, a_n can be seen as a perturbation of $f'(0) - b_0$. In this case, we can write

$$a_n(x_0) = -b_0 + f'(0) + h(x_0) \tag{3.2}$$

where $h(x_0)$ is a small quantity which goes to zero when $x_0 \to 0$.

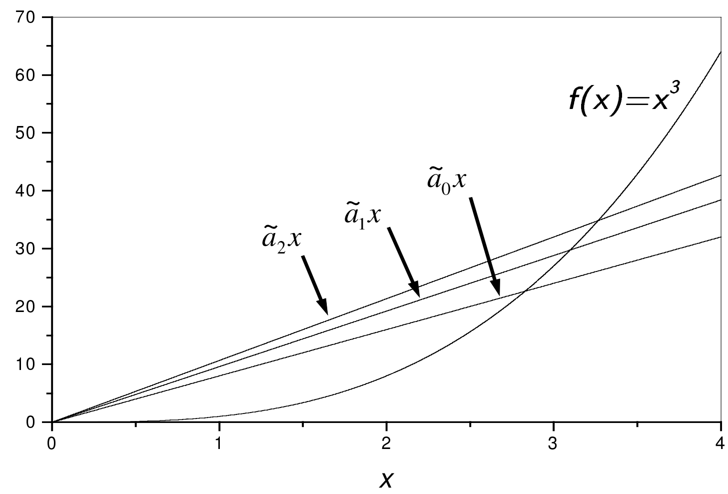

Figure 1. An illustration of the area approximation property

It is also possible to show that the derivative at any point $x = \beta$ of the nonlinear vector field can be approximated using a_n. Let us define a modified version of a_n by

$$c_n(x_0, \beta) = -b_0 + \frac{n+2}{x_0^{n+2}} \int_0^{x_0} x^n \left[f(x+\beta) - f(\beta) \right] dx \tag{3.3}$$

with $f(0) = 0$. The derivative of f at the point β can be computed by

$$f^{'}(\beta) = \lim c_n(x_0, \beta) + b_0; x_0 \to 0 \tag{3.4}$$

This also can be proven using L'Hopital's rule. We have when $x_0 \to 0$

$$\lim c_n(x_0, \beta) = \lim \left[-b_0 + \frac{f(x_0+\beta) - f(\beta)}{x_0} \right] = -b_0 + f^{'}(\beta) \tag{3.5}$$

This means that if x_0 is very small, then we can write

$$c_n(x_0, \beta) = -b_0 + f^{'}(\beta) + h(x_0) \tag{3.6}$$

where $h(x_0)$ is a small quantity which goes to zero when $x_0 \to 0$. The relationship of the method with the classical linearization shows the evidence of the method and the possibility to use $a_n(x_0)$ to study the stability of the origin in a similar way to the Fréchet derivative when x_0 is small. Furthermore, the method can be used for the numerical calculations of the derivative. This result can be extended easily to systems of order k. We define the following matrices ($n = 0, 1, 2...$)

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_n & -b_1 & -b_2 & \cdots & -b_{k-1} \end{bmatrix} \tag{3.7}$$

where a_n are given by (2.11). Matrices A_n approximate the Jacobian matrix at the origin when the initial state is small. Furthermore, matrices

$$B_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_n & -b_1 & -b_2 & \cdots & -b_{k-1} \end{bmatrix} \tag{3.8}$$

approximate the Jacobian matrix at the point $X = [\beta, x_2, ..., x_k]^T$, where c_n are given by (3.3).

4 Order of the approximation

The classical linearization is well-known to be a first order linearization method. This is not the case for the family of linearizations suggested here. To show this property we consider the particular case where the nonlinear function is written under the form of a polynomial, i.e.,

$$\begin{aligned} f(x) &= c_1 x + c_2 x^2 + c_3 x^3 + ... + c_k x^k \\ &= \sum_{j=1}^{k} c_j x^j \end{aligned} \tag{4.1}$$

The family of approximations gives

$$a_n = \sum_{j=1}^{k} \frac{(n+2)}{n+j+1} c_j x_0^{j-1} \tag{4.2}$$

and the linearized system is the following

$$\dot{x} = a_n x = \left[\sum_{j=1}^{k} \frac{(n+2)}{n+j+1} c_j x_0^{j-1} \right] x \tag{4.3}$$

It is clear that system (4.3) has the same order k as the nonlinear function, and the order does not depend on n. Equation (4.3) shows the classical linearization (where the only term kept is the term in c_1) as a particular case of the family of linearizations introduced here.

5 Relation with the asymptotic stability

In this section, we discuss, the relation of the method with the asymptotic stability of the equilibrium point and we focus on the case when the equilibrium is non-hyperbolic. In general, a non-hyperbolic equilibrium point is characterized by at least a zero eigenvalue or a pair of eigenvalues with zero real parts. Since the nonlinear function is scalar, we focus on the scalar case. Let us rewrite system (1.6) under the following form

$$\begin{aligned} \tfrac{dx}{dt} &= g(x) \\ x(t_0) &= x_0 \end{aligned} \tag{5.1}$$

with

$$g(x) = b_0 x + f(x) \tag{5.2}$$

In the scalar case, a non-hyperbolic equilibrium point is characterized by $g'(x_{eq}) = 0$. (obviously, x_{eq} is hyperbolic when $g'(x_{eq}) \neq 0$). Usually the stability of a nonlinear system near its equilibrium point is deduced from linearization at the equilibrium point. The following theorem states condition under which linear and nonlinear systems are equivalent.

Theorem 1. *(Hartman-Grobman)*
If the linearization leads to hyperbolic equilibrium point, then the nonlinear system near an equilibrium point is topologically equivalent its linearized system.

As a result to this theorem, the stability type of a hyperbolic equilibrium point can be deduced from linearization. However, this is not the case with non-hyperbolic equilibrium points, and the stability depends on higher order terms. The classical linearization fails in this case. Our aim is to show that the suggested method can be used to overcome this problem. The linearized system is characterized by

$$a_n = \frac{2+n}{x^{2+n}} \int_0^{x_0} x^n g(x)\, dx \tag{5.3}$$

We assume for simplicity and without loss of generality that the origin is an equilibrium point for the nonlinear system, which means that $g(0) = 0$. A non-hyperbolic equilibrium point situated at the origin is characterized by $g'(0) = 0$. $a_n(x_0)$ can be used in the same way as $g'(0)$ to determine the nature of an equilibrium point. For the linearization near the origin, x_0 is chosen small. In this section we consider a_0 only. Note that according to the sign of $a_0(x_0)$ we have

1. $a_0(x_0) < 0$ for $x_0 \in [-\delta, \delta]$, where δ is a small real number.

2. $a_0(x_0) > 0$ for $x_0 \in [-\delta, \delta]$.

3. $a_0(x_0) > 0$ for $x_0 \in (0, \delta]$ (or $[-\delta, 0)$), and $a_0(x_0) < 0$, for $x_0 \in [-\delta, 0)$ (or $(0, \delta]$).

Our aim is to show the possibility of construction of Lyapunov function using the linearization. For $n = 0$, let's define

$$v(x) = \frac{1}{2} x^2 a_0(x) \tag{5.4}$$

Clearly, $v(x)$ has the same sign as $a_0(x)$ with $v(0) = 0$.

Proposition 1. *If $v(x) < 0$ for $x \in [-\delta, \delta] - \{0\}$, where δ is a small real number, then $v(x)$ is a Lyapunov function and the origin is asymptotically stable.*

Proof. In this case, $v(x) < 0$ for $x \neq 0$ and $v(0) = 0$, which means that $v(x)$ is negative definite candidate Lyapunov function. The time derivative of $v(x)$ is given by $\frac{\partial v(x)}{\partial t} = g^2(x)$, which is positive, thus $v(x)$ is a Lyapunov function, and the origin is asymptotically stable. ■

Proposition 2. *If $v(x) > 0$ for $x \in [-\delta, \delta] - \{0\}$, where δ is a small real number, then the origin is unstable.*

Proof. In this case, $v(x) > 0$ for $x \neq 0$ and $v(0) = 0$, which means that $v(x)$ is positive definite. The time derivative of $v(x)$ is given by $\frac{\partial v(x)}{\partial t} = g^2(x)$, which is positive, thus the origin is unstable. ■

Case (1) corresponds to negative definite Lyapunov function, with positive derivative with respect to time, the origin is asymptotically stable. Case (2) corresponds to positive definite Lyapunov function with positive derivative with respect to time, the origin is unstable. In case (3), the origin is half stable, where the stability depends on the sign of x_0. The case of half stable equilibrium exists only for non-hyperbolic equilibria.
The construction of Lyapunov function using $a_0(x)$ is valid for both hyperbolic and non-hyperbolic equilibria, however it is more useful in the case of non-hyperbolic equilibria. We restricted ourselves to the case of $n = 0$ because the Lyapunov function constructed using $a_n(x)$ for higher values of n results in complicated functions. An illustration is given in the next section.

Example. In this example we illustrate the construction of Lyapunov function for a non-hyperbolic equilibrium point. Consider the following nonlinear system

$$\begin{cases} \frac{dx(t)}{dt} = rx - \alpha x^3 \\ x(t_0) = x_0 \end{cases} \tag{5.5}$$

with $\alpha > 0$. This system undergoes a pitchfork bifurcation at $r = 0$. We are interested in studying the stability of the origin at the bifurcation value ($r = 0$), since the origin is a non-hyperbolic point at this value. In this case the classical linearization fails as a result of Hartman-Grobman theorem. Using the suggested method we get (for $r = 0$)

$$a_0 = -\frac{2}{4}\alpha x_0^2 \tag{5.6}$$

The construction of the Lyapunov function from (5.4) gives us

$$v(x) = -\frac{1}{4}\alpha x^4 \tag{5.7}$$

Clearly, we have:
(i) $v(0) = 0$
(ii) $v(x) < 0$ for $x \neq 0$.
Since conditions (i) and (ii) are satisfied, $v(x)$ is a candidate Lyapunov function. The time derivative of $v(x)$ is given by

$$\frac{\partial v(x)}{\partial t} = \left(\alpha x^3\right)^2 \tag{5.8}$$

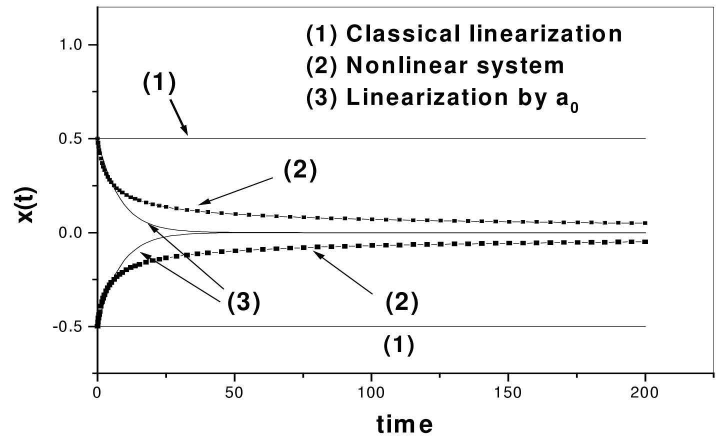

Figure 2. Solution for system (5.5) and its linearized systems for $r = 0$

The time derivative of $v(x)$ is positive, which means that the origin is asymptotically stable. The solutions for $r = 0$ of the nonlinear system (5.5), the classical linearization and linearization given by a_0 for $x_0 = \pm 0.5$ are shown in figure 2. Clearly, the classical linearization does not reflect the nonlinear behavior. However there exists good agreement between the linearization given by a_0 and the nonlinear system. Note that in the case where $\alpha < 0$, a_0 is positive (which corresponds to case (2)) and $v(x)$ is positive definite with positive derivative with respect to time, thus the origin is unstable in this case.

The property which allows us to determine the nature of non-hyperbolic equilibria using a_n is the order of the method. In fact a_n has the same order as the nonlinearity as we mentioned previously.
A comparison between the classical linearization and the suggested linearization when $g(x)$ is anlytic function can be shown as follows:
Let $g(x)$ be an analytic function, then we can write (the origin is an equilibrium point)

$$g(x) = g'(0)\,x + \frac{g^{(2)}(0)}{2!}x^2 + ... + \frac{g^{(m)}(0)}{m!}x^m + ... \tag{5.9}$$

The classical linearization is given by

$$\frac{dx}{dt} = g'(0)\,x \tag{5.10}$$

and fails when $g'(0) = 0$. We get for $a_0(x_0)$

$$a_0(x_0) = g'(0) + \frac{g^{(2)}(0)}{2!}\frac{2x_0}{3} + \frac{g^{(3)}(0)}{3!}\frac{2x_0^2}{4} + ... + \frac{g^{(m)}(0)}{m!}\frac{2x_0^{m-1}}{m+1} + ... \tag{5.11}$$

where x_0 is near the origin. Thus, even when $g'(0) = 0$, $a_0(x_0)$ does not vanish, since higher order terms are included in the expression of a_0 as a function of the initial state. The generalization to equations of order k is simple in this case, since the nonlinear function is scalar.

The main drawback of the method appears for high values of n, where it becomes difficult to find a_n in closed form for many nonlinear functions. In this situation, numerical integration becomes necessary to compute a_n.

The next section is to test the method by considering a numerical example for a second order system.

6 Numerical example

For the application, we consider a nonlinear system modeling an overdamped bead on rotating hoop [1]. The system is modeled by the following equation

$$mr\frac{d^2\phi}{dt^2}+b\frac{d\phi}{dt}=-mg\sin(\phi)+mr\omega^2\sin(\phi)\cos(\phi) \tag{6.1}$$

where ϕ is an angle, b is the damping constant, ω is a constant angular velocity and r is a radius. If we put $\varepsilon=\frac{m^2gr}{b^2}$, $\gamma=\frac{t\omega^2}{g}$, $\tau=\frac{mg}{b}$, we get following the dimensionless system

$$\frac{d^2\phi}{dt^2}+\frac{1}{\varepsilon}\frac{d\phi}{dt}=\frac{1}{\varepsilon}\left[-\sin(\phi)+\gamma\sin(\phi)\cos(\phi)\right] \tag{6.2}$$

with

$$f(\phi)=\frac{1}{\varepsilon}\left[-\sin(\phi)+\gamma\sin(\phi)\cos(\phi)\right] \tag{6.3}$$

We suggest to linearize the system of equation (6.2) near the origin, the closed form solution for a_0 is given by

$$a_0=\frac{2}{\phi_0^2}\left[\frac{1}{\varepsilon}\left(\cos(\phi_0)-1\right)+\frac{\gamma}{\varepsilon}\left(\frac{1}{2}-\frac{1}{2}\cos^2(\phi_0)\right)\right] \tag{6.4}$$

Observe that a_0 depends on the nonlinear model parameters γ and ε. For $\gamma=0.1$, $\varepsilon=0.2$, $\phi(t_0)=\frac{\pi}{2}$ and $\frac{d\phi(t_0)}{dt}=1$, we get for the following table for different values of n

n	0	1	2	3	4	10
a_n	−3.8502	−3.7182	−3.62974	−3.5663	−3.5186	−3.3751

Table 1: values of a_n for $n=0,1,2,3,4$ and 10. Where a_n are obtained by a simple integration of the nonlinear vector field. The classical linearized system is given by

$$DF(0)=\begin{bmatrix}0 & 1\\ f'(0) & -\frac{1}{\varepsilon}\end{bmatrix}=\begin{bmatrix}0 & 1\\ \frac{1}{\varepsilon}(\gamma-1) & -\frac{1}{\varepsilon}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4.5 & -5\end{bmatrix} \tag{6.5}$$

The solutions for $\phi(t)$ and $\frac{d\phi(t)}{dt}$ for a_0, a_1, a_2 and $DF(0)$ and the nonlinear system are depicted in figures 3 and 4, respectively. The absolute error due to the approximations $\|x_{nonlinear}-x_{linear}\|$ is depicted in figure 5. The error goes to zero uniformly when time goes to infinity. We observe that near the initial state, the approximation is better for

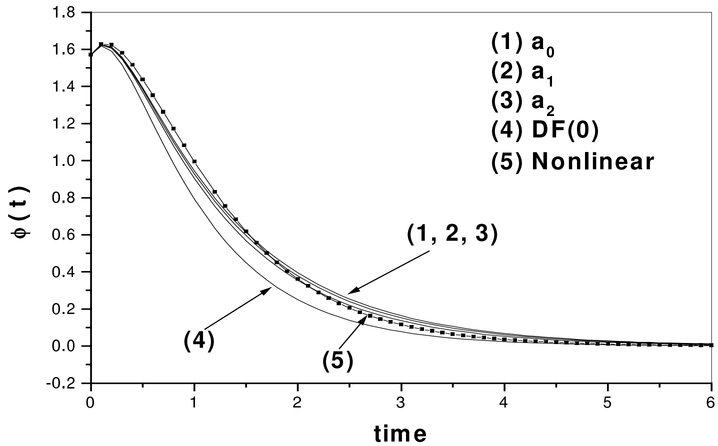

Figure 3. Solution for $\phi(t)$

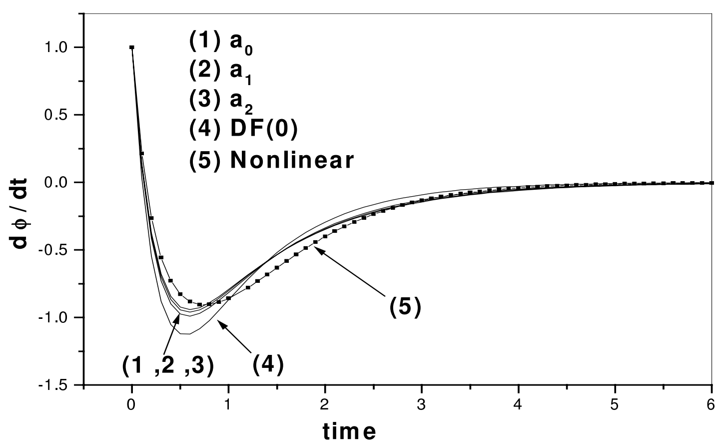

Figure 4. Solution for $\frac{d\phi(t)}{dt}$

higher values of n, and all a_n are better than the classical linearization. In the neighborhood of the origin, the classical linearization becomes the best. This confirms that the classical linearization is the best approximation near the origin.

ϕ_0	$\pi/3$	$\pi/4$	$\pi/5$	$\pi/6$	$\pi/7$
a_0	-4.2175	-4.3429	-4.4001	-4.4488	-4.4493
a_1	-4.1603	-4.3113	-4.3800	-4.4169	-4.4391
a_2	-4.1221	-4.2910	-4.3665	-4.4076	-4.4323

Table 2: values of a_n for $n = 0, 1, 2$ for different values of the initial state ϕ_0.

Table 2 shows a_n for $n = 0$, 1, and 2 for different initial states. It is clear that all the a_n go to $f'(0)$ when the initial state x_0 becomes smaller. This shows the ability of a_n to approximate $f'(0)$ when $x_0 \to 0$, and confirms our previous result about the relation between the approximation and the derivative of the nonlinear vector field at the origin.

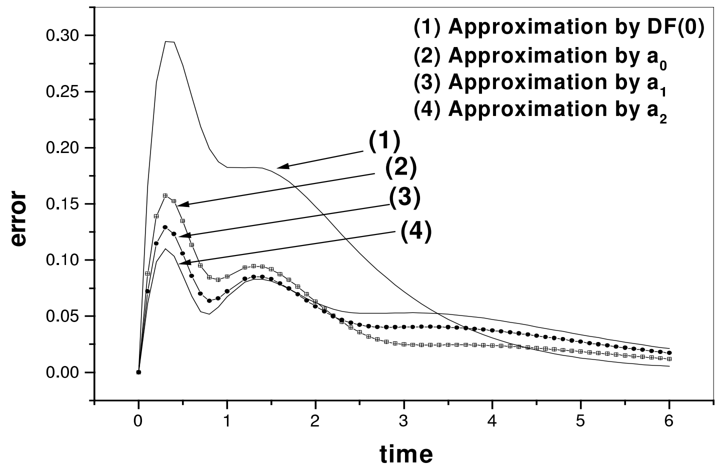

Figure 5. Error due to the approximations

7 Conclusion

In this paper, we presented a method for the linearization of a class of nonlinear autonomous systems. The method consists of a family of linearizations which are time independent but depend on the initial state. It is shown that the method can approximate the derivative of the nonlinear vector field. A relation between the method and the stability type of a non-hyperbolic equilibrium point is elaborated. Using numerical experiments, it is shown that the method presents good agreement with the nonlinear system. Furthermore, the family of linearizations present better performance in terms of the error in the neighborhood of the initial state.

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